Mini-Course on String Theory (Sunil Mukhi)

Exercise Sheet, No. 1

1. Show that the action

$$S = \tfrac{1}{2}\int dt\,\left(\frac{1}{\sqrt{g}}\,\dot{X}^\mu \dot{X}_\mu - m^2\sqrt{g}\right)$$

is invariant under

$$\begin{aligned} X^\mu(t) &\;\to\; X'^\mu(t') = X^\mu(t) \\ g(t) &\;\to\; g'(t') = \left(\frac{dt}{dt'}\right)^2 g(t) \end{aligned}$$

Also write the infinitesimal form of this transformation and check invariance under that.

2. Show that on solving the equations of motion for g and substituting in the action, one finds the new action:

$$S = -m\int dt\,\sqrt{-\dot{X}^\mu \dot{X}_\mu}$$

Show that this action is also invariant under reparametrisations of t.

3. Derive the equations of motion for the bosonic string:

$$\begin{aligned} \frac{\delta S}{\delta g^{ab}(\sigma,t)} &= 0 \;\Rightarrow\; \partial_a X^\mu \partial_b X_\mu - \tfrac{1}{2}\,g_{ab}\,g^{cd}\,\partial_c X^\mu \partial_d X_\mu = 0 \\ \frac{\delta S}{\delta X^\mu(\sigma,t)} &= 0 \;\Rightarrow\; \partial_a\left(\sqrt{-\|g\|}\,g^{ab}\,\partial_b X_\mu\right) = 0 \end{aligned}$$

Note that if we contract the first equation with g^{ab}, we get 0, but only after using the fact that the string worldsheet is *two-dimensional*. What is the reason for this traceless property, and why only in two dimensions?

4. Consider the flat worldsheet action obtained in conformal gauge:

$$S = -\frac{T}{2}\int d\sigma\,dt\,\partial_a X^\mu \partial^a X_\mu$$

Show directly that the conserved current for translation invariance is:

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu - \tfrac{1}{2}\,\eta_{ab}\,\eta^{cd}\,\partial_c X^\mu \partial_d X_\mu, \qquad \partial^a T_{ab} = 0$$

5. Recalling that the dimension of α' is length^2, examine the mode expansion:

$$X^\mu_R(t+\sigma) = \tfrac{1}{2}x^\mu + \alpha'\, p^\mu\,(t+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\alpha^\mu_n\, e^{-2in(t+\sigma)}$$

and compute the dimensions that are assigned to all the quantities appearing on the RHS.

6. Derive the classical relation

$$L_n = \tfrac{1}{2}\sum_{m=-\infty}^{\infty} \alpha_{-m}\cdot\alpha_{n+m}$$

7. Show that the mode expansion for open strings with DN boundary conditions is:

$$X^\mu(\sigma,t) = c^\mu - \sqrt{2\alpha'}\sum_{r\in Z+\frac{1}{2}}\frac{1}{r}\,\alpha^\mu_r\, e^{-irt}\sin r\sigma$$

What do we get for ND boundary conditions?