Topological Matrix Models

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ÉCOLE DE PHYSIQUE
LES HOUCHES

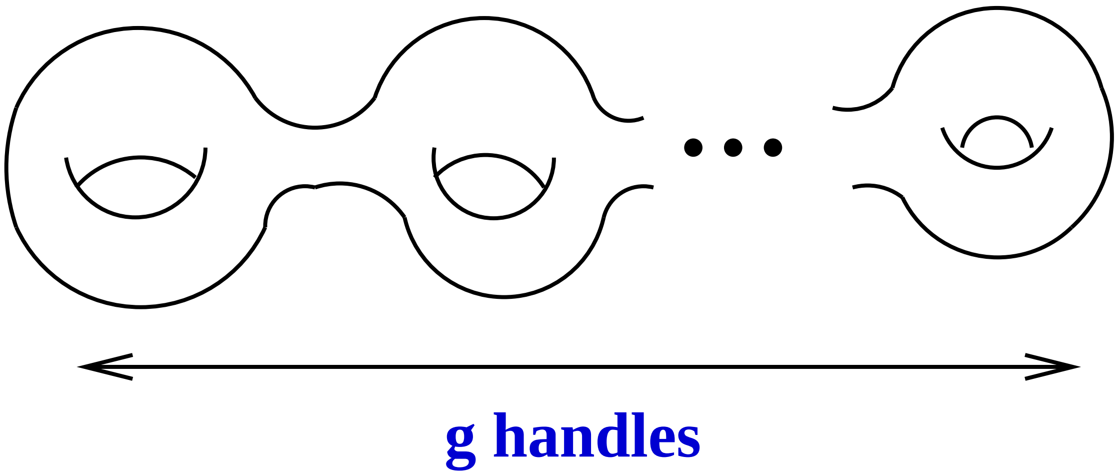
Some of this material, as well as references, can be found in:
Topological Matrix Models, Liouville Matrix Model and $c = 1$ *String Theory, (SM, hep-th/0310287)*

Contents

Moduli Space of Riemann Surfaces and its Topology

A Riemann surface is a 1-complex-dimensional manifold. We will deal with manifolds that are compact and without boundary.

Topologically these are classified by the genus or number of handles:



These manifolds admit a many-parameter family of complex structures (ways to define complex coordinates that are analytically inequivalent).

The moduli space of a compact Riemann surface of genus g and n punctures, $\mathcal{M}_{g,n}$, is the space of inequivalent complex structures that one can put on the surface.

It is known to be a (singular) complex manifold of complex dimension $3g-3+n$ (whenever this number is ≥ 0).

It arises as the quotient of a covering space, the Teichmüller space $\mathcal{T}_{g,n}$, by a discrete group, the mapping class group $MC_{g,n}$:

$$\mathcal{M}_{g,n} = \frac{\mathcal{T}_{g,n}}{MC_{g,n}}$$

This action typically has fixed points, hence the moduli space $\mathcal{M}_{g,n}$ has "orbifold" singularities.

A simple example is $\mathcal{M}_{0,n}$, the moduli space of the sphere ($g = 0$) with n punctures.

This has complex dimension $n - 3$.

For the simplest case of $n = 3$ one can fix all the punctures at arbitrary locations using the $SL(2, C)$ invariance of the sphere, so the moduli space is a point.

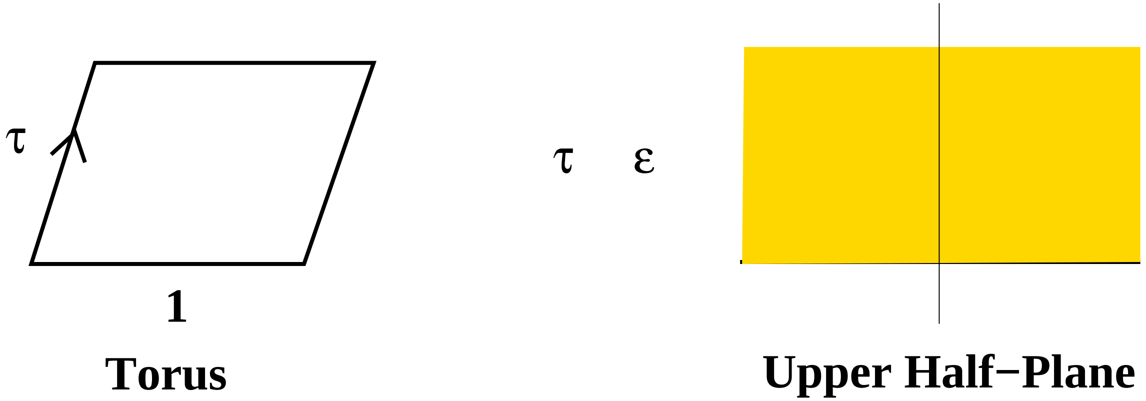
That point in turn is fixed under the action of the mapping class group S_3 that permutes the punctures.

Locally, $\mathcal{M}_{0,n}$ has the structure of $n - 3$ copies of the complex plane, but with a singularity whenever a pair of punctures coalesces on the original sphere.

Another example: $g = 1, n = 0$.

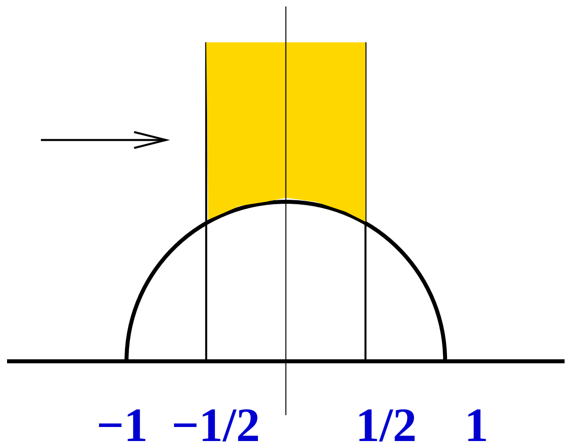
$$\dim(\mathcal{M}_{1,0}) = 1$$

In this case, the Teichmüller space $\mathcal{T}_{1,0}$ is the upper half plane.



And the mapping class group is $PSL(2, Z) : \tau \to \frac{a\tau+b}{c\tau+d}$.

The quotient space is $\mathcal{M}_{1,0}$:



The remaining $\mathcal{M}_{g,n}$ are much more complicated.

Mathematicians would like to know their topological invariants.

What is the simplest topological invariant of $\mathcal{M}_{g,n}$?

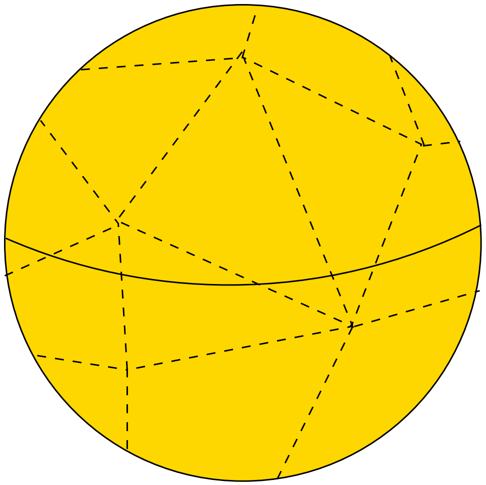
For a smooth manifold, we can define the the Euler characteristic χ. Make a simplicial decomposition (triangulation) $\mathcal{S}$ of the manifold, and evaluate:

$$\chi = \sum_{I \in \mathcal{S}} (-1)^{d_I}$$

where d_I is the dimension of the I th simplex, and the sum is over all the simplices in the complex $\mathcal{S}$.

This is a topological invariant, independent of how we triangulate the manifold.

Example:



For a two-dimensional surface, a triangulation is really made of triangles, and

$$\chi = \#(\text{vertices}) - \#(\text{edges}) + \#(\text{faces}) = 2 - 2g$$

Topological invariance implies that the answer is completely independent of the triangulation.

In general dimensions, a simplicial complex involves “solid triangles” of all dimensions upto the maximum.

In the presence of orbifold singularities, the natural quantity to define is the virtual Euler characteristic χ_V.

Here each term in the sum over simplices is divided by the order of a discrete group Γ_I that fixes the I th simplex.

Thus:

$$\chi_V = \sum_{I \in \mathcal{S}} \frac{(-1)^{d_I}}{\#(\Gamma_I)}$$

Using combinatoric methods, it was found by Harer and Zagier that the virtual Euler characteristic of $\mathcal{M}_{g,n}$ is:

$$\chi_V(\mathcal{M}_{g,n}) = (-1)^n \frac{(n+2g-3)!(2g-1)}{n!(2g)!} B_{2g}$$

where B_{2g} are the Bernoulli numbers.

Quadratic Differentials and Fatgraphs

The above results were obtained by triangulating the moduli space of punctured Riemann surfaces in terms of quadratic differentials. This was done by Harer, using a theorem due to Strebel, as follows.

On a Riemann surface with a finite number of marked points, one can define a meromorphic quadratic differential

$$\eta = \eta_{z,z}(z)dz^2$$

with poles at the marked points.

Under a change of coordinates $z \to z'(z)$, a quadratic differential transforms as:

$$\eta'_{z',z'}(z') = \left(\frac{\partial z}{\partial z'}\right)^2 \eta_{z,z}(z)$$

For a fixed complex structure on the surface, such a differential (with certain extra properties) is unique upto multiplication by a positive real number.

This differential can be used to invariantly define the length of a curve γ on the Riemann surface:

$$|\gamma|_\eta = \int_\gamma \sqrt{|\eta(z)|}|dz|$$

Indeed, defining a new coordinate via

$$dw = \sqrt{\eta(z)}dz$$

we see that this length is the ordinary length of the curve in the Euclidean sense, in the w coordinate.

Now consider a geodesic curve under the metric defined above.

At any point, such a curve will be called horizontal if η is real and positive along it, and vertical if η is real and negative.

The horizontal curves define flows along the Riemann surface.

The flow pattern is regular except at zeroes and poles of η. Here the flows exhibit interesting properties.

At an nth-order zero of the quadratic differential, precisely $n+2$ horizontal curves meet at a point.

To see this, consider the differential near this zero and along the radial direction:

$$\eta \sim z^n (dz)^2 \sim e^{i(n+2)\theta} dr^2$$

As we encircle the zero, there are precisely $n+2$ values of the angle θ at which this differential is positive.

At a double pole, if the coefficient is real and negative, the flows form concentric circles around the point.

We see that near the pole, and along the angular direction, the differential looks like:

$$\eta \sim -c\frac{dz^2}{z^2} \sim c\,d\theta^2$$

Thus, in the θ direction, the differential is positive, or horizontal, at all points surrounding the double pole.

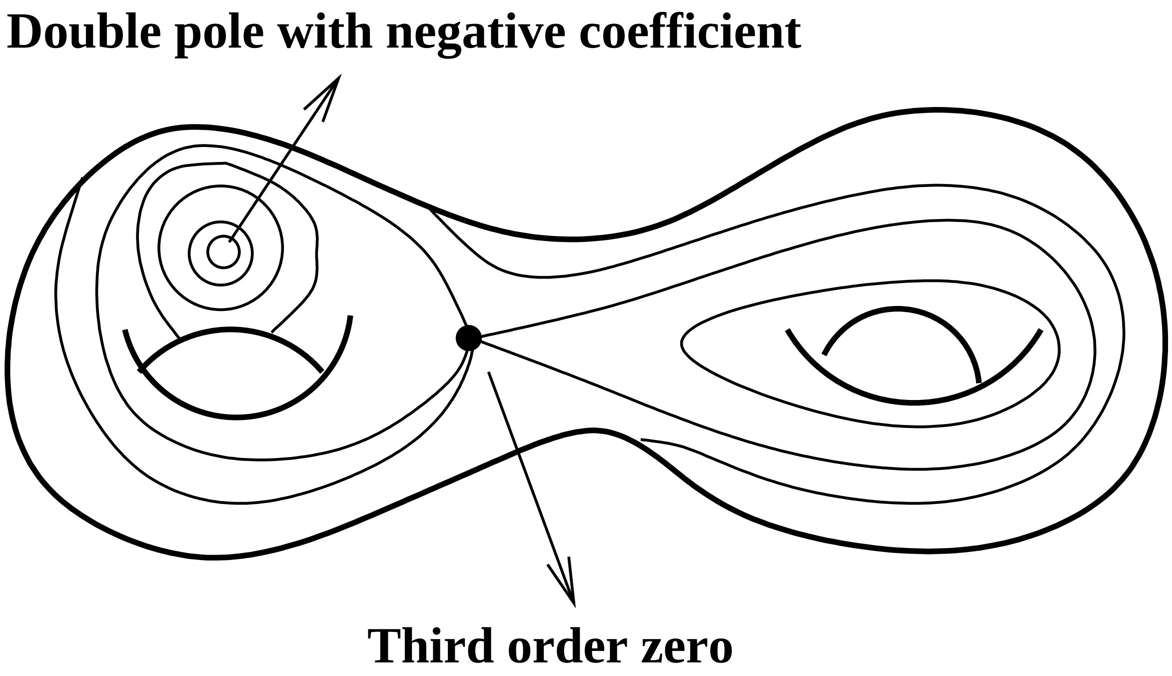
Other behaviours are possible at poles other than double poles, or if the coefficient of η at a double pole is complex.

But we will restrict our attention to quadratic differentials with a double pole at a point P, with the coefficient c being real and negative.

We also require that all smooth horizontal trajectories (i.e., those that do not pass through zeroes of η) form closed curves.

Quadratic differentials satisfying all these conditions exist, and are called horocyclic .

Example:



Riemann surface with the flow pattern of a horocyclic quadratic differential.

The vertex has five lines meeting at a point, indicating a third-order zero.

Strebel's theorem: on every Riemann surface of genus g with 1 puncture, for fixed complex structure, there exists a unique horocyclic quadratic differential with a double pole at the puncture.

(The uniqueness is upto multiplication by a real positive number).

Thus, by studying how these quadratic differentials vary as we vary the moduli, we get information about the moduli space $\mathcal{M}_{g,1}$ of a once-punctured Riemann surface.

Similar considerations apply for $\mathcal{M}_{g,n}$.

We can now see the emergence of “fatgraphs” and hence random matrices.

Most of the flows are closed and smooth, but there are singular ones that branch into $n+2$-point vertices at nth order zeroes of η.

We can think of these singular flows as defining a Feynman diagram, whose vertices are the branch points, and whose edges are the singular flow lines.

Each double pole of η is a point around which the flows form a loop. Hence the number of loops of the diagram is the number of double poles, which is the number of punctures of the original Riemann surface.

Finally, because the flows that do not pass through a zero are closed and smooth, each singular flow can be “thickened” into a smooth ribbon in a unique way, and we arrive at a fatgraph.

The fatgraphs with a single loop triangulate the moduli space $\mathcal{M}_{g,1}$ in the following way.

Consider the lengths of each edge of a fatgraph, as computed in the metric defined earlier.

Scaling the whole Riemann surface clearly does not change the complex structure. So to vary the complex structure, we must change the lengths of the different edges keeping the total length fixed.

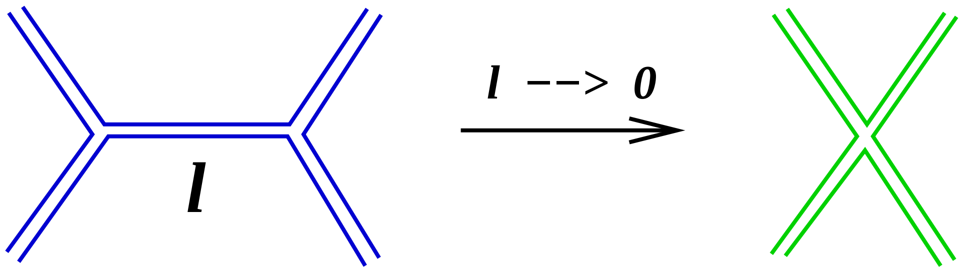
This sweeps out a region of the moduli space of the Riemann surface. The (real) dimensionality of this region will be $E - 1$ where E is the number of edges of the graph.

This region is a simplex of the moduli space.

In a simplicial decomposition, at the boundary of a simplex we find a lower-dimensional simplex.

In terms of fat graphs, a boundary occurs whenever a length goes to zero and two vertices meet.

Example:



Now the virtual Euler characteristic of $\mathcal{M}_{g,n}$ can be defined directly in terms of fatgraphs.

We consider the set of all fatgraphs of a given genus g and a single puncture. Call the set $\mathcal{S}$, and label each distinct graph by an integer $I \in \mathcal{S}$.

Let Γ_I be the automorphism group of a fatgraph. We will define it more precisely later.

Then, defining $d_I = (E-1)_I$, we claim that:

$$\chi_V(\mathcal{M}_{g,1}) = \sum_{I \in \mathcal{S}} \frac{(-1)^{d_I}}{\#(\Gamma_I)}$$

This is analogous to the original definition of χ_V, except that now the sum is over fatgraphs rather than over simplices.

In particular, the automorphism group of the fatgraph is the same as the group that fixes the corresponding simplex.

Let us check how this correspondence between fatgraphs and quadratic differentials works out in practice.

The fatgraphs we have been considering have V vertices, E edges and 1 face. These integers satisfy:

$$V - E + 1 = 2 - 2g$$

where g is the genus of the Riemann surface on which the graph is drawn.

We also have the relations:

$$V = \sum_k v_k, \quad E = \frac{1}{2} \sum_k k v_k$$

where v_k is the number of k-point vertices. From these relations, we get:

$$\sum_k (k-2) v_k = 4g - 2$$

All integer solutions of this equation, i.e. all choices of the set $\{v_k\}$ for fixed g, are valid graphs that correspond to simplices in the triangulation of $\mathcal{M}_{g,1}$.

Let us recast the above equation as

$$\sum_k (k-2)v_k - 2 = 4g - 4$$

Since $k-2$ is the order of the zero for a k-point vertex, the first term on the left is the total number of zeroes (weighted with multiplicity) of the quadratic differential corresponding to the given fatgraph.

Moreover, the differential has precisely one double pole, so the second term is minus the (weighted) number of poles.

Thus this result agrees with the theorem that for meromorphic quadratic differentials on a Riemann surface of genus g,

$$\#(\text{zeroes}) - \#(\text{poles}) = 4g - 4$$

A particular solution that is always available is

$$v_3 = V, \quad v_k = 0, k \geq 4$$

This gives the maximum possible number of vertices, and therefore also of edges.

In this case,

$$V = 4g - 2, \qquad E = \frac{3}{2}V = 6g - 3$$

Thus the dimension of the space spanned by varying the lengths of the graph keeping the overall length fixed, is:

$$E - 1 = 6g - 4 = 2(3g - 3 + 1)$$

which is the real dimension of $\mathcal{M}_{g,1}$.

Thus, graphs with only cubic vertices span a top-dimensional simplex in moduli space.

All other graphs arise by collapse of one or more lines, merging two or more 3-point vertices to create higher n-point vertices. These correspond to simplices of lower dimension in the moduli space.

Example: $\chi_V(\mathcal{M}_{1,1})$

To conclude this part, let us see how $\chi_V(\mathcal{M}_{1,1})$ is obtained from fatgraphs.

From the Harer-Zagier formula, we expect to find:

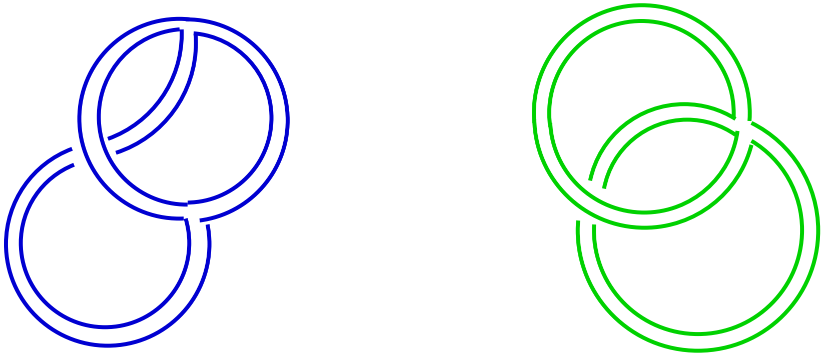
$$\chi_V(\mathcal{M}_{1,1}) = -\frac{1}{2}B_2 = -\frac{1}{12}$$

In genus 1, there are two possible ways to satisfy

$$\sum_k (k-2)v_k = 4g - 2 = 2$$

namely $v_3 = 2$ or $v_4 = 1$. In the first case we find $V = 2, E = 3$ and in the second, $V = 1, E = 2$.

The graphs are:



and we see explicitly that they have genus 1.

Exercise: show that the automorphism groups of these graphs are of order 6 and 4 respectively. Then, $\chi_V(\mathcal{M}_{1,1}) = \frac{(-1)^2}{6} + \frac{(-1)^1}{4} = -\frac{1}{12}$.

The Penner model

In 1986, Penner constructed a model of random matrices that provides a generating functional for $\chi_V(\mathcal{M}_{g,s})$.

The Penner model is defined in terms of $N \times N$ matrices whose fatgraphs are precisely the ones described in the previous subsection.

The free energy $\mathcal{F} = \log \mathcal{Z}$ of this model therefore must have the expansion:

$$\mathcal{F} = \sum_g \mathcal{F}_g = \sum_{g,n} \chi_{g,n} \, N^{2-2g} \, t^{2-2g-n}$$

where t is a parameter of the model. The term $n = 0$ is not present in the sum.

The model is given by an integral over Hermitian random matrices:

$$\begin{aligned} \mathcal{Z}_{\textsf{Penner}} &= \mathcal{N}_P \int [dQ]\, e^{-Nt\, \mathbf{tr} \sum_{k=2}^{\infty} \frac{1}{k}\, Q^k} \\ &= \mathcal{N}_P \int [dQ]\, e^{Nt\, \mathbf{tr} \left(\log(1-Q) + Q \right)} \end{aligned}$$

where $\mathcal{N}_P$ is a normalisation factor given by:

$$\mathcal{N}_P^{-1} = \int [dQ]\, e^{-Nt\, \mathbf{tr}\, \frac{1}{2} Q^2}$$

and the matrix measure $[dQ] \equiv \prod_i dQ_{ii} \prod_{i<j} dQ_{ij}\, dQ^*_{ij}$ as usual.

This action has all powers of the random matrix appearing in it!

The model is to be considered as a perturbation series around $Q \sim 0$.

To show that this model is correct, we must show that its fatgraphs are in one-to-one correspondence with those arising from quadratic differentials.

Thus the free energy must be a sum over connected fatgraphs of a fixed genus g and number of faces n, multiplied by the weighting factor

$$\frac{(-1)^{E-n}}{\#(\Gamma_I)} N^{2-2g} t^{2-2g-n} = \frac{1}{\#(\Gamma_I)} (-Nt)^V (Nt)^{-E} (N)^n$$

Here Γ_I, the automorphism group, is the collection of maps of a given fatgraph to itself such that:

(i) the set of vertices is mapped onto itself,

(ii) the set of edges is mapped to itself,

(iii) the cyclic ordering of each vertex is preserved.

A key result due to Penner is that the order of Γ_I is given by:

$$\frac{1}{\#(\Gamma_I)} = C \times \prod_k \left(\frac{1}{k}\right)^{v_k} \frac{1}{v_k!}$$

where C is the combinatoric factor labelling how many distinct contractions lead to the same graph.

Now this is exactly the factor that arises if we obtain our fatgraphs by expanding the Penner matrix integral:

$\frac{1}{v_k!}$	: order of expansion of the kth term in the exponent
$\frac{1}{k}$	: weight per vertex appearing in the action
C	: combinatoric factor from contractions
$(-Nt)^V$	: from weight of each vertex
$(Nt)^{-E}$	: from each propagator
N^n	: from the index sum on each face

This proves that the Penner model computes the desired quantity, $\chi_V(\mathcal{M}_{g,n})$.

In his paper, Penner constructed the orthogonal polynomials for this model. They turn out to be Laguerre polynomials.

Using the above facts, Penner was able to deduce, directly from his matrix model, that

$$\chi_V(\mathcal{M}_{g,n}) = (-1)^n \frac{(n+2g-3)!(2g-1)}{n!(2g)!} B_{2g}$$

where B_{2g} are the Bernoulli numbers.

Penner Model and Matrix Gamma Function

Recall the definition of the Penner matrix integral:

$$\mathcal{Z}_{\textsf{Penner}} = \mathcal{N}_P \int [dQ]\, e^{Nt\, \mathbf{tr}\, \big(\log(1-Q) + Q \big)}$$

Let us make the following change of variables:

$$Q = 1 - \frac{t+1}{t} M, \qquad t = -1 + \frac{\nu}{N}$$

This replaces the original matrix Q and parameter t by a new matrix M and parameter ν. The Penner action becomes:

$$Nt\, \mathbf{tr}\, \big(\log(1-Q) + Q \big) = \; \mathbf{tr}\, \big((\nu - N) \log M - \nu M \big) + \text{constant}$$

The additive constant depends on ν, N.

Thus we can write:

$$\mathcal{Z}_{\textsf{Penner}} = \mathcal{N}'_P \int [dM]\, e^{\,\mathbf{tr}\,\big((\nu - N)\log M - \nu M\big)}$$

where the new normalisation $\mathcal{N}'_P$ has absorbed the constant factors in the exponential and also the simple Jacobian.

For a 1×1 matrix $M = m$, the integral is just the Euler Γ-function:

$$\int dm\, m^{\nu - 1}\, e^{-\nu m} = \Gamma(\nu)$$

as long as we choose the correct limits $m \in (0, \infty)$.

Hence we make the same restriction on the matrix M in $\mathcal{Z}_{\textsf{Penner}}$ above, namely its eigenvalues must be positive.

It can then be called the Matrix Γ-Function.

We can remove the positivity restriction on M by defining:

$$M = e^{\boldsymbol{\Phi}}$$

where Φ is a generic Hermitian matrix. In this case there is a nontrivial Jacobian:

$$[dM] = (\det e^{\boldsymbol{\Phi}})^N \, [d\Phi]$$

Writing this equivalently as:

$$[d\Phi] = [dM](\det M)^{-N} = [dM] e^{-N \, \mathbf{tr} \, \log M}$$

we see that the Penner integral takes its simplest form:

$$\mathcal{Z}_{\textbf{Penner}} = \mathcal{N}'_P \int [d\Phi] \, e^{\nu \, \mathbf{tr} \, (\Phi - e^{\Phi})}$$

which we call the Liouville Matrix Model.

The Liouville matrix model:

$$\mathcal{Z}_{\textbf{Penner}} = \mathcal{N}'_P \int [d\Phi]\, e^{\nu\, \mathbf{tr}\, \left(\Phi - e^{\Phi}\right)}$$

has some intriguing properties that are familiar from string theory.

The integral is like a matrix version of the Liouville path integral occurring in string theory, when restricted to the constant mode of the Liouville field.

It converges at $\Phi \to +\infty$ because of the exponential term, and at $\Phi \to -\infty$ because of the linear term.

It has an N-independent coefficient ν, suggestive of D-brane actions in string theory, if ν is interpreted as the inverse string coupling.

We will see later that this interpretation of ν does hold in a string theory setting of this model.

The Kontsevich Model

Another interesting topological problem associated to $\mathcal{M}_{g,n}$ is the following.

It is known that $\mathcal{M}_{g,n}$ can be compactified, and the resulting space is called $\overline{\mathcal{M}}_{g,n}$. Topological invariants can then be defined as integrals of cohomology classes on $\overline{\mathcal{M}}_{g,n}$.

The problem of intersection theory on moduli space can then be defined as follows.

Let $\mathcal{L}_i, i = 1, 2, \ldots, n$ be line bundles on $\overline{\mathcal{M}}_{g,n}$. The fibre for each i is the cotangent space to the Riemann surface at the puncture.

Each such bundle has its associated top Chern class $c_1(\mathcal{L}_i)$. This is a two-form (intuitively, the field strength associated to the $U(1)$ connection on this bundle).

Now construct the integral

$$\int_{\overline{\mathcal{M}}_{g,n}} c_1(\mathcal{L}_1)^{d_1} \wedge \cdots \wedge c_1(\mathcal{L}_n)^{d_n}$$

where $d_i \geq 0$ are a set of integers satisfying:

$$\sum_{i=1}^{n} d_i = 3g - 3 + n$$

This means that the integrand is a $6g - 6 + 2n$ form, equal in degree to the real dimension of $\overline{\mathcal{M}}_{g,n}$.

So the integral is well-defined and is a topological invariant of the moduli space.

Next we give this invariant a suggestive name:

$$\int_{\overline{\mathcal{M}}_{g,n}} c_1(\mathcal{L}_1)^{d_1} \wedge \cdots \wedge c_1(\mathcal{L}_n)^{d_n} = \langle \tau_{d_1} \cdots \tau_{d_n} \rangle$$

as if it is a correlation function of some field theory. (We define the RHS to be 0 if $\sum d_i \neq 3g - 3 + n$ for any integer g.)

This is actually the case, and the field theory (due to Witten) is called topological 2d gravity. But we won't need to know this here.

Let us now define a generating functional for these invariants by summing them up.

$$\begin{aligned} F(t_0, t_1, \cdots) &\equiv \left\langle \exp\Big(\sum_{i=0}^{\infty} t_i \tau_i\Big) \right\rangle = \sum_{k_0, k_1, \cdots} \langle \tau_0^{k_0} \tau_1^{k_1} \cdots \rangle \prod_{i=0}^{\infty} \frac{t_i^{k_i}}{k_i!} \\ &= \sum_{n=1}^{\infty} \sum_{\{d_i\}} \frac{1}{n!} \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle \, t_{d_1} \ldots t_{d_n} \end{aligned}$$

It is known that:

$$U(t_0, t_1, \cdots) \equiv \frac{\partial^2 F}{\partial t_0^2}(t_0, t_1, \cdots)$$

satisfies the KdV equation:

$$\frac{\partial U}{\partial t_1} = U \frac{\partial U}{\partial t_0} + \frac{1}{12} \frac{\partial^3 U}{\partial t_0^3}$$

Also, the series $\exp(F)$ in terms of the variables

$$T_{2i+1} \equiv \frac{1}{(2i+1)!!} t_i$$

is a τ-function of the KdV hierarchy.

This constitutes the “solution” of the problem.

Kontsevich proposed a matrix model whose connected fatgraphs generate the function $F(t_0, t_1, \cdots)$.

Clearly the model must depend on infinitely many parameters t_i. However, these are encoded in a nontrivial way.

Introduce an $N \times N$ positive-definite Hermitian matrix Λ and let:

$$t_i = -(2i-1)!!\, \mathbf{tr}\, \Lambda^{-(2i+1)}$$

Clearly the t_i obtained in this way are not all independent of each other if the rank of Λ is finite.

Only as $N \to \infty$ can they be chosen independently.

This is a new role for the large-N limit!

The Kontsevich matrix model, depending on the fixed matrix Λ, is:

$$\mathcal{Z}_{\textsf{Kontsevich}}(\Lambda) = \mathcal{N}_K(\Lambda) \int [dX]\, e^{\,\mathbf{tr}\,\left(-\frac{1}{2}X^2\Lambda + \frac{i}{6}X^3\right)}$$

where X is an $N \times N$ Hermitian random matrix, and:

$$\mathcal{N}_K(\Lambda) = \left\{\int [dX]\, e^{\,\mathbf{tr}\,\left(-\frac{1}{2}X^2\Lambda\right)}\right\}^{-1}$$

By a change of variables, the above model can also be written:

$$\mathcal{Z}_{\textsf{Kontsevich}}(\tilde{\Lambda}) = \mathcal{N}'_K(\tilde{\Lambda}) \int [d\tilde{X}]\, e^{i\,\mathbf{tr}\,\left(\frac{1}{3}\tilde{X}^3 - \tilde{X}\tilde{\Lambda}\right)}$$

Comparing this with the Airy Function:

$$\mathcal{A}(\lambda) = \int_{-\infty}^{\infty} dx\, e^{i\,\left(\frac{1}{3}x^3 - x\lambda\right)}$$

we see that the Kontsevich model is a Matrix Airy Function.

Without loss of generality, the fixed matrix Λ can be taken to be diagonal:

$$\Lambda = \text{diag}(\Lambda_1, \Lambda_2, \cdots, \Lambda_N)$$

Then, using

$$\mathbf{tr}\, X^2\Lambda = \frac{1}{2}\sum_{i,j}(\Lambda_i + \Lambda_j)X_{ij}X_{ji}$$

we see that the matrix propagator in this model is:

$$\langle X_{ij}X_k{}^l\rangle = \delta_{jk}\delta_{li}\,\frac{2}{\Lambda_i + \Lambda_j}$$

The vertices, unlike in the Penner model, are all cubic.

In his paper, Kontsevich showed that:

$$F(t_0, t_1, \cdots) \equiv \left\langle \exp\left(\sum_{i=0}^{\infty} t_i \tau_i\right) \right\rangle = \log \mathcal{Z}_{\textsf{Kontsevich}}(\Lambda)$$

He also showed that $\mathcal{Z}_{\textsf{Kontsevich}}(\Lambda)$ is a τ-function of the KdV hierarchy.

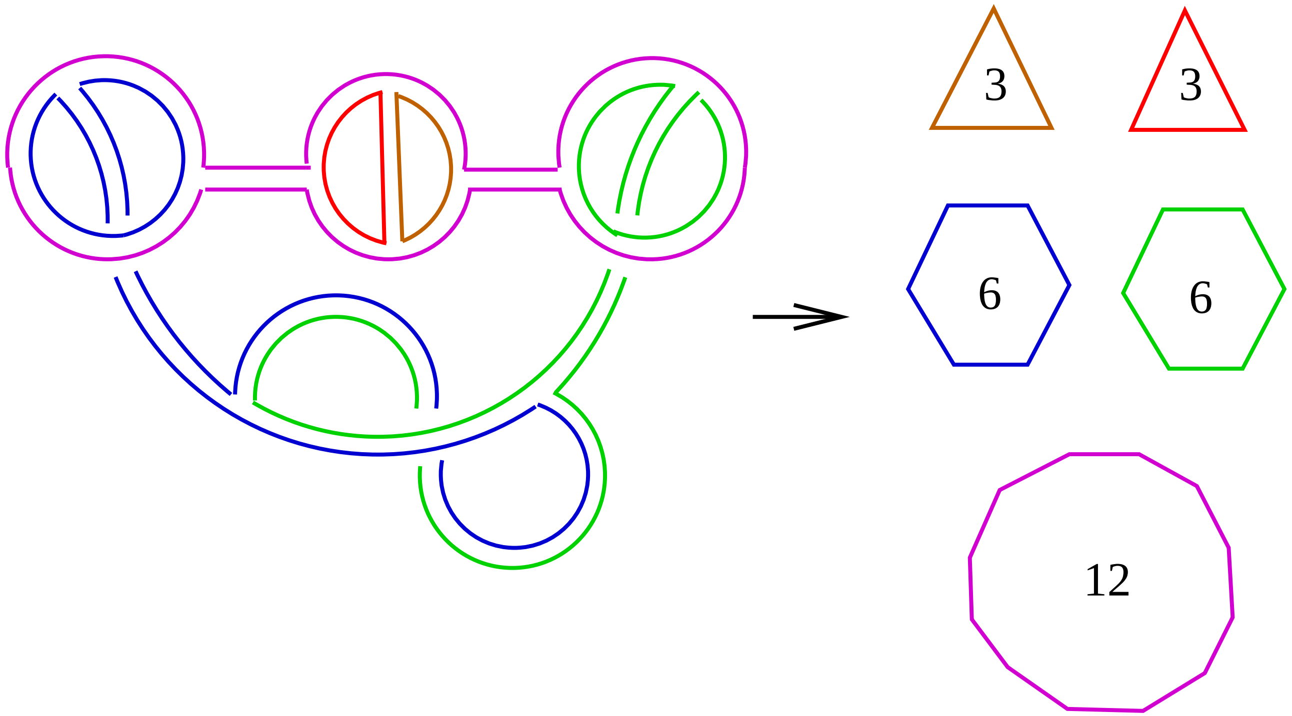
Let's sketch the derivation. From the definition of $F(t_0, t_1, \cdots)$ and the change of variables

$$t_{d_i} = -(2d_i - 1)!! \sum_{j=1}^{N} \frac{1}{\Lambda_j^{2d_i+1}}$$

we see that:

$$\begin{aligned} F(t_0, t_1, \cdots) &= \sum_{n=1}^{\infty} \sum_{\{d_i\}} \frac{1}{n!} \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle \, t_{d_1} \ldots t_{d_n} \\ &= \sum_{n=1}^{\infty} \sum_{\{d_i\}} \frac{(-1)^n}{n!} \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle \sum_{\{j_i\}} \prod_{i=1}^{n} \frac{(2d_i - 1)!!}{\Lambda_{j_i}^{2d_i+1}} \end{aligned}$$

Now given a 3-valent graph, we first “unravel” it into polygons:



On the polygons, we associate lengths l_a via the metric induced from the horocyclic quadratic differentials.

The unravelling defines a map:

$$\mathcal{M}_{g,n} \times R_+^n \to \left(\text{space of polygons with marked lengths}\right)^n$$

Next, for each polygon we define a 2-form:

$$\omega_i \sim \sum_{a<b} dl_a \wedge dl_b$$

and pull the form back to $\mathcal{M}_{g,n} \times R_+^n$.

Kontsevich then proves that the resulting 2-form projects to a 2-form on $\mathcal{M}_{g,n}$, and is in fact just equal to $c_1(\mathcal{L}_i)$.

This sets up a correspondence between the desired Chern classes and properties of fatgraphs.

From this he then shows that:

$$\sum_{n=1}^{\infty} \sum_{\{d_i\}} \frac{(-1)^n}{n!} \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle \sum_{\{j_i\}} \prod_{i=1}^{n} \frac{(2d_i - 1)!!}{\Lambda_{j_i}^{2d_i+1}} = \sum_{\substack{\textbf{3-valent} \\ \textbf{graphs } I}} \frac{\left(\frac{i}{2}\right)^V}{\#(\Gamma_I)} \prod_{\substack{\textbf{edge} \\ \langle i,j \rangle}} \frac{2}{\Lambda_i + \Lambda_j}$$

The RHS is the graphical expansion of $F(t_0, t_1, \cdots) = \log \mathcal{Z}_{\textbf{Kontsevich}}(\Lambda)$.

Finally, Kontsevich provides an asymptotic expansion of the Matrix Airy Function using the famous Harish-Chandra formula:

$$\int [dX]\, \mathbf{tr}\, p(X)\, e^{-i\, \mathbf{tr}\, X\Lambda} = \mathsf{C} \int \prod_i dx_i \prod_{i<j} \frac{(x_i - x_j)}{(\Lambda_i - \Lambda_j)} \sum_i p(x_i)\, e^{-i \sum_i x_i \Lambda_i}$$

He then identifies this with the asymptotic expansion of the τ-function of the KdV hierarchy.

This proves that $\mathcal{Z}_{\textsf{Kontsevich}}$ is a KdV τ-function.

Applications to String Theory

Kontsevich Model

Topological gravity was introduced by Witten as an alternative way to understand the noncritical closed-string theories that were solved around 1990 using double-scaled matrix models.

The string theories corresponded to $c < 1$ conformal field theories coupled to two-dimensional (Liouville) gravity.

"Pure" topological gravity describes the simplest of these theories, the $(p, q) = (2, 1)$ minimal model with central charge $c = -2$. In matrix model language, one gets this theory by not going to any critical point.

The theory is non-trivial (though its critical exponents are trivial), and its operators are the τ_i mentioned before.

By construction, the Kontsevich model gives us all its correlators in every genus.

However, the entire chain of $(2, q)$ minimal models coupled to gravity, for all odd q, can be studied using the same model.

As Witten argued, to go to higher q, one only has to give an expectation value to some of the t_i.

Thus, the Kontsevich model expanded around different "vacua" i.e. choices of expectation values $\langle t_i \rangle$ generates all $(2, q)$ minimal models coupled to gravity.

For $(p, 1)$ noncritical strings with $p > 2$, one needs a model proposed by Adler-van Moerbeke and indepedently by Kharchev-Marshakov-Mironov-Morozov-Zabrodin:

$$\mathcal{Z}_{\textbf{AvM-KMMMZ}}(\tilde{\Lambda}) = \mathcal{N}'_{\textbf{AvM-KMMMZ}}(\tilde{\Lambda}) \int [d\tilde{X}]\, e^{i\, \mathbf{tr}\left(\frac{1}{p+1}\tilde{X}^{p+1} - \tilde{X}\tilde{\Lambda}\right)}$$

and again one recovers the (p, q) case by going to suitable critical points.

Recently in a remarkable paper, Gaiotto and Rastelli obtained the Kontsevich model by evaluating the action of open-string field theory on the physical states:

$$\int \left(\frac{1}{2}\Psi Q\Psi + \frac{1}{3}\Psi * \Psi * \Psi\right) \rightarrow \ \mathbf{tr}\left(\frac{1}{2}\Lambda X^2 + \frac{1}{3}X^3\right)$$

But the latter action describes closed-string states! So this is an example of open-closed duality.

Penner Model

In 1990, it was noticed by Distler and Vafa that starting with the Penner free energy:

$$\begin{aligned} \mathcal{F} &= \sum_g \mathcal{F}_g = \sum_{g,n} \chi_{g,n}\, N^{2-2g}\, t^{2-2g-n} \\ \chi_{g,n} &= \frac{(-1)^n (2g-3+n)!(2g-1)}{(2g)!\, n!} B_{2g} \end{aligned}$$

one can perform the sum over n explicitly, to get:

$$\mathcal{F}_g = \frac{B_{2g}}{2g(2g-2)} (Nt)^{2-2g} \left(\left(1+\frac{1}{t}\right)^{2-2g} - 1 \right)$$

For $g > 1$, they took the limit $N \to \infty$ and $t \to t_c = -1$, keeping fixed the product $N(1+t) = \nu$. This led to the simpler result:

$$\mathcal{F}_g = \frac{B_{2g}}{2g(2g-2)}\, \nu^{2-2g}$$

But this, for $g > 1$, is precisely the virtual Euler characteristic of unpunctured Riemann surfaces!

Thus the Penner model, originally designed to study the moduli space of punctured Riemann surfaces, describes unpunctured ones too. This happens in the special double-scaling limit above.

More remarkably, we see that its free energy in the double scaling limit:

$$\mathcal{F} = \sum_g \frac{B_{2g}}{2g(2g-2)}\,\nu^{2-2g}$$

is almost identical to a well-known quantity in string theory: the free energy of the $c = 1$ string compactified at self-dual radius:

$$\mathcal{F} = \sum_g \frac{|B_{2g}|}{2g(2g-2)}\,\mu^{2-2g}$$

However, there is an issue of alternating signs. We have:

$$|B_{2g}| = (-1)^{g-1} B_{2g}$$

Therefore if we define $\mu = i\nu$, we can write:

$$\begin{aligned} \mathcal{F}(\nu)_{c=1} &= \sum_{g=0}^{\infty} \frac{B_{2g}}{2g(2g-2)} \nu^{2-2g} \\ &= \sum_{g=0}^{\infty} \chi_g \, \nu^{2-2g} \end{aligned}$$

Thus the genus g contribution to the free energy of the $c = 1, R = 1$ string at imaginary cosmological constant is the (virtual) Euler characteristic of genus-g moduli space, which in turn is the Penner free energy after double-scaling.

If the Penner model is associated to the $c = 1$ string, it should describe correlators of its observables: the so-called "discrete tachyons" $\mathcal{T}_k$.

But it does not depend on the necessary (infinitely many) parameters.

However, there is a deformation of the model that does precisely this job. This was constructed in 1995 (Imbimbo and SM) starting with the generating functional for all tachyon correlators to all genus.

Such a functional $\mathcal{F}(t, \bar{t})$ depends on couplings $t_k, \bar{t}_k$ such that:

$$\langle \mathcal{T}_{k_1} \ldots \mathcal{T}_{k_n} \mathcal{T}_{-l_1} \ldots \mathcal{T}_{-l_m} \rangle = \frac{\partial}{\partial t_{k_1}} \cdots \frac{\partial}{\partial t_{k_n}} \frac{\partial}{\partial \bar{t}_{l_1}} \cdots \frac{\partial}{\partial \bar{t}_{l_m}} \mathcal{F}(t, \bar{t}) \Big|_{t = \bar{t} = 0}$$

where on the LHS we have connected amplitudes.

In the same spirit as Kontsevich, start by defining a constant $N \times N$ matrix A that satisfies:

$$t_k = \frac{1}{\nu k} \,\mathbf{tr}\, A^{-k}$$

This matrix can encode infinitely many parameters t_k in the limit $N \to \infty$.

However, we do not perform a similar transformation on $\bar{t}_k$, rather we expect the model to depend directly on these parameters.

Using matrix quantum mechanics at $R = 1$, it was shown by Dijkgraaf, Moore and Plesser that $\mathcal{Z}(t, \bar{t}) = e^{\mathcal{F}(t,\bar{t})}$ satisfies the W_∞ equation:

$$\frac{1}{(-\nu)} \frac{\partial \mathcal{Z}}{\partial \bar{t}_n} = \frac{1}{(-\nu)^n} (\det A)^{\nu} \,\mathbf{tr}\, \left(\frac{\partial}{\partial A} \right)^n (\det A)^{-\nu} \,\mathcal{Z}(t, \bar{t})$$

where $\nu = -i\mu$ and μ is the cosmological constant.

Let us postulate that $\mathcal{Z}(t,\bar{t})$ is an integral over Hermitian matrices M of the form:

$$\mathcal{Z}(t,\bar{t}) = (\det A)^{\nu} \int [dM]\, e^{\,\mathbf{tr}\, V(M,A,\bar{t})}$$

for some $V(M,A,\bar{t})$.

The function V is determined by imposing the above differential equation:

$$\left[\frac{1}{(-\nu)}\frac{\partial}{\partial \bar{t}_n} - \frac{1}{(-\nu)^n}\,\mathbf{tr}\left(\frac{\partial}{\partial A}\right)^n\right] \int [dM]\, e^{\,\mathbf{tr}\, V(M,A,\bar{t})} = 0$$

This determines:

$$V(M,A,\bar{t}) = -\nu\Big(MA + \sum_{k=1}^{\infty} \bar{t}_k M^k\Big) + f(M)$$

where $f(M)$ is a function independent of $A, \bar{t}$ that we determine using a boundary condition.

From conservation of the tachyon momentum, we know that $Z(t,0)$ must be independent of t_k. Using:

$$Z(t,0) = (\det A)^{\nu} \int [dM]\, e^{-\nu\, \mathbf{tr}\, MA + \,\mathbf{tr}\, f(M)}$$

and changing variables $M \to MA^{-1}$, we have

$$[dM] \to (\det A)^{-N} [dM]$$

Then:

$$\begin{aligned} Z(t,0) &= (\det A)^{\nu - N} \int [dM]\, e^{-\nu\, \mathbf{tr}\, M + \,\mathbf{tr}\, f(MA^{-1})} \\ &= \int [dM]\, e^{-\nu\, \mathbf{tr}\, M + \,\mathbf{tr}\, f(MA^{-1}) + (\nu - N)\, \mathbf{tr}\, \log A} \end{aligned}$$

This uniquely determines:

$$f(M) = (\nu - N) \log M$$

In summary, we have found that the generating function of all tachyon amplitudes in the $c = 1, R = 1$ string theory is:

$$
\begin{aligned}
\mathcal{Z}(t,\bar{t}) &= (\det A)^{\nu} \int [dM]\, e^{\,\mathbf{tr}\,(-\nu M A + (\nu - N)\log M - \nu \sum_{k=1}^{\infty} \bar{t}_k M^k)} \\
&= \int [dM]\, e^{\,\mathbf{tr}\,(-\nu M + (\nu - N)\log M - \nu \sum_{k=1}^{\infty} \bar{t}_k (M A^{-1})^k)}
\end{aligned}
$$

Note the re-appearance of the Penner model, from a completely independent starting point, and now with infinitely many parameters $A, \bar{t}_k$.

This deformed Penner model can be called the W_∞ model.

As we saw before, the matrix M (more precisely, its eigenvalues) must be positive semidefinite.

Comments

1. The deformed Penner model is universal, in the following sense.

Setting $A, \bar{t}_k = 0$ we recover the original Penner model.

Setting $\nu = N, \bar{t}_3 = const, \bar{t}_i = 0 \ (i \neq 3)$ we recover the matrix Airy function, or Kontsevich model.

Setting $\nu = N, \bar{t}_{p+1} = const, \bar{t}_i = 0 \ (i \neq p+1)$ we recover the p-th AvM-KMMMZ model.

Setting $\nu = N, A = 0, \bar{t}_k = 0, k > m$ we recover the polynomial 1-matrix model of any degree m.

2. The W_∞ model has a genus expansion governed entirely by μ, not N. Thus even at finite N, it has a genus expansion!

This tells us the generating function for the case where most of the t_k are not independent. So for a given correlator, you only need to go to the value of N large enough for the desired t_k to be independent.

A similar property holds for the Kontsevich model. There, a given correlator only has a contribution in a definite genus.

3. There is a different (2-matrix) model that also describes the $c = 1$ string at selfdual radius (and other radii) - due to Alexandrov, Kazakov, Kostov.

This is the normal matrix model, for a complex matrix Z satisfying:

$$[Z, Z^\dagger] = 0$$

The matrix integral (at selfdual radius) is:

$$\mathcal{Z}(t,\bar{t})_{NMM} = \int [dZ\, dZ^\dagger]\, e^{\,\mathbf{tr}\left(-\nu Z Z^\dagger + (\nu - N)\log Z Z^\dagger - \nu \sum_{k=1}^{\infty}(t_k Z^k + \bar{t}_k Z^{\dagger\, k})\right)}$$

This is different from the W_∞ model and yet describes the same correlation functions. Also it has no Kontsevich-type constant matrix in it. What is its relation to the W_∞ model?

4. We have seen that all these topological matrix models describe special noncritical string theories.

This is nice, but not too surprising, because integration over $\mathcal{M}_{g,n}$ is central in perturbative string theory,

Do they capture nonperturbative effects in string theory? Not known.

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