M-theory and Membranes

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Outline

M-theory: Motivation and background

- Superstring theory, which has been studied intensely for over 25 years, has the following properties:
  (i) It is a consistent high-energy completion of physically relevant low-energy field theories including gauge theories and gravity.
  (ii) It naturally unifies gauge fields, the spacetime metric as well as matter fields into a single type of string.
  (iii) It is well-defined only in 10 dimensions but by compactification it can be reduced to any dimension $d < 10$ as well.
  (iv) It has a perturbative expansion in terms of Riemann surfaces of increasing genus.

- In these lectures we are going to discuss a closely related theory called M-theory.
- It has similar properties as we listed for string theory, with some changes:
  (iii') It is well-defined only in 11 dimensions but by compactification it can be reduced to any dimension $d < 11$ as well.
  (iv') It has no perturbative expansion.
- Interestingly if we compactify M-theory from 11d to 10d, we recover type IIA string theory.
- Thus M-theory seems more basic than string theory!
- Unfortunately it is not so well-understood. However even at the level at which we understand it, it "explains" many interesting features of string theory.

- In string theory, the basic objects are strings. However the theory also has many other stable extended objects called branes, which play an important role in dynamics.
- These include BPS NS-branes:

$$F1, \; NS5 \qquad \text{type IIA and IIB}$$

  and BPS D-branes:

$$\begin{array}{ll} D0, \; D2, \; D4, \; D6, \; D8 & \text{type IIA} \\ D1, \; D3, \; D5, \; D7, \; D9 & \text{type IIB} \end{array}$$

- In uncompactified M-theory there are only two stable BPS branes:

$$M2, \; M5$$

- We will see that these two branes can be used to explain the origin of all the branes of string theory – including the fundamental string!

- What is the fundamental object in M-theory?
- Strings appear to be "fundamental" in string theory mainly because the perturbation expansion is defined in terms of them.
- Since there is no perturbation expansion in M-theory, we cannot precisely identify a fundamental object in that theory.
- Nevertheless, it is the membrane that seems closest to being a fundamental object. In fact, M-theory stands for Membrane theory among other things.

- In string theory, the starting point is quantisation of fundamental strings. Then we show that the massless degrees of freedom, for closed strings, couple according to 10d supergravity.
- However quantisation of membranes is still an unsolved problem.
- So M-theory must be studied by combining what we know about string theory with the knowledge of 11d supergravity. This is also how it was originally discovered.

Outline

11d Supergravity

- One reason to believe that M-theory is a very basic theory is that 11 is the highest allowed number of dimensions for supersymmetry, if we don't allow "spin > 2" fields.
- Indeed, in 11 uncompactified dimensions we can only have supergravity, i.e. local rather than global supersymmetry. The field content is as follows.
- Supergravity obviously has a graviton G_{MN} with $M, N = 0, 1, \cdots, D-1$.
- In D dimensions this has $\frac{(D-1)(D-2)}{2} - 1$ on-shell degrees of freedom.
- This counting comes from the fact that the little group is $SO(D-2)$. The symmetric traceless representation of this group has the given dimension.

- The superpartner of the graviton must be a gravitino $\Psi_{M\,\alpha}$. This is a fermion with a vector and a spinor index, $M = 0, 1, \cdots, D-1$ and $\alpha = 1, 2, \cdots, \tilde{D}$.
- Here $\tilde{D}$ is the dimension of the irreducible spinor representation, which depends in a complicated way on D.
- The gravitino $\Psi_{M,\alpha}$ has $\frac{(D-3)\tilde{D}}{2}$ on-shell degrees of freedom.
- To see this, note that a simple spinor of $\tilde{D}$ components has $\frac{\tilde{D}}{2}$ components on-shell.
- A vector of D components has $D-2$ components on-shell.
- Finally, $\frac{\tilde{D}}{2}$ components are removed by imposing Γ-tracelessness:

$$\Gamma^M \Psi_{M,\alpha} = 0$$

- Let us now make a table for various dimensions:

Dim D	Spinor dim $\tilde{D}$	Graviton $\frac{(D-1)(D-2)}{2} - 1$	Gravitino $\frac{(D-3)\tilde{D}}{2}$	Diff.
4	4	2	2	0
5	8	5	8	3
6	8	9	12	3
7	16	14	32	18
8	16	20	40	20
9	16	27	48	21
10	16	35	56	21
11	32	44	128	84
12	64	54	288	234

- The deficit can be made up by adding new bosons to the theory. However, from $D \geq 12$ there are so many bosons needed that we inevitably encounter "spin > 2" fields.

- For $D = 11$ we need to add 84 bosonic fields to obtain a matching of on-shell degrees of freedom.
- Luckily there is an irreducible representation of the little group $SO(9)$ that has precisely this dimension.
- It is the antisymmetric 3-form C_{MNP}.
- This has $\frac{(D-2)(D-3)(D-4)}{6}$ on-shell degrees of freedom. For $D = 11$ this is precisely 84!
- Thus we may hope to find an 11d supergravity theory containing the massless fields:

$$G_{MN}, \; C_{MNP}, \; \Psi_{M,\alpha}$$

- Indeed, the following action is supersymmetric:

$$S_{11d} = \frac{2\pi}{(2\pi\ell_p)^9}\left[\int d^{11}x\sqrt{-||G||}\left(R - \frac{1}{2}|F|^2\right) - \frac{1}{6}\int C\wedge F\wedge F\right]$$

$$+ \text{ fermionic terms}$$

- The quantities in the above are defined as follows:

$$\begin{aligned} \ell_p : &\quad \text{11-dimensional "Planck length"} \\ ||G|| : &\quad \text{determinant of the metric} \\ R : &\quad \text{Ricci scalar} \\ F : &\quad F_{LMNP} = \partial_{[L}C_{MNP]} \end{aligned}$$

- Besides general coordinate invariance, the above action is invariant upto a total derivative under the "gauge symmetry":

$$\delta C = d\Lambda$$

where Λ is a 2-form.

- It is also invariant under the supersymmetry transformations:

$$
\begin{aligned}
\delta E_M^A &= \bar{\epsilon}\,\Gamma^A \Psi_M \\
\delta C_{MNP} &= -3\,\bar{\epsilon}\,\Gamma_{[MN}\Psi_{P]} \\
\delta \Psi_M &= \nabla_M \epsilon + \frac{1}{12}\Big(\frac{1}{4!}\Gamma_M F_{PQRS}\,\Gamma^{PQRS} - \\
&\qquad \frac{1}{2} F_{MQRS}\,\Gamma^{QRS}\Big)\epsilon
\end{aligned}
$$

Here:

E_M^A : frames satisfying $E_M^A E_N^A = G_{MN}$

$\Gamma^{P_1\cdots P_n}$: $\Gamma^{[P_1}\Gamma^{P_2}\ldots\Gamma^{P_n]}$

$\nabla_M \epsilon$: $\partial_M \epsilon + \frac{1}{4}\omega_M{}^{AB}\,\Gamma_{AB}\epsilon$

$\omega_M{}^{AB}$: spin connection

- Since we did not write the fermion terms in the action, we cannot check supersymmetric invariance.
- However if we assume it holds, then we don't really need to know the fermionic terms in the action. The same information is contained in the supersymmetry variation!

Outline

M-branes as black branes

- In string theory, branes can be understood in two ways:
  - (i) As gravitating classical solutions that preserve supersymmetry ("black branes").
  - (ii) As D-branes on which open strings end.
- Since the former description requires only a supergravity theory, we can use it in 11d supergravity.
- Are there stable black branes in 11d supergravity?
- And do they preserve supersymmetry?

- The condition for a background $G_{MN}, C_{MNP}, \Psi_{M,\alpha}$ to preserve supersymmetry is that there should be some spinor ϵ such that the supersymmetry variations on the background vanish.
- Since we consider bosonic backgrounds, we only have to check supersymmetry variations of the fermions:

$$\begin{aligned} \delta\Psi_M &= \nabla_M \epsilon + \frac{1}{12}\Big(\frac{1}{4!}\Gamma_M (dC)_{PQRS}\,\Gamma^{PQRS} - \\ &\qquad\qquad \frac{1}{2}(dC)_{MQRS}\,\Gamma^{QRS}\Big)\epsilon \\ &= 0 \end{aligned}$$

- It turns out that for the case of interest to us (maximal supersymmetry), vanishing of these equations is sufficient to guarantee a classical solution of the full supergravity equations.
- And being first order, these are much easier to solve.

- Supersymmetric configurations will be stable because of the BPS bound.
- What kind of solutions should we look for? Stability suggests that the solutions will be supported by a flux.
- The only possible flux in 11d supergravity comes from the 3-form C_{MNP} whose field strength is the 4-form:

$$F_{LMNP} = \partial_{[L} C_{MNP]}$$

- This is like a magnetic field. The dual electric field is the 7-form:

$$\tilde{F}_{LMNPQRS} = \frac{1}{4!} \epsilon_{LMNPQRS}{}^{ABCD} F_{ABCD}$$

- A magnetically charged classical solution should emit a flux F_{LMNP} through a 4-sphere that encloses it.
- It is easy to check that in D dimensions, a d-sphere encloses a $D-d-2$ dimensional object. In 4 dimensions this is familiar as the fact that:
  (i) A 2-sphere S^2 encloses a point.
  (ii) A circle S^1 encloses an infinitely extended string.
- We conclude that a magnetic object in 11 dimensions which sources a 4-form flux must extend along $11-4-2=5$ dimensions. It is called an $M5$-brane.
- Therefore we should look for 5-branes which satisfy:

$$\int_{\Sigma_4} F = Q_{\mathrm{M5}}$$

where Q_{M5} is their magnetic charge (that will be quantised in certain units) and Σ_4 a 4-sphere enclosing the 5-brane.

- Similarly we conclude that an electric object in 11 dimensions (which sources the 7-form flux $\tilde{F}$) must extend along $11-7-2=2$ dimensions. It is called an $M2$-brane.
- In this case we will have:

$$\int_{\Sigma_7} \tilde{F} = Q_{\text{M2}}$$

  where Q_{M2} will be a quantised electric charge.
- A priori we do not expect to find any other stable objects in 11 dimensions!
- In particular there will be no stable strings. So 11d supergravity is not a string theory.

- For the $M2$-brane we take the coordinates on the brane to be $y^a = (y^0, y^1, y^2)$ while the coordinates transverse to the brane will be $x^I = (x^1, x^2, \cdots, x^8)$.
- We assume a symmetry $SO(2,1) \times SO(8)$ and also translational invariance in the y-coordinates.
- This fixes the metric to be of the form:

$$ds^2 = f_{(1)}(r)\, dy^a dy^a + f_{(2)}(r)\, dx^I dx^I$$

while the electric flux is nonvanishing only for the component:

$$F_{012r} = f_{(3)}(r)$$

Here r is the radial distance from the brane:

$$r = \sqrt{(x^1)^2 + (x^2)^2 + \cdots + (x^8)^2}$$

- Thus we only need to find the three functions $f_{(1)}(r), f_{(2)}(r)$ and $f_{(3)}(r)$.
- These functions are all determined by a single harmonic function:

$$H(r) = 1 + \left(\frac{r_{\text{M2}}}{r}\right)^6$$

  where r_{M2} will be related to the charge and tension of the M2-brane and $\partial_I \partial_I H(r) = 0$.
- In terms of this function we have:

$$\begin{aligned} f_{(1)}(r) &= H(r)^{-\frac{2}{3}} \\ f_{(2)}(r) &= H(r)^{\frac{1}{3}} \\ f_{(3)}(r) &= -\frac{\partial}{\partial r}\left(H(r)^{-1}\right) \end{aligned}$$

- We can now go on to evaluate its charge and mass in terms of the parameter r_{M2}.
- For the charge, we easily find that:

$$\tilde{F}_{J_1 J_2 \cdots J_7} = 6(r_{\mathsf{M2}})^6 \, \epsilon_{I J_1 J_2 \cdots J_7} \, \frac{x^I}{r^8}$$

from which:

$$Q_{\mathsf{M2}} = \int_{\Sigma_7} \tilde{F} = 6(r_{\mathsf{M2}})^6 \, \Omega_7 = 2\pi^4 r_{\mathsf{M2}}^6$$

where $\Omega_7 = \pi^4/3$ is the volume of a unit 7-sphere.

- By comparing the metric with Newton's law for weak fields, we get a relation between r_{M2} and the brane tension.
- The general formula for a p-brane of M-theory is:

$$g_{00} \sim -\left(1 - \frac{(2\pi\ell_p)^9}{2\pi} \frac{n_p T_p}{9\,\Omega_{9-p}} \frac{1}{r^{8-p}}\right)$$

where n_p is the (integer) number of branes described by the solution.

- Applying this for M2-branes, we find:

$$r_{\mathsf{M2}}^6 = \frac{(2\pi\ell_p)^9}{2\pi} \frac{n_{\mathsf{M2}} T_{\mathsf{M2}}}{6\,\Omega_7} = \frac{(2\pi\ell_p)^9}{2\pi} \frac{n_{\mathsf{M2}} T_{\mathsf{M2}}}{2\pi^4}$$

- We can now combine the results we have found to derive a relation between the tension and charge of a single M2-brane:

$$Q_{\mathsf{M2}} = \frac{(2\pi\ell_p)^9}{2\pi} T_{\mathsf{M2}}$$

- For the $M5$-brane we take the coordinates on the brane to be $y^a = (y^0, y^1, \cdots, y^5)$ while the coordinates transverse to the brane will be $x^I = (x^1, x^2, \cdots, x^5)$.
- We assume a symmetry $SO(5,1) \times SO(5)$ and also translational invariance in the y-coordinates.
- This fixes the metric to be of the form:

$$ds^2 = g_{(1)}(r)\, dy^a dy^a + g_{(2)}(r)\, dx^I dx^I$$

while the magnetic flux is specified by giving the component:

$$F_{012345r} = g_{(3)}(r)$$

Here r is the radial distance from the brane:

$$r = \sqrt{(x^1)^2 + (x^2)^2 + \cdots + (x^5)^2}$$

- Thus we again need to find three functions, $g_{(1)}(r), g_{(2)}(r)$ and $g_{(3)}(r)$.
- These functions are again determined by a single harmonic function:

$$H'(r) = 1 + \left(\frac{r_{\mathsf{M5}}}{r}\right)^3$$

  where r_{M5} will be related to the charge and tension of the 5-brane, and $\partial_I \partial_I H'(r) = 0$.
- In terms of this function we have:

$$\begin{aligned}
g_{(1)}(r) &= H'(r)^{-\frac{1}{3}} \\
g_{(2)}(r) &= H'(r)^{\frac{2}{3}} \\
g_{(3)}(r) &= -\frac{\partial}{\partial r}\left(H'(r)^{-1}\right)
\end{aligned}$$

- This time we find that r_{M5} is related to the charge by:

$$Q_{\text{M5}} = 8\pi^2 r_{\text{M5}}^3$$

- Using the Newtonian approximation, we also find the relation between r_{M5} and the 5-brane tension T_{M5} to be:

$$r_{\text{M2}}^3 = \frac{(2\pi\ell_p)^9}{2\pi}\frac{n_{\text{M2}}T_{\text{M2}}}{3\,\Omega_4} = \frac{(2\pi\ell_p)^9}{2\pi}\frac{n_{\text{M5}}T_{\text{M5}}}{8\pi^2}$$

Combining the two we have, for a single M5-brane:

$$Q_{\text{M5}} = \frac{(2\pi\ell_p)^9}{2\pi}T_{\text{M5}}$$

- We now invoke the Dirac quantisation condition on charges, which says that:

$$Q_{\mathrm{M2}} Q_{\mathrm{M5}} = \frac{(2\pi\ell_p)^9}{2\pi}\, 2\pi n$$

where n is an integer.

- It follows that the tensions of the $M2$ and $M5$ branes are related by:

$$T_{\mathrm{M2}} T_{\mathrm{M5}} = \frac{(2\pi)^2}{(2\pi\ell_p)^9}\, n$$

- We will see that the tensions satisfy the above relation with $n = 1$.

- The tension of the branes we have discussed is just their mass per unit worldvolume, with units of $(\text{length})^{-p-1}$.
- Since M-theory has only one dimensional parameter ℓ_p, we can predict that:

$$\mathcal{T}_{M2} \sim \frac{1}{\ell_p^3}, \quad \mathcal{T}_{M5} \sim \frac{1}{\ell_p^6}$$

- We will shortly argue that the correct answers are:

$$\mathcal{T}_{M2} = \frac{2\pi}{(2\pi\ell_p)^3}, \quad \mathcal{T}_{M5} = \frac{2\pi}{(2\pi\ell_p)^6}$$

- Thus we have argued that there is some supersymmetric theory defined in 11 flat spacetime dimensions which has massless fields including a graviton, as well as stable 2-branes and 5-branes.
- We refer to this as M-theory.
- There are two kinds of limitations in our knowledge of M-theory:
  (i) We have formulated it in a fixed spacetime background and it is not clear how to study it in a background-independent way (the brane solutions are specific to flat spacetime).
  (ii) It is not obvious that it has a consistent ultraviolet completion.

- The first issue is also a problem in string theory.
- However the second one is new. In string theory, using the perturbative expansion, ultraviolet finiteness can be quite convincingly demonstrated. The stringy nature cuts off UV infinities.
- We may suspect that something similar holds in M-theory. Its brane excitations could perhaps provide an ultraviolet cutoff.
- So far, we don't know whether this is true and if so, which brane is responsible. But the most logical possibility is that the M2-brane, or membrane, governs the consistency of M-theory.

Outline

Compactification to 10d

- In contrast to 11d, where supergravity (in flat spacetime) is unique, in 10d there are three distinct classical supergravity actions: type IIA, type IIB and type I.
- Each of them is associated to a superstring theory: type IIA, type IIB and type I/heterotic.
- Now, compactifying 11d supergravity on a circle must lead to 10d supergravity, therefore to one of the theories listed above.
- This is very interesting and suggests a close relation between M-theory in 11d and superstring theory in 10d.

- To compactify 11d supergravity, we have to split the 11d metric into components:

$$G_{MN} \to G_{\mu\nu},\ G_{\mu\,10},\ G_{10\,10}$$

where $\mu, \nu = 0, 1, \cdots, 9$.

- Thus the 11d metric gives rise to a metric, a vector field and a scalar in 10d.
- Similarly the 3-form gives:

$$C_{MNP} \to C_{\mu\nu\rho},\ C_{\mu\nu\,10}$$

and gives rise to a 3-form as well as a 2-form in 10d.

- This bosonic spectrum is identical to that of type IIA supergravity.
- Since circle compactification does not break supersymmetry, we can be quite sure that the 10d theory thus obtained is indeed going to be type IIA supergravity.
- To see this more explicitly, we must parametrise the 11d metric properly.

- Let us first look at the action of type IIA supergravity in a canonical normalisation:

$$S_{IIA} = \frac{2\pi}{(2\pi\ell_s)^8}\left[\int d^{10}x\sqrt{-||g||}\, e^{-2\Phi}\left(R + 4\,\partial_\mu\Phi\partial^\mu\Phi - \tfrac{1}{2}|H_3|^2\right)\right.$$

$$\left. -\tfrac{1}{2}\int d^{10}x\sqrt{-||g||}\left(|F_2|^2 + |F_4 + A\wedge H_3|^2\right) - \tfrac{1}{2}\int B_2\wedge F_4\wedge F_4\right]$$

where:

Φ : dilaton, $\quad e^{\langle\Phi\rangle} = g_s$

$A : A_\mu\, dx^\mu,$ Ramond-Ramond 1-form

$B_2 : B_{\mu\nu}\, dx^\mu\wedge dx^\nu,$ NS-NS 2-form

$A_3 : A_{\mu\nu\rho}\, dx^\mu\wedge dx^\nu\wedge dx^\rho,$ Ramond-Ramond 3-form

$F_2 = dA,\; H_3 = dB_2,\; F_4 = dA_3$

- A useful trick is to instead work with the 11d vielbein E_M^A. Let us start by parametrising it as:

$$E_M^A = \begin{pmatrix} e_\mu^a & 0 \\ 0 & e^\Phi \end{pmatrix}$$

where e_μ^a is the 10d vielbein, Φ is the 10d scalar and we are temporarily setting the 10d 1-form A_μ to 0.

- With the above parametrisation:

$$||E||\, R(E) \to e^\Phi ||e|| \left(R(e) + 4\, \partial\Phi\, \partial\Phi \right)$$

- Comparing with the type IIA action we see that this is not what we want, so we perform a Weyl rescaling:

$$\begin{pmatrix} e_\mu^a & 0 \\ 0 & e^\Phi \end{pmatrix} \to e^{\gamma\,\Phi} \begin{pmatrix} e_\mu^a & 0 \\ 0 & e^\Phi \end{pmatrix}$$

  which has the effect:

$$\left\| \begin{matrix} e_\mu^a & 0 \\ 0 & e^\Phi \end{matrix} \right\| \to e^{11\gamma\,\Phi} \left\| \begin{matrix} e_\mu^a & 0 \\ 0 & e^\Phi \end{matrix} \right\|, \quad R \to e^{-2\gamma\,\Phi} R$$

  so the RHS gets multiplied by $e^{9\gamma\,\Phi}$.

- Thus we require:

$$9\gamma + 1 = -2$$

- It follows that $\gamma = -1/3$.

- Thus the correct decomposition is:

$$E_M^A = e^{-\Phi/3} \begin{pmatrix} e_\mu^a & 0 \\ e^\Phi A_\mu & e^\Phi \end{pmatrix}$$

  where we have now included the 10d 1-form as well.

- From this we easily find that:

$$G_{MN} = e^{-2\Phi/3} \begin{pmatrix} g_{\mu\nu} + e^{2\Phi} A_\mu A_\nu & e^{2\Phi} A_\mu \\ e^{2\Phi} A_\nu & e^{2\Phi} \end{pmatrix}$$

- On the other hand, the M-theory 3-form becomes, on dimensional reduction:

$$\begin{aligned} C_{MNP} &\rightarrow C_{\mu\nu\rho} = A_{\mu\nu\rho} \\ &\rightarrow C_{\mu\nu\,10} = B_{\mu\nu} \end{aligned}$$

- With these identifications, we can compactify 11d supergravity and compare with the Lagrangian of 10d type IIA supergravity.
- Notice that the former after compactifying has two parameters, ℓ_p and R_{10}. On the other hand, the latter has two parameters, ℓ_s and $e^{\langle \Phi \rangle} = g_s$.
- Thus we should find a relation between the two pairs of parameters. This will provide us the physical interpretation of the result.

- Comparing Lagrangians, we right away find:

$$2\pi R_{10}\,\frac{2\pi}{(2\pi\ell_p)^9} = \frac{1}{g_s^2}\frac{2\pi}{(2\pi\ell_s)^8}$$

- Additionally the relation between metrics tells us that:

$$\ell_p = g_s^{1/3}\ell_s$$

- Inserting the latter in the former, we find:

$$R_{10} = g_s\ell_s$$

- This is a truly striking result! It relates the radius of a compact dimension to the string length and coupling.

- The proposed interpretation is as follows.
- When we compactify M-theory on a circle of radius R_{10}, it becomes type IIA string theory in the limit $R_{10} \to 0$.
- Conversely, if we start with 10d type IIA string theory and take the coupling $g_s \to \infty$, the string description breaks down. In this limit, a new space dimension opens up and we get M-theory.
- This is a duality.

Outline

Branes and dualities from M-theory

- Let us now try to justify the proposal that M-theory and type IIA string theory are related as claimed:

$$\text{M-theory} \quad \underset{\xleftarrow{\text{strong coupling}}}{\xrightarrow{\text{compactification}}} \quad \text{type IIA string theory}$$

- First we review the spectrum of branes in string theory.
- In a classic calculation of the force between D-branes using open string theory, it was shown that:

$$\mathcal{T}_{Dp} = \frac{1}{g_s} \frac{2\pi}{(2\pi\ell_s)^{p+1}}$$

- We also have:

$$\mathcal{T}_{F1} = \frac{2\pi}{(2\pi\ell_s)^2}, \qquad \mathcal{T}_{NS5} = \frac{1}{g_s^2} \frac{2\pi}{(2\pi\ell_s)^6}$$

- Let us now try to derive these results starting with M-branes.
- In principle this might not be possible at all!
- The tensions of string theory branes were calculated at weak coupling. One might expect them to be renormalised.
- However, the fact that these are supersymmetric branes saves us.
- It can be argued that the tension of supersymmetric branes is exact. This is an example of a non-renormalisation theorem.
- Therefore we can compare the tensions of M-theory branes with type IIA branes, and we will now do this.

- When we compactify on a circle, the M2-brane can be either wrapped on the circle or transverse to the circle.
- In the first case it looks (as $R_{10} \to 0$) like a string or 1-brane. In the second case it is a 2-brane.
- Doing the same thing for an M5-brane we get a 4-brane or a 5-brane.
- To match with the branes in string theory, the only possibilities are:

$$\text{wrapped M2} \to \text{F1}, \qquad \text{transverse M2} \to \text{D2}$$
$$\text{wrapped M5} \to \text{D4}, \qquad \text{transverse M5} \to \text{NS5}$$

- This is a definite set of predictions!
- Start with the M2-brane. We had proposed that its tension is:

$$\mathcal{T}_{M2} = \frac{1}{4\pi^2 \ell_p^3}$$

- Wrapping on the circle, the tension of the resulting brane is:

$$\begin{aligned} \mathcal{T}_{M2 \text{ wrapped}} &= \mathcal{T}_{M2} \times 2\pi R_{10} \\ &= \frac{1}{2\pi \ell_s^2} \end{aligned}$$

  which is the correct tension for the fundamental string.
- But this result really serves to fix the tension of the M2-brane, which we had not determined previously.

- Now consider the transverse M2-brane. Its tension is:

$$\begin{aligned}
\mathcal{T}_{M2} &= \frac{1}{4\pi^2\ell_p^3} \\
&= \frac{1}{4\pi^2(g_s^{1/3}\ell_s)^3} \\
&= \frac{1}{g_s}\frac{1}{4\pi^2\ell_s^3} \\
&= \mathcal{T}_{D2}
\end{aligned}$$

- This is a truly remarkable agreement!

- For the M5-brane, things work out as follows.
- We have proposed that its tension is:

$$\mathcal{T}_{M5} = \frac{1}{32\pi^5 \ell_p^6}$$

- Wrapping on the circle, the tension of the resulting brane is:

$$\begin{aligned}
\mathcal{T}_{M5\ \text{wrapped}} &= \mathcal{T}_{M5} \times 2\pi R_{10} \\
&= \frac{g_s \ell_s}{16\pi^4 g_s^2 \ell_s^6} \\
&= \frac{1}{g_s} \frac{2\pi}{(2\pi \ell_s)^5} \\
&= \mathcal{T}_{D4}
\end{aligned}$$

which is correct, but again can be thought of as a determination of $\mathcal{T}_{M5}$.

- Finally, the transverse M5-brane gives:

$$
\begin{aligned}
\mathcal{T}_{M5} &= \frac{1}{32\pi^5 \ell_p^6} \\
&= \frac{1}{g_s^2} \frac{1}{32\pi^5 \ell_s^6} \\
&= \frac{1}{g_s^2} \frac{2\pi}{(2\pi \ell_s)^6} \\
&= \mathcal{T}_{NS5}
\end{aligned}
$$

which is again a remarkable confirmation of the equivalence between M-theory and type IIA string theory.

- This still leaves the D0 and D6 branes.
- Note that the mass of a D0 brane is:

$$\mathcal{T}_0 = \frac{1}{g_s \ell_s} = \frac{1}{R_{10}}$$

- What mode of M-theory can have this mass?
- We will argue that it is the mode with one unit of momentum along the compact direction.

- Indeed, on a compact dimension of length L, the momentum is quantised in integers as:

$$p = \frac{2\pi n}{L}$$

- For massless particles in 11d, we have:

$$E^2 = p_1^2 + \cdots p_9^2 + p_{10}^2$$

- After compactification, a fixed value of p_{10} will appear as a mass.
- Since $L = 2\pi R_{10}$, we have:

$$\text{mass in 10d} = |p_{10}| = \frac{n}{R_{10}}$$

- Thus a single D0-brane ($n = 1$) is a single unit of momentum along x^{10}.
- But we have a new prediction. For every integer n, there should be a bound state of n D0-branes!.
- This is a statement about string theory that we did not know before the discovery of M-theory! And it has now been verified directly within string theory.

- Let us see how the charge of the D0-brane works out.
- Note that in 10 dimensions, a 0-brane is surrounded by S^8. Thus its charge is the integral of the spatial components of an 8-form.
- The Poincaré dual of this 8-form in 10d is a 2-form which must be the field strength of the Ramond-Ramond 1-form A_μ.
- Therefore the D0-brane is electrically charged under A_μ. From the M-theory point of view, the latter is a Kaluza-Klein gauge field.

- As we have seen, it is possible to have a dual object which is surrounded by S^2 and is a magnetic source for the same field strength.
- Such an object must be a 6-brane. Indeed it is known that in type IIA string theory, the 6-brane is the magnetic dual of the 0-brane.
- In M-theory, A_μ is a Kaluza-Klein gauge field. Therefore a magnetically charged object must be a Kaluza-Klein monopole.

- Let us first discuss Kaluza-Klein monopoles abstractly.
- Consider the metric in 4 Euclidean dimensions:

$$ds^2_{\text{Taub-NUT}} \quad = \quad V(\vec{x})\, d\vec{x} \cdot d\vec{x} + \frac{1}{V(\vec{x})}\Big(dy + \vec{A} \cdot d\vec{x}\Big)^2$$

  where $\vec{A}$ is the vector potential for a magnetic monopole in 3 dimensions:

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

  and $V(\vec{x})$ is a harmonic function in 3d determined by:

$$\vec{\nabla} V = -\vec{B}$$

- This metric solves the 4d Euclidean Einstein equation without sources.

- We choose a specific harmonic function V depending on a real number R, namely:

$$V(\vec{x}) = 1 + \frac{R}{2r}$$

where $r = |\vec{x}|$.

- Thus the magnetic field is:

$$\vec{B} = \frac{R}{2}\frac{\vec{x}}{r^3}$$

- As $r \to 0$ this metric is apparently singular due to the terms:

$$\frac{R}{2r}\, dr^2 + \frac{2r}{R}\, dy^2$$

- The singularity can be avoided as follows. Define:

$$\tilde{r} = \sqrt{2rR}$$

- The dangerous terms then become:

$$d\tilde{r}^2 + \frac{\tilde{r}^2}{R^2}\, dy^2$$

- Now the second term is non-singular only if y is an angle with periodicity $2\pi R$.

- Being a non-singular metric with a monopole charge, this is called a Kaluza-Klein monopole (if we add $-dt^2$ to make it a particle).
- The monopole is located at the core near $r \to 0$, where the Kaluza-Klein circle shrinks to zero size.
- Let us now embed this solution in M-theory by taking the $\vec{x}$ directions to be x^7, x^8, x^9 and the KK direction y to be x^{10} with periodicity $2\pi R_{10}$.
- The resulting object is translational invariant along $x^1, x^2, \cdots, x^6$ so it is a 6-brane.
- And it is magnetically charged under the Kaluza-Klein gauge field arising from compactification of x^{10}.

- So we have a candidate for the D6-brane of type IIA string theory.
- To compute the tension, we just integrate the energy density $\vec{\nabla}^2 V$ along the four dimensions in which the monopole is embedded.
- Since V is independent of the compact direction, we get:

$$\begin{aligned} \mathcal{T}_{KK6} &= \frac{2\pi}{(2\pi\ell_p)^9} \times 2\pi R_{10} \int d^3x\, \vec{\nabla}^2 V \\ &= \frac{2\pi}{(2\pi\ell_p)^9} \times (2\pi R_{10})^2 \\ &= \frac{1}{g_s}\frac{2\pi}{(2\pi\ell_s)^7} = \mathcal{T}_{D6} \end{aligned}$$

- Success!!

- We know that branes in type IIB string theory can be obtained from those of type IIA by circle compactification and T-duality.
- It is easy to check that this reproduces the tensions of all the branes of type IIB: D1,D3,D5,D7 and NS5.
- However it gives us some more information.
- Recall that in type IIB there are two types of strings:

F-strings of tension: $\dfrac{1}{2\pi\ell_s^2}$

D-strings of tension: $\dfrac{1}{g_s}\dfrac{1}{2\pi\ell_s^2}$

- It has been argued that type IIB string theory has S-duality:

$$g_s \to \frac{1}{g_s}, \quad \ell_s \to \sqrt{g_s}\,\ell_s$$

- Under this symmetry, the F-string and D-string are interchanged. One can easily check that their tensions get interchanged.
- It has also been shown that p F-strings and q D-strings form stable bound states called (p, q) strings, if p, q are co-prime.
- These have tension:

$$\mathcal{T}_{p,q} = \sqrt{p^2 + \frac{q^2}{g_s^2}}\,\frac{1}{2\pi\ell_s^2}$$

- We will now see that M-theory explains both these facts in a beautiful way.

- Suppose we compactify M-theory on two circles x^{10}, x^9 of radii R_{10}, R_9 to get type IIA string theory in 9 dimensions.

  M2-brane wrapped on x^{10} $\rightarrow$ type IIA F-string

  M2-brane wrapped on x^9 $\rightarrow$ D2-brane wrapped on x^9

- Now let us perform a T-duality on x^9:

  type IIA F-string $\rightarrow$ type IIB F-string

  D2-brane wrapped on x^9 $\rightarrow$ type IIB D-string

- It follows that:

  F-string $\leftrightarrow$ D-string (IIB) $\Longleftrightarrow$ $x^9 \leftrightarrow x^{10}$ (M-theory)

- But the latter is part of Lorentz invariance and is a manifest geometrical symmetry of M-theory! This "proves" S-duality.

- In fact we easily find that:

$$g_s \text{ (IIB)} = \frac{R_{10}}{R_9}, \qquad \ell_s \text{ (IIB)} = \sqrt{\frac{\ell_p^3}{R_{10}}}$$

- Next, suppose in the same compactification, we wrap an M2-brane p times along x^{10} and q times along x^9.
- The result, after T-dualising on x^9, is a string in type IIB theory that has p units of F-string charge as well as q units of D-string charge.
- Its tension will be:

$$\begin{aligned} \mathcal{T}_{M2\,\text{wrapped}} &= \mathcal{T}_{M2}\sqrt{p(2\pi R_{10})^2 + q(2\pi R_9)^2} \\ &= \sqrt{p^2 + \frac{q^2}{g_s^2}}\,\frac{1}{2\pi\ell_s^2} = \mathcal{T}_{p,q} \end{aligned}$$

reducing the existence of (p, q) strings to Pythagoras theorem!

Outline

M2-brane field theory: Motivation

- In string theory, the discovery of D-branes brought about a revolution in our understanding of quantum field theory.
- Like any soliton (e.g. monopole, cosmic string), a D-brane possesses degrees of freedom that are bound to it.
- These can be found by considering the light modes of the bulk theory expanded about the brane solution.
- They are usually non-gravitational degrees of freedom, namely gauge fields, scalar fields and fermions.

- The Dirichlet description of branes, in terms of open string endpoints, provides an explicit construction of these degrees of freedom - by quantising open strings.
- It also provides information about their interactions.
- Thus we have a worldvolume field theory on D-branes.
- This is generically a non-renormalisable field theory with arbitrarily high-derivative operators suppressed by the string scale ℓ_s.

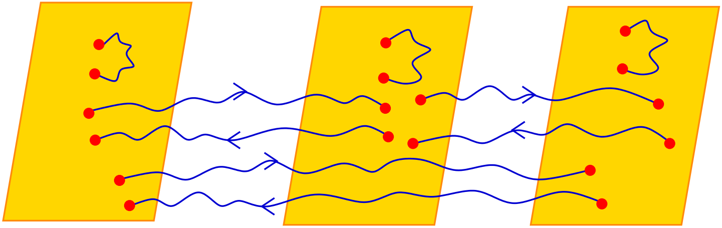
- For D-branes, this field theory is in principle completely calculable in string perturbation theory.
- On taking the string length $l_s = \sqrt{\alpha'} \to 0$ it reduces to a conventional Yang-Mills type field theory.
- Since the D-branes are supersymmetric, the field theory will also be supersymmetric.
- By taking different brane configurations in different backgrounds, a variety of field theories can be "engineered" in this way.

- The simplest special case arises when we take superstring theory in flat 10d spacetime with a stack of N parallel Dp-branes.
- The result (as $l_s \to 0$) is maximally supersymmetric Yang-Mills theory:

$$\mathcal{L} = \frac{1}{g^2_{YM}} \mathrm{tr}\Big\{ -\frac{1}{4} \boldsymbol{F}_{\mu\nu} \boldsymbol{F}^{\mu\nu} - \frac{1}{2} D_\mu \boldsymbol{X}^i D^\mu \boldsymbol{X}^i - \frac{1}{4} [\boldsymbol{X}^i, \boldsymbol{X}^j]^2 + \text{fermions} \Big\}$$

  where $i = 1, 2, \cdots, 9 - p$.
- Here $\boldsymbol{A}_\mu, \boldsymbol{X}^i, \boldsymbol{\psi}$ are all in the adjoint of $U(N)$.
- The diagonal components of the scalars parametrise the transverse directions to the brane.

- The group-theory structure has a nice pictorial representation in terms of open strings, shown here for the case of $U(3)$.



- This picture has it all: Cartan subalgebra, positive roots, negative roots, simple roots...
- Using orientifolds (orientation-reversing hyperplanes) one can get the other classical gauge groups, $SO(N), Sp(N)$.
- Using orbifolds (which do not reverse orientation) one can get direct product gauge groups and bi-fundamental matter.

- Many features of the field theory can be understood using branes along with the underlying superstring theory:
  - Nonabelian gauge symmetry (from stretched open strings).
  - Higgs mechanism (from transverse motions of the branes).
  - Supersymmetry (from spatial alignment).
  - Duality (from duality of string theory).
  - Monopoles (from D-strings ending on D3-branes).
  - Conformal invariance for D3 branes (from constancy of dilaton).

- Conversely the field theory explains many aspects of the underlying string theory:
    - M(atrix) theory.
    - AdS/CFT correspondence (here the field theory *is* the entire string theory!).

- We would like to have a similar understanding for the worldvolume theory on M-branes.
- As we have seen, M-theory has two kinds of stable branes:
  - M2-branes (membranes)
  - M5-branes
- Besides the above motivations, one additional motivation is that one may be able to use the M-brane field theory to give a precise definition of M-theory.
- In the following lectures I will describe some recent progress in understanding the field theory on multiple M2-branes, which remained unknown for a decade.

- We have seen that type IIA string theory lifts to M-theory as $g_s \to \infty$.
- In the process D2-branes lift to M2-branes.
- The field theory on N D2-branes is just maximally supersymmetric or $\mathcal{N} = 8$ Yang-Mills theory in $(2+1)$d, which has 7 transverse scalars.
- This is a super-renormalisable theory that inherits its coupling from the string coupling g_s:

$$g_{\mathsf{YM}} = \sqrt{\frac{g_s}{l_s}}$$

- Therefore in the M-theory limit, $g_{YM} \longrightarrow \infty$.
- For a super-renormalisable theory, this is the infrared limit.
- Thus we may define:

$$\mathcal{L}_{M2} = \lim_{g_{YM} \to \infty} \frac{1}{g_{YM}^2} \mathcal{L}_{D2}$$

  and the problem is to find an explicit form for this limiting theory.
- The limiting theory, if interacting, must be an infrared fixed point and therefore conformal invariant.
- Also, if the brane interpretation is to make sense, the field theory should have 8 scalars with an SO(8) global symmetry, describing transverse motion.

- Thus we are looking for a $(2+1)$d field theory with:
  - $\mathcal{N}=8$ supersymmetry.
  - $SO(8)$ global symmetry.
  - Superconformal invariance.
- Possible application also to quantum critical points in Condensed Matter physics!

Outline

Single M2 brane

- For a single D2-brane the theory is relatively simple.
- It has been known for a long time how to relate it to a single M2-brane.
- In the limit $\ell_s \to 0$ (the Yang-Mills limit) the D2-brane is described by a free Abelian gauge theory:

$$\mathcal{L}_{\text{single } D2} = \frac{1}{g_{\mathsf{YM}}^2}\Big\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu X^i \partial^\mu X^i + \text{ fermions}\Big\}$$

where $X^i, i = 1, 2, \cdots, 7$ parametrise the 7 transverse directions to the D2-brane.

- The M2-brane action has to be different, because the brane now has 8 transverse directions so there should be 8 scalar fields.
- The 8th scalar arises by a duality transformation as follows:

$$-\frac{1}{4g_{\mathsf{YM}}^2}F_{\mu\nu}F^{\mu\nu} \to \frac{1}{2}\varepsilon^{\mu\nu\lambda}B_\mu F_{\nu\lambda} - \frac{1}{2}g_{\mathsf{YM}}^2 B_\mu B^\mu$$

- Integrating out the auxiliary field B_μ gives:

$$B_\mu = \frac{1}{2g_{\mathsf{YM}}^2}\epsilon_{\mu\nu\lambda}F^{\nu\lambda} = \frac{1}{g_{\mathsf{YM}}^2}\tilde{F}_\mu$$

  where $\tilde{F}_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda}F^{\nu\lambda}$ is the dual of $F_{\mu\nu}$.
- Integrating out A_μ instead gives:

$$\partial_\mu B_\nu - \partial_\nu B_\mu = 0 \implies B_\mu = \frac{1}{g_{\mathsf{YM}}^2}\partial_\mu X^8$$

  where X^8 is a new scalar field, whose Lagrangian is then:

$$-\frac{1}{2g_{\mathsf{YM}}^2}\partial_\mu X^8 \partial^\mu X^8$$

- The field X^8 is actually a compact scalar. To see this, note that the quantisation law for electric charges in the theory is:

$$\int_{\Sigma_1} \tilde{F} = 2\pi n g_{YM}$$

- Let us take the minimum quantum of charge, $n = 1$. From the relation between B_μ and $\tilde{F}_\mu$ we have:

$$\int_{\Sigma_1} B = \frac{2\pi}{g_{YM}}$$

It follows that:

$$\int_{\Sigma_1} dX^8 = 2\pi g_{YM}$$

which means X^8 has periodicity $2\pi g_{YM}$.

- Note that in our normalisation, X^I have dimensions of length^{-1}. If we want to go to standard conventions where their dimension is length$^{\frac{1}{2}}$ we must multiply by $\ell_p^{\frac{3}{2}}$.
- Performing this on X^8 we find that the periodicity becomes:

$$2\pi g_{\mathsf{YM}} \times \ell_p^{\frac{3}{2}} = 2\pi\sqrt{\frac{g_s}{\ell_s}} \times g_s^{\frac{1}{2}}\ell_s^{\frac{3}{2}} = 2\pi g_s \ell_s = 2\pi R_{10}$$

  which is the periodicity of the ambient space.
- In the limit $g_{\mathsf{YM}} \to \infty$, equivalent to $R_{10} \to \infty$, the scalar X decompactifies, and we can relabel it X^8 to find the theory for a single M2 brane:

$$\mathcal{L}_{\text{single } M2} = -\frac{1}{2}\partial_\mu X^I \partial^\mu X^I + \text{ fermions}$$

  where $I = 1, 2, \cdots, 8$.

- This theory satisfies all the desired criteria. However that was relatively easy, since it's a free field theory.
- As we know, multiple D-branes form an interacting Yang-Mills theory.
- In this case the problem of finding the corresponding multiple M2-brane theory is much more difficult.

Outline

Bagger-Lambert theory

- Let us first give arguments for some general properties of the multiple membrane theory.
- Since membranes have 8 transverse directions, the field theory needs to have 8 scalar fields. By supersymmetry, it also must have 4 two-component fermions.
- In $(2+1)$d the canonical dimensions of fields are as follows:

$$[X] = \frac{1}{2}, \quad [\Psi] = 1$$

- Then scale invariance restricts the interactions to be of the form:

$$(X)^6 \qquad \text{and} \qquad (X)^2 \bar{\Psi}\Psi$$

- A key insight was that the bosonic field content can, and should, include a nondynamical (Chern-Simons) gauge field with a Lagrangian:

$$S_{CS} = \frac{k}{2\pi}\mathrm{tr}\int \epsilon^{\mu\nu\lambda}\left(\boldsymbol{A}_\mu \partial_\nu \boldsymbol{A}_\lambda + \frac{2}{3}\boldsymbol{A}_\mu \boldsymbol{A}_\nu \boldsymbol{A}_\lambda\right)$$

- This is special to $(2+1)$d. This action is a topological invariant and does not have any local gauge-invariant degrees of freedom.
- Therefore the gauge field does not contribute to the dynamical degrees of freedom, though it can help to close the supersymmetry algebra.

- The first successful attempt to find an interacting CFT satisfying all the requirements was made by Bagger and Lambert (a crucial step was provided independently by Gustavsson). This is called the Bagger-Lambert A_4 theory.
- It relies on a mathematical structure called a Euclidean 3-algebra.
- This involves generators T^A, a "three-bracket", and a totally antisymmetric 4-index structure constant f^{ABCD} satisfying:

$$[T^A, T^B, T^C] = f^{ABC}{}_D T^D$$

- A generalised "trace" over the three-algebra indices provides a "metric" on the 3-algebra:

$$h^{AB} = \mathrm{tr}(T^A, T^B)$$

- The structure constants satisfy the 'fundamental identity':

$$f^{AEF}{}_G\, f^{BCDG} - f^{BEF}{}_G\, f^{ACDG} + f^{CEF}{}_G\, f^{ABDG} - f^{DEF}{}_G\, f^{ABCG} = 0$$

- The scalars X^I and fermions Ψ are three-algebra valued and the interactions are:

$$\sim \mathrm{Tr}\Big([X^A, X^B, X^C]^2\Big) \quad \text{and} \quad \sim \mathrm{Tr}\Big([\bar{\Psi}^A, X^B, \Psi^C] X^D\Big)$$

- And there is a gauge field A_μ^{AB} with minimal couplings to the scalars and fermions, and a Chern-Simons interaction:

$$k\varepsilon^{\mu\nu\lambda}\Big(f_{ABCD} A_\mu{}^{AB} \partial_\nu A_\lambda{}^{CD} + \tfrac{2}{3} f_{AEF}{}^G\, f_{BCDG}\, A_\mu{}^{AB} A_\nu{}^{CD} A_\lambda{}^{EF}\Big)$$

where k is the quantised level.

- Here f^{ABCD} has been left abstract, but it was later shown that there is only one consistent solution of the fundamental identity:

$$f^{ABCD} = \epsilon^{ABCD}, \quad A, B, C, D = 1, \cdots, 4$$

- By taking suitable linear combinations of A_μ^{AB} one finds a pair of $SU(2)$ gauge fields $\boldsymbol{A}_\mu, \tilde{\boldsymbol{A}}_\mu$.
- The Chern-Simons term reduces to the difference of two actions:

$$k \ \mathrm{tr}\left(\boldsymbol{A} \wedge d\boldsymbol{A} + \tfrac{2}{3}\boldsymbol{A} \wedge \boldsymbol{A} \wedge \boldsymbol{A} - \tilde{\boldsymbol{A}} \wedge d\tilde{\boldsymbol{A}} - \tfrac{2}{3}\tilde{\boldsymbol{A}} \wedge \tilde{\boldsymbol{A}} \wedge \tilde{\boldsymbol{A}}\right)$$

- The scalars and fermions are bi-fundamentals:

$$X^I_{a\dot{a}} \quad \text{and} \quad \Psi_{a\dot{a}}$$

and, e.g.

$$D_\mu X = \partial_\mu X - \boldsymbol{A}_\mu X + X\tilde{\boldsymbol{A}}_\mu$$

- In this way the theory reduces to a conventional gauge theory.
- Importantly we see that parity is preserved if we require $\boldsymbol{A} \leftrightarrow \tilde{\boldsymbol{A}}$ under parity.

- To complete this discussion let us write down the action of Bagger-Lambert theory in the "bi-fundamental" notation:

$$
\begin{aligned}
S_{\mathcal{A}_4} \;=\; \frac{k}{2\pi}\int d^3x\,\mathrm{Tr}\Big[& -(\tilde{D}^\mu X^I)^\dagger \tilde{D}_\mu X^I + i\,\bar{\Psi}^\dagger \tilde{\Gamma}^\mu \tilde{D}_\mu \Psi \\
& - \tfrac{8}{3}\, X^{IJK\dagger}\, X^{IJK} \\
& -\tfrac{1}{3} i\, \bar{\Psi}^\dagger \tilde{\Gamma}_{IJ}[X^I, X^{J\dagger}, \Psi] + \tfrac{1}{3} i\, \bar{\Psi}\tilde{\Gamma}_{IJ}[X^{I\dagger}, X^J, \Psi^\dagger] \\
& +\tfrac{1}{2}\,\epsilon^{\mu\nu\lambda}\Big(A_\mu \partial_\nu A_\lambda + \tfrac{2}{3} A_\mu A_\nu A_\lambda \\
& - \widetilde{A}_\mu \partial_\nu \widetilde{A}_\lambda - \tfrac{2}{3}\widetilde{A}_\mu \widetilde{A}_\nu \widetilde{A}_\lambda\Big)\Big]
\end{aligned}
$$

where:

$$
\begin{aligned}
X^{IJK} &= X^{[I} X^{J\dagger} X^{K]} \\
[X^I, X^{J\dagger}, \Psi] &= X^{[I} X^{J]\dagger}\Psi - X^{[I}\Psi^\dagger X^{J]} + \Psi X^{[I\dagger} X^{J]}
\end{aligned}
$$

Outline

Interpretation of BL theory

- One should now check if this theory really describes M2-branes.
- It should be noted that there is no coupling constant in the theory. This is as expected since M-theory also has no coupling constant.
- However the level k of the Chern-Simons actions acts as a coupling.
- This is a puzzle: what is k doing here?

- The interpretation of the Bagger-Lambert theory becomes clearer when we consider the Higgs mechanism.
- Take $k = 1$ to start with. If we give a vev v to one component of the bi-fundamental fields, then at energies below this vev, the Lagrangian becomes:

$$L_{BL}\Big|_{vev\ v} = \frac{1}{v^2} L_{SYM}^{U(2)} + \mathcal{O}\left(\frac{1}{v^3}\right)$$

and one $SU(2)$ gauge field has become dynamical!
- This is an unusual result. In Yang-Mills with gauge group G, when we give a vev to one component of an adjoint scalar, at low energy the Lagrangian becomes:

$$\frac{1}{g_{\mathsf{YM}}^2} L_{SYM}^{(G)}\Big|_{vev\ v} = \frac{1}{g_{\mathsf{YM}}^2} L_{SYM}^{(G' \subset G)}$$

where G' is the subgroup that commutes with the vev.

- Let's give a quick derivation of this novel Higgs mechanism:

$$\begin{aligned} L_{CS} &= \mathrm{tr}\left(\boldsymbol{A}\wedge d\boldsymbol{A} + \tfrac{2}{3}\boldsymbol{A}\wedge\boldsymbol{A}\wedge\boldsymbol{A} - \tilde{\boldsymbol{A}}\wedge d\tilde{\boldsymbol{A}} - \tfrac{2}{3}\tilde{\boldsymbol{A}}\wedge\tilde{\boldsymbol{A}}\wedge\tilde{\boldsymbol{A}}\right) \\ &= \mathrm{tr}\left(\boldsymbol{A}_-\wedge\boldsymbol{F}_+ + \tfrac{1}{6}\boldsymbol{A}_-\wedge\boldsymbol{A}_-\wedge\boldsymbol{A}_-\right) \end{aligned}$$

  where $\boldsymbol{A}_\pm = \boldsymbol{A}\pm\tilde{\boldsymbol{A}}$, $\qquad \boldsymbol{F}_+ = d\boldsymbol{A}_+ + \frac{1}{2}\boldsymbol{A}_+\wedge\boldsymbol{A}_+$.

- Also the covariant derivative on a scalar field is:

$$D_\mu X = \partial_\mu X - \boldsymbol{A}_\mu X + X\tilde{\boldsymbol{A}}_\mu$$

- If $\langle X\rangle = v\,\mathbf{1}$ then:

$$-(D_\mu X)^2 \sim -v^2(\boldsymbol{A}_-)_\mu(\boldsymbol{A}_-)^\mu + \cdots$$

- Thus, $\boldsymbol{A}_-$ is massive – but not dynamical. Integrating it out gives us:

$$-\frac{1}{4v^2}(\boldsymbol{F}_+)_{\mu\nu}(\boldsymbol{F}_+)^{\mu\nu} + \mathcal{O}\left(\frac{1}{v^3}\right)$$

  so $\boldsymbol{A}_+$ becomes dynamical.

- One can check that the bi-fundamental X^I reduces to an adjoint under $\boldsymbol{A}_+$. The rest of $\mathcal{N} = 8$ SYM assembles itself correctly.
- But how should we physically interpret this?

$$L_{BL}\Big|_{vev\ v} = \frac{1}{v^2} L_{SYM}^{U(2)} + \mathcal{O}\left(\frac{1}{v^3}\right)$$

- It seems like the M2 branes are becoming a pair of D2 branes with $g_{\mathsf{YM}} = v$.
- Have we somehow compactified the spacetime theory? This is not really possible because we have not done anything to the bulk spacetime.

- The resolution is to note that for any finite v, there are corrections to the Yang-Mills action. These decouple only as $v \to \infty$. So at best we can say that:

$$L_{BL}\Big|_{vev\ \ v\to\infty} = \lim_{v\to\infty} \frac{1}{v^2} L_{SYM}^{U(2)}$$

- The RHS is by definition the theory on two $M2$-branes! So this is more like a "proof" that the original Chern-Simons theory really is the theory on $M2$-branes.
- More precisely, this is the case far out on the moduli space (at large Higgs vev v).

- However once we introduce the Chern-Simons level k then the analysis is different:

$$L_{BL}\Big|_{vev\ v} = \frac{k}{v^2} L_{SYM}^{U(2)} + \mathcal{O}\left(\frac{k}{v^3}\right)$$

- If we take $k \to \infty, v \to \infty$ with $v^2/k = g_{\mathsf{YM}}$ fixed, then in this limit the RHS actually becomes:

$$\frac{1}{g_{\mathsf{YM}}^2} L_{SYM}^{U(2)}$$

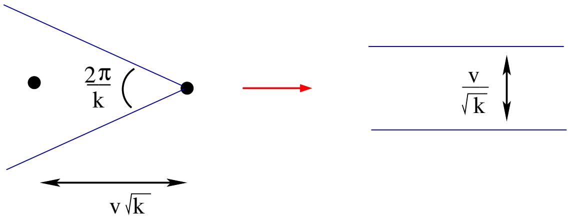
  and this is definitely the Lagrangian for two $D2$ branes at finite coupling.
- So this time we have compactified the theory! How can that be?

- It has been proposed that the level k corresponds to the order of an orbifold group.
- In this proposal, the branes described by Bagger-Lambert theory are not transverse to $\mathbb{R}^8$, but to $\mathbb{R}^8/Z_k$ for some action of the group Z_k.
- This is a potentially nice explanation since the level k is an integer which fits well with the fact that the order of a finite group is also an integer.
- The proposed action of the orbifold on the complex coordinates of the 8-dimensional space is:

$$(z_1, z_2, z_3, z_4) \rightarrow (\omega z_1, \omega z_2, \omega z_3, \omega z_4)$$

  where $\omega = \exp \frac{2\pi i}{k}$. Thus the space is better thought of as $\mathbb{C}/Z_k$.

- Now we can try to explain the apparent "compactification".
- In other similar field theories in 4d, it is known that by taking the order of an orbifold group very large and simultaneously moving the branes far away, one effectively sees a compact space.



- As $v \to \infty, k \to \infty$ the brane moves further away from the orbifold and the opening angle also becomes very small.
- In the limit, the brane sees the space as a cylinder having a compact circular direction.

- Puzzle: for $k > 2$ the standard orbifold $\mathbb{C}^4/Z_k$ has $\mathcal{N} = 6$ supersymmetry and $SU(4)$ R-symmetry.
- However, the Bagger-Lambert field theory has $\mathcal{N} = 8$ supersymmetry and $SO(8)$ R-symmetry.
- Therefore, though the basic ideas above seem to be correct, the Bagger-Lambert theory does not describe M2-branes on a standard orbifold.
- It remains possible that there is a non-standard orbifold ("M-fold") with $\mathcal{N} = 8$ supersymmetry and $SO(8)$ R-symmetry.
- This is an open problem.

- Beside the problem of interpreting the Bagger-Lambert theory, there is also the problem of finding generalisations.
- It has been proved mathematically that the 3-algebra arising in Bagger-Lambert theory is essentially unique.
- In other words there is no generalisation of the $SU(2) \times SU(2)$ theory to $SU(N) \times SU(N)$!
- How can there be a theory of 2 membranes and no theory of $N > 2$ membranes?

Outline

ABJM theory

- Subsequently a new class of field theories in $(2+1)$d were proposed (by Aharony, Bergman, Jafferis and Maldacena) to be the multiple membrane field theories.
- In this work the starting point was reversed.
- The authors started from a configuration of M2-branes at a $\mathbb{C}^4/Z_k$ orbifold:

$$(z_1, z_2, z_3, z_4) \to (\omega z_1, \omega z_2, \omega z_3, \omega z_4)$$

  where $\omega = \exp \frac{2\pi i}{k}$.
- They accepted the idea that this orbifold breaks supersymmetry (in the bulk) down to $\mathcal{N} = 6$. Therefore they looked for a field theory on the branes with $\mathcal{N} = 6$ supersymmetry.

- Theories in $(2+1)$d with this amount of supersymmetry have $SU(4)$ global symmetry rather than $SO(8)$.
- Therefore the scalar fields of the theory are naturally described as $C_I, I = 1, 2, 3, 4$.
- The technique of brane constructions in string theory was used to derive the worldvolume CFT with $\mathcal{N} = 6$ supersymmetry and $SU(4)$ R-symmetry.
- This theory has a $U(N) \times U(N)$ Chern-Simons sector for any $\mathcal{N}$, and the scalars and fermions are bi-fundamentals $(N, \bar{N})$.

- The ABJM action looks remarkably like the Bagger-Lambert action.
- There is a sextic potential for the bi-fundamental scalars C^I:

$$V(C, C^\dagger) \sim \mathrm{tr}\Big(- C^I C^{[I\,\dagger} C^J C^{J\,\dagger} C^K C^{K]\,\dagger} + C^I C^{[I\,\dagger} C^J C^{K\,\dagger} C^K C^{J]\,\dagger} \Big)$$

- The fermionic interactions are of the type:

$$C C^\dagger \bar{\Psi} \Psi$$

- And finally there is a difference of two Chern-Simons terms, with the gauge group being $U(N) \times U(N)$.

- For this theory the moduli space was found to be:

$$\frac{(\mathbb{R}^8/Z_k)^N}{S_N}$$

  which is the right moduli space for N membranes at a Z_k orbifold.

- This can be compared with the Bagger-Lambert moduli space:

$$\frac{\mathbb{R}^8 \times \mathbb{R}^8}{D_{2k}}$$

  where D_{2k} is the dihedral group.

- The ABJM moduli space for $N = 2$ does not agree with the Bagger-Lambert moduli space, confirming that the latter does not describe branes at a conventional orbifold.

- So it seems that the ABJM theory does describe multiple membranes at a conventional Z_k orbifold.
- However for the special case $k = 1$ there is no orbifold, just flat spacetime. In this case the ABJM theory should have $\mathcal{N} = 8$ supersymmetry, and translational invariance but neither of these is visible!
- The most practical use of the ABJM theory has been in the limit of large k, where it is weakly coupled.
- In this limit it was argued that there is a new AdS/CFT correspondence.

- Recall that in his original paper, Maldacena already argued that N M2-branes are dual to M-theory on $AdS_4 \times S^7$.
- This argument used the solitonic description of M2-branes. But because of the lack of knowledge of M2-brane field theory, one side of the duality could not be explored at all.
- Are we in a better situation now?
- The ABJM theory at level k is dual to M-theory on $AdS_4 \times S^7/Z_k$.
- Moreover for $k \to \infty$ the ABJM theory is weakly coupled so we should be able to study it reliably.

- But what happens to M-theory on $AdS_4 \times S^7/Z_k$ in the same limit?
- S^7 can be thought of as a $U(1)$ fibration over CP^3. The Z_k action reduces the radius of the fibre by a factor k.
- In the limit of large k, the fibre effectively shrinks to zero size. Then we are left with type IIA string theory on $AdS_4 \times CP^3$.
- This is a nice new AdS/CFT duality but unfortunately it involves type IIA string theory rather than M-theory!

- The only hope we might have to do something in M-theory is to study the ABJM theory at finite, small k.
- But in this case, notably for $k = 1, 2$, ABJM theory fails to have the most basic properties one would expect.
- It does not have $\mathcal{N} = 8$ supersymmetry and $SO(8)$ global symmetry.
- And at $k = 1$ where there is really no orbifold, it fails to explicitly exhibit translation invariance.
- Probably this does not mean the theory is wrong, but simply that the properties are hidden and not manifest.

- Anyway it seems that ABJM theory simply is not useful to study M2-branes in flat spacetime.
- In that sense we are more or less back where we were about two years ago!
- Still a lot of features of M2-branes are much clearer now and there are many possible directions to follow.
- And perhaps it's good that there is much more to be done in the future!

- To conclude the discussion, there are a few interesting points to discuss.

  (i) Do we need 3-algebras any more?

- The ABJM paper makes no reference to 3-algebras.
- And since their theory has only $\mathcal{N} = 6$ supersymmetry, it cannot be based on the Bagger-Lambert type of 3-algebra (which always gives $\mathcal{N} = 8$).
- Nevertheless, later on it was shown that the ABJM theory is an example of a more general type of 3-algebra in which some of the conditions are relaxed.

- In the more general 3-algebra, the structure constants f^{ABCD} are not taken to be real. In general they are complex.
- They satisfy:

$$f^{ABCD} = -f^{BACD} = -f^{ABDC} = f^{*\,CDAB}$$

- The fundamental identity holds, but in the form:

$$f^{EFG}{}_{B} f^{CBA}{}_{D} + f^{FEA}{}_{B} f^{CBG}{}_{D} + f^{*\,GAF}{}_{B} f^{CEB}{}_{D} + f^{*\,AGE}{}_{B} f^{CFB}{}_{D} = 0$$

- This structure is shown to guarantee $\mathcal{N} = 6$ supersymmetry. In the special case where the f^{ABCD} are real, this is enhanced to $\mathcal{N} = 8$.
- The ABJM models can be reproduced from this kind of 3-algebra.

- An additional support for 3-algebras comes from the study of the first nontrivial ℓ_p corrections, which were computed recently for the Lorentzian 3-algebra theory.
- It was found that these too are expressed in terms of 3-algebra quantities such as the 3-bracket.
- For example, the potential gets corrected by:

$$-\tfrac{1}{72}\,\ell_p^3\,\mathrm{STr}\Big(X^{IJM}X^{KLM}X^{IKN}X^{JLN} + \tfrac{1}{4}X^{IJK}X^{IJK}X^{LMN}X^{LMN}\Big)$$

  where we recall that:

$$X^{IJK} = X_+^I[\boldsymbol{X}^J,\boldsymbol{X}^K] + X_+^J[\boldsymbol{X}^K,\boldsymbol{X}^I] + X_+^K[\boldsymbol{X}^I,\boldsymbol{X}^J]$$

- As expected, this correction (being 12th order in scalars) is non-renormalisable, but is still expressed in terms of the 3-algebra.

(ii) Entropy of black branes

- Over a decade ago, it was argued by analogy with black holes and their horizon area, that the entropy of black branes scales as:

$$S \sim N^{\frac{D-2}{D-3-p}}$$

  for the conformal invariant branes, namely D3, M2 and M5.

- Evaluating this for the three cases we find:

$$\text{D3:}\ \ S \sim N^2, \quad \text{M2:}\ \ S \sim N^{3/2}, \quad \text{M5:}\ \ S \sim N^3$$

- The first one is understandable from a count of the N^2 degrees of freedom of a gauge theory with adjoint fields.

- The entropy growth for M2 and M5-branes still await a microscopic explanation.
- An important test of the usefulness of an M2-brane field theory would be to provide some insight on why there are $N^{3/2}$ degrees of freedom.
- This is an open problem.
- The M5-brane is also an open problem!

謝謝