Mini-Course on String Theory (Sunil Mukhi)

Exercise Sheet, No. 3

1. In a two-dimensional spacetime with Minkowski signature, find a basis of gamma-matrices in which a Majorana fermion ψ_α, $\alpha = 1, 2$ becomes

$$\psi_\alpha = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}$$

where $\psi_\pm$ are both real. Show that a Lorentz boost with parameter Λ transforms these components as

$$\psi_\pm \to e^{\pm\Lambda/2}\psi_\pm$$

This exercise reveals the truism that the irreducible representations of an Abelian group ($SO(1,1)$ in this case) are always one-dimensional. In particular, it is unnecessary to use gamma-matrices in two dimensions. Also, it is consistent (with Lorentz invariance) to have a real one-component fermion ψ_+ without any ψ_-. What are the analogues of these results in two Euclidean dimensions?

2. Write the open fermionic string worldsheet action

$$\frac{T}{2}\int d\sigma dt\left(i\bar{\psi}^\mu \rho^a \partial_a \psi_\mu\right)$$

in terms of the components $\psi_\pm$ in the gamma-matrix basis of the previous problem. Write the canonical anticommutators for this action. Using the NS sector mode expansion

$$\text{NS sector}: \ \psi^\mu_-(\sigma, t) = \sqrt{\alpha'}\sum_{r\in Z+\frac{1}{2}} b^\mu_r\, e^{-2ir(t-\sigma)}$$

show that the canonical anticommutators are implied by the anticommutation relations

$$\{b^i_r,\, b^j_s\} = \delta_{r+s,0}\,\delta^{ij}$$

3. Show that for an open fermionic string, the variation of the action (as written in terms of $\psi_\pm$, see the previous exercise) leads to boundary terms proportional to:

$$\int dt\left[\delta\psi^\mu_+\,\psi_{+\mu} - \delta\psi^\mu_-\,\psi_{-\mu}\right]_0^\pi$$