Mini-Course on String Theory (Sunil Mukhi)

Exercise Sheet, No. 2

1. For the open string with NN boundary conditions, use the mode expansion

$$X^\mu(\sigma,t) = x^\mu + 2\alpha' p^\mu\, t + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\,\alpha_n^\mu\, e^{-int}\cos n\sigma$$

and $P_\mu = \frac{1}{2\pi\alpha'}\dot{X}^\mu$ to show that the canonical commutator

$$[P_\mu(\sigma,t), X^\nu(\sigma',t)] = -i\delta(\sigma-\sigma')\delta_\mu^\nu$$

is implied by the commutation relations

$$[\alpha_m^\mu, \alpha_n^\nu] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}, \quad [x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad [\alpha_m^\mu, x^\nu] = [\alpha_m^\mu, p^\nu] = 0$$

2. Derive the equation:

$$\tfrac{1}{2}\sum_{m=-\infty}^{\infty}\alpha_{-m}^i\,\alpha_m^i = \tfrac{1}{2}\sum_{m=-\infty}^{\infty} :\alpha_{-m}^i\,\alpha_m^i: \; + \frac{D-2}{2}\sum_{m=1}^{\infty} m$$

where : : represents normal ordering, with annihilation operators $(\alpha_m, m>0)$ to the right of creation operators $(\alpha_m, m<0)$.

3. Defining

$$L_n = \tfrac{1}{2}\sum_{m\in Z} :\alpha_{-m}^\mu \alpha_{\mu,n+m}: - \;\delta_{n,0}$$

show that

$$L_n\left(\zeta_\mu(k)\,\alpha_{-1}^\mu|k\rangle\right) = 0$$

for all $n \geq 0$ implies the conditions

$$k_\mu k^\mu\,\zeta_\nu(k) = 0, \qquad k^\mu\,\zeta_\mu(k) = 0$$