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Mini-Course on String Theory

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Mumbai, October 2003

Contents:

1. Classical Bosonic Strings

(i) Review of particle mechanics

We are going to formulate and study the physics of relativistic superstrings.

How does this differ from the more traditional physics of relativistic point particles? For particles, we have two options:

(a) Relativistic particle mechanics (single particle, first quantised):

- Write the worldline action of a relativistic particle.
- It has reparametrisation invariance on the worldline.
- Gauge-fix this invariance.
- Quantise the particle and study free propagation.
- Introduce interactions and compute scattering amplitudes.

Amplitudes are computed in perturbation theory. The background in which the particle propagates is treated as external and fixed.

(b) Relativistic quantum field theory:

- Write down an interacting (nonlinear) action for classical fields.
- If the action has gauge symmetries, fix them.
- Quantise the fields.
- Compute scattering amplitudes.

Amplitudes can be computed in perturbation theory , but also non-perturbative effects may be studied, e.g. using semi-classical methods (solitons, instantons). The background in which the amplitudes are computed is obtained by solving the classical field equations.

Both approaches work only if the interactions are of renormalisable type. The second approach, however, is not tied to perturbation theory or fixed backgrounds. Therefore it is much more powerful.

In string theory, the analogue of the first approach (relativistic string mechanics) is rather well-understood, while the analogue of the second (relativistic string field theory) remains rather complicated and abstruse.

Therefore we will develop the first approach, by analogy with particle mechanics. This calls for a review of the relativistic point particle.

This is described by spacetime coordinates $X^\mu(t)$, $\mu = 0, 1, \ldots, D-1$, along with an auxiliary variable $g(t)$. The worldline action is:

$$S = \frac{1}{2} \int dt \left(\frac{1}{\sqrt{g}} \dot{X}^\mu \dot{X}_\mu - m^2 \sqrt{g} \right)$$

Under reparametrisations $t \to t'(t)$, we declare that

$$\begin{aligned} X^\mu(t) &\to X'^\mu(t') = X^\mu(t) \\ g(t) &\to g'(t') = \left(\frac{dt}{dt'} \right)^2 g(t) \end{aligned}$$

Then the action is invariant under reparametrisations of the worldline.

The equations of motion are:

$$\frac{\delta S}{\delta g(t)} = 0 \;\Rightarrow\; g^{-1}\dot{X}^{\mu}\dot{X}_{\mu} + m^2 = 0$$

$$\frac{\delta S}{\delta X^{\mu}(t)} = 0 \;\Rightarrow\; \frac{d}{dt}\left(\frac{1}{\sqrt{g}}\dot{X}_{\mu}\right) = 0$$

The canonical momentum is:

$$p_{\mu} = \frac{1}{\sqrt{g}}\dot{X}_{\mu}$$

The $g(t)$ equation of motion then says that

$$p_{\mu}p^{\mu} = -m^2$$

which means the particle has mass m.

The $X^{\mu}(t)$ equation tells us that

$$\dot{p}_{\mu} = 0$$

so the momentum is conserved.

From the first equation we can solve for $g(t)$:

$$g(t) = \frac{-\dot{X}^\mu \dot{X}_\mu}{m^2}$$

Substituting into the action we find:

$$S = -m \int dt \sqrt{-\dot{X}^\mu \dot{X}_\mu}$$

This action is the invariant length of the particle's trajectory. We see that reparametrisations of t are still a symmetry.

To quantise this theory, it will be necessary to make a gauge choice. A convenient choice is

$$X^0(t) = t$$

This fixes the reparametrisation invariance and leads to

$$S = -m \int dt \sqrt{1 - \dot{X}_i \dot{X}_i} \sim -m + \frac{1}{2} m \int dt\, (\dot{X}_i)^2, \qquad \dot{X}_i \ll 1$$

In this gauge, the momentum satisfying the equations of motion:

$$\dot{p}_\mu = 0$$

takes the familiar form:

$$p_0 = -\frac{m}{\sqrt{1-(\dot{X}_i)^2}}$$

$$p_i = \frac{m\dot{X}_i}{\sqrt{1-(\dot{X}_i)^2}}$$

We see that the gauge choice $X^0(t) = t$, while it breaks manifest Lorentz invariance, is a good one to provide physical insight.

An alternative approach is to not eliminate $g(t)$. Instead, we gauge-fix by using the worldline reparametrisation invariance to set $g(t) = 1$.

Unlike the previous gauge, this one does not break Lorentz invariance, and is called a covariant gauge. The action in this gauge is

$$S = \frac{1}{2} \int dt \, \left(\dot{X}^\mu(t) \dot{X}_\mu(t) - m^2 \right)$$

The action is simple, but we must keep track of the equations of motion that followed from the original action. These reduce to:

$$\begin{aligned} \dot{X}^\mu \dot{X}_\mu + m^2 &= 0 \\ \ddot{X}_\mu &= 0 \end{aligned}$$

The second of these equations follows from the gauge-fixed action, but the first one does not. We must impose it as a constraint.

Thus the velocities $\dot{X}^\mu(t)$ are not all independent.

The second equation is solved by writing

$$X^{\mu}(t) = X_0^{\mu} + p^{\mu}\, t$$

Thus the complete solution of the the classical relativistic free particle is given by solutions of the above equation, subject to the constraint.

This constraint is the price we pay for choosing a covariant gauge. However, there are two important benefits:

- The action is quadratic, with no square roots.
- The action and constraints both have a sensible $m \longrightarrow 0$ limit.

(ii) The closed bosonic string

Proceeding by analogy, we define a string through its spacetime coordinates $X^\mu(\sigma, t)$ as well as an auxiliary worldsheet field $g_{ab}(\sigma, t)$, the worldsheet metric.

Here σ is a coordinate along the string, whose range is $0 \leq \sigma \leq \pi$, while $a, b = 0, 1$ label the worldsheet directions: $0 \leftrightarrow t,\ 1 \leftrightarrow \sigma$.

For the string to be closed, we require that

$$X^\mu(\sigma + \pi, t) = X^\mu(\sigma, t)$$

The action is chosen to be:

$$S = -\frac{T}{2} \int d\sigma\, dt\, \sqrt{-\|g\|}\, \left(g^{ab}\, \partial_a X^\mu \partial_b X_\mu + \Lambda\right)$$

where g^{ab} is the inverse and $\|g\|$ is the determinant of the metric.

The constant T has dimensions of length^{-2} which is the same (when $\hbar = c = 1$) as mass/length. It is called the string tension.

Λ is the worldsheet cosmological constant.

This action has reparametrisation invariance in two dimensions:

$$
\begin{aligned}
t &\longrightarrow t'(\sigma, t) \\
\sigma &\longrightarrow \sigma'(\sigma, t) \\
X^\mu(\sigma, t) &\longrightarrow X'^\mu(\sigma', t') = X^\mu(\sigma, t) \\
g_{ab}(\sigma, t) &\longrightarrow g'_{ab}(\sigma', t') = \frac{\partial \xi^c}{\partial \xi'^a} \frac{\partial \xi^d}{\partial \xi'^b} \, g_{cd}(\sigma, t)
\end{aligned}
$$

where $(\xi^0, \xi^1) = (t, \sigma)$.

The action is very similar to the point particle action that we wrote earlier.

Note that with a metric on the worldsheet, our theory is rather like two-dimensional gravity. However, there is no kinetic term for the metric.

In particular, the Einstein term $\int \sqrt{-\|g\|}\, R$ is a total derivative in two dimensions and does not lead to propagation of the metric field. The string action could be thought of as non-dynamical worldsheet gravity coupled to matter.

We will henceforth set $\Lambda = 0$, analogous to the case of a massless particle.

On doing this, we find that the action develops a new invariance, under:

$$g_{ab}(\sigma,t) \rightarrow e^{\rho(\sigma,t)}\, g_{ab}(\sigma,t)$$

with no change in worldsheet or spacetime coordinates. This is called Weyl invariance.

The equations of motion are straightforward to write down:

$$\frac{\delta S}{\delta g^{ab}(\sigma,t)} = 0 \;\Rightarrow\; \partial_a X^\mu \partial_b X_\mu - \frac{1}{2} g_{ab}\, g^{cd}\, \partial_c X^\mu \partial_d X_\mu = 0$$

$$\frac{\delta S}{\delta X^\mu(\sigma,t)} = 0 \;\Rightarrow\; \partial_a \left(\sqrt{-\|g\|}\, g^{ab}\, \partial_b X_\mu \right) = 0$$

The first equation states the vanishing of the energy-momentum tensor:

$$T_{ab} \equiv -\frac{2}{T} \frac{1}{\sqrt{-\|g\|}} \frac{\delta S}{\delta g^{ab}} = 0$$

From reparametrisation invariance, this tensor is conserved: $\partial^a T_{ab} = 0$.

The equation $T_{ab} = 0$ is solved by writing:

$$g_{ab} = e^{\phi(\sigma,t)}\, \partial_a X^\mu \partial_b X_\mu$$

This is a solution for arbitrary $\phi(\sigma, t)$, because of Weyl invariance.

Inserting this into the action we get:

$$S = -T \int d\sigma\, dt\, \sqrt{-\|\partial_a X^\mu \partial_b X_\mu\|}$$

This action is equal to the invariant area of the worldsheet. It is still invariant under reparametrisations of σ, t as one can easily check. However, it is highly nonlinear in the string coordinates.

Now it is time to fix a gauge. By analogy with the point particle, we could choose:

$$X^0(\sigma, t) = t, \qquad X^1(\sigma, t) = \sigma$$

which is called the static gauge. The action remains nonlinear in the remaining X^i, and is hard to quantise.

A more convenient gauge is the one analogous to $g(t) = 1$ for a point particle. Thus, we again do not eliminate g_{ab} from the equations of motion, but instead fix two of its three independent components using reparametrisation invariance:

$$g_{ab} = e^{\phi(\sigma,t)}\, \eta_{ab}$$

This is called the conformal gauge. This is a covariant gauge since it preserves Lorentz invariance.

Because of Weyl invariance, the function $\phi(\sigma, t)$ decouples from the action, which becomes very simple:

$$S = -\frac{T}{2}\int d\sigma\, dt\, \partial_a X^\mu \partial^a X_\mu$$

In conformal gauge, the energy-momentum tensor can be re-derived as the conserved current for worldsheet translation invariance:

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\eta_{ab}\,\eta^{cd}\,\partial_c X^\mu \partial_d X_\mu, \qquad \partial^a T_{ab} = 0$$

As before, after choosing a gauge we must make sure that all the original equations of motion are satisfied. The g_{ab} equation of motion is

$$T_{ab} = 0$$

and has to be implemented as a constraint.

This simplifies if we use light-cone coordinates on the worldsheet:

$$\xi^{\pm} = t \pm \sigma$$

$$\partial_{\pm} = \partial/\partial\xi^{\pm} = \frac{1}{2}(\partial_t \pm \partial_\sigma)$$

We find that the equation

$$T_{+-} = 0$$

is identically satisfied, while the remaining equations are:

$$T_{++} = \partial_+ X^\mu \, \partial_+ X_\mu = 0$$

$$T_{--} = \partial_- X^\mu \, \partial_- X_\mu = 0$$

These are called Virasoro constraints.

The X^μ equation of motion in conformal gauge becomes:

$$\partial_a \partial^a X^\mu \sim \partial_- \partial_+ X^\mu = 0$$

This is just the two-dimensional Klein-Gordon equation following from the conformal gauge action.

In particular, this implies that the Virasoro constraints satisfy:

$$\partial_- T_{++} = \partial_+ T_{--} = 0$$

which is the conservation equation in these coordinates.

The Klein-Gordon equation is solved by writing:

$$X^\mu(\sigma, t) = X_L^\mu(t + \sigma) + X_R^\mu(t - \sigma)$$

where X_L, X_R are arbitrary functions of one argument.

This essentially solves the classical bosonic closed string. The string coordinates are arbitrary functions of $t + \sigma$ or $t - \sigma$, subject to the requirements of periodicity of X^μ in σ and the Virasoro constraints.

We can solve the periodicity requirement by making a mode expansion.

First, it is convenient to define a new dimensional parameter α' by:

$$T = \frac{1}{2\pi\alpha'}$$

Now, an arbitrary periodic function of σ can be expanded in modes:

$$X_L^\mu(t+\sigma) = \frac{1}{2}x^\mu + \alpha'\, p^\mu\,(t+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\tilde{\alpha}_n^\mu\, e^{-2in(t+\sigma)}$$

$$X_R^\mu(t-\sigma) = \frac{1}{2}x^\mu + \alpha'\, p^\mu\,(t-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\alpha_n^\mu\, e^{-2in(t-\sigma)}$$

Because of the linear term in σ, X_L and X_R are not separately periodic in σ, but their sum is periodic.

Reality of X_L and X_R imposes the requirement that:

$$(\tilde{\alpha}_n^\mu)^* = \tilde{\alpha}_{-n}^\mu, \qquad (\alpha_n^\mu)^* = \alpha_{-n}^\mu$$

In terms of these modes, we have:

$$\partial_+ X^\mu = \partial_+ X_L^\mu = \sqrt{2\alpha'} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_n^\mu \, e^{-2in(t+\sigma)}$$

$$\partial_- X^\mu = \partial_- X_R^\mu = \sqrt{2\alpha'} \sum_{n=-\infty}^{\infty} \alpha_n^\mu \, e^{-2in(t-\sigma)}$$

where we have defined $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}\, p^\mu$.

Let us also expand the energy-momentum tensor in terms of modes:

$$T_{++} = 4\alpha' \sum_{n=-\infty}^{\infty} \tilde{L}_n e^{-2in(t+\sigma)}$$

$$T_{--} = 4\alpha' \sum_{n=-\infty}^{\infty} L_n e^{-2in(t-\sigma)}$$

Then the Virasoro constraints amount to:

$$L_n = \tilde{L}_n = 0, \quad n \in \mathbb{Z}$$

It is clear that the bilinear relations

$$T_{\pm\pm} = \partial_\pm X^\mu \, \partial_\pm X_\mu \equiv \partial_\pm X \cdot \partial_\pm X$$

impose a bilinear relation between L_n and α_n^μ, and similarly for the left movers.

This relation is easily seen to be:

$$L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{-m} \cdot \alpha_{n+m}, \qquad \widetilde{L}_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \widetilde{\alpha}_{-m} \cdot \widetilde{\alpha}_{n+m}$$

Thus, in terms of Fourier modes, the Virasoro constraints that must be satisfied by the closed string coordinates are expressed as:

$$\widetilde{L}_n = L_n = 0$$

with $\widetilde{L}_n, L_n$ given as above.

(iii) The open bosonic string

For open strings, $X^\mu(\sigma,t)$ is no longer periodic in σ. Instead, the string must "end" at $\sigma = 0, \pi$.

At each end, we need to specify boundary conditions for the coordinate X^μ or its derivatives.

For the open string, the equations of motion follow from varying the action only if we also require the absence of boundary terms. From:

$$S = -\frac{T}{2}\int_0^\pi d\sigma \int dt\ \partial_a X^\mu \partial^a X_\mu$$

we see that

$$\delta S = T\int_0^\pi d\sigma \int dt\ \delta X^\mu\,\partial_a\,\partial^a X_\mu - T\int dt\left[\delta X^\mu\,\partial_\sigma X_\mu\right]_0^\pi$$

To make the second term vanish, we must require:

$$\delta X^\mu(0,t)\,\partial_\sigma X_\mu(0,t) = \delta X^\mu(\pi,t)\,\partial_\sigma X_\mu(\pi,t) = 0$$

Thus, at $\sigma = 0$, we can impose one of the following two boundary conditions on each of the spacetime coordinates X^μ:

$$\partial_\sigma X^\mu(0,t) = 0 \qquad \text{(Neumann)}$$

$$X^\mu(0,t) = c^\mu \qquad \text{(Dirichlet)}$$

where c^μ is an arbitrary constant.

At the other end $\sigma = l$, we must also independently choose one of these conditions.

Thus an open string can be Neumann-Neumann (NN), Dirichlet-Dirichlet (DD) or Neumann-Dirichlet (ND), with respect to each of its spacetime coordinates.

The D boundary condition violates translation and Lorentz invariance. In fact, it is the statement that the end of the string is stuck at a particular location.

The locus on which the end is stuck is called a Dirichlet brane or D-brane for short.

In the DD case, the two ends can be stuck at the same location c^μ or at two different locations c^μ, d^μ. In one case, the string starts and ends on the same brane, while in the other, it stretches between two different branes.

With open-string boundary conditions, the mode expansion is different from that for the closed string.

A wave travelling one way on the string hits the end, and gets reflected back. So there is only one set of modes rather than two.

For NN boundary conditions, we find:

$$X^\mu(\sigma,t) = x^\mu + 2\alpha' p^\mu\, t + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\,\alpha_n^\mu\, e^{-int}\cos n\sigma$$

and

$$\partial_\pm X^\mu(\sigma,t) = \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\alpha_n^\mu\, e^{-in(t\pm\sigma)}$$

where $\alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu$.

The Virasoro constraints reduce to the vanishing of a single set of modes:

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}}\alpha_{-m}\cdot\alpha_{n+m} = 0$$

For DD boundary conditions the result is:

$$X^\mu(\sigma,t) = c^\mu\left(1-\frac{\sigma}{\pi}\right) + d^\mu\sigma - \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\,\alpha_n^\mu\, e^{-int}\sin n\sigma$$

Here c^μ, d^μ specify the locations of the D-branes on which the ends of the string are fixed. As one would expect, there are no translational zero modes x^μ, p^μ in this case.

For DN and ND strings, the mode expansion involves half-integer modes, as one can easily check.

2. Quantising the Bosonic String

(i) Open strings: the role of the constraints

Quantisation of the bosonic string would be extremely simple if it were not for the constraints.

Let us define the canonical momentum conjugate to X^μ by

$$P_\mu(\sigma,t) = \frac{\delta S}{\delta(\partial_t X^\mu(\sigma,t))} = T\,\partial_t X_\mu(\sigma,t)$$

and impose equal-time canonical commutators on P_μ, X^μ:

$$[P_\mu(\sigma,t), X^\nu(\sigma',t)] = -i\delta(\sigma-\sigma')\delta_\mu^\nu$$

$$[X^\mu(\sigma,t), X^\nu(\sigma',t)] = [P^\mu(\sigma,t), P^\nu(\sigma',t)] = 0$$

Now consider for definiteness the open string with NN boundary conditions. The canonical commutators define commutation relations among the modes α_n, x^μ, p^μ:

$$[\alpha_m^\mu, \alpha_n^\nu] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}, \quad [x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad [\alpha_m^\mu, x^\nu] = [\alpha_m^\mu, p^\nu] = 0$$

From the classical relation $(\alpha_n^\mu)^* = \alpha_{-n}^\mu$, it follows that the corresponding operators satisfy

$$(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu$$

By analogy with the harmonic oscillator, the operators α_n^μ are creation operators for $n < 0$ and annihilation operators for $n > 0$.

Thus the ground state $|0\rangle$ of the string would be defined by

$$\alpha_n^\mu |0\rangle = 0, \;\; n > 0$$

Physical states would then be constructed by acting with the oscillators α_{-n}^μ on the ground state $|0\rangle$. These would correspond to excited states of the string.

However, we can see right away that something is wrong.

Consider the state $\alpha^{\mu}_{-n}|0\rangle$. Assuming the ground state to be normalised, this excited state would have the norm:

$$\|\alpha^{\mu}_{-n}|0\rangle\|^2 = \langle 0|\alpha^{\mu}_{n}\,\alpha^{\mu}_{-n}|0\rangle = n\,\eta^{\mu\mu}$$

Thus for $\mu = 0$ (the time direction) we have negative-norm states, which are unacceptable in any physical theory.

The constraints play the essential role of eliminating such states.

There are several ways to implement the constraints. Broadly they fall into two categories:

(I) Solve the constraints by singling out some spacetime directions from others. As a result some oscillators become dependent on others and are not to be independently quantised. The Hilbert space then manifestly has positive norm, but Lorentz invariance is not manifest.

(II) Quantise all the oscillators, but then impose the constraints on the Hilbert space. Only those states satisfying the constraints will be treated as physical. Then the full Hilbert space will have negative-norm states, but the constrained subspace will be positive. In this procedure, manifest Lorentz invariance can be maintained.

Instead of choosing one of these two approaches, we will exhibit the flavour of both. Both approaches have important merits.

(ii) Open strings: light-cone gauge

We may try to directly solve the constraint:

$$\partial_- X^\mu \partial_- X_\mu = 0$$

by eliminating, say, X^0 in favour of the other coordinates. However, this will introduce square roots and the equations become very hard to solve.

A more convenient way to solve the constraints is to use the light-cone gauge. Pick two of the D spacetime coordinates μ, for example $\mu = 0$ and $\mu = D-1$, and define:

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1})$$

If the remaining space coordinates are denoted X^i, $i = 1, 2, \ldots, D-2$, then the Virasoro constraint above becomes:

$$-2\,\partial_- X^+\,\partial_- X^- + \partial_- X^i\,\partial_- X^i = 0$$

In conformal gauge, we still have the freedom to reparametrise the worldsheet coordinates $\xi^{\pm}$ by:

$$\xi^{+} \to \xi'^{+}(\xi^{+}), \qquad \xi^{-} \to \xi'^{-}(\xi^{-})$$

since this preserves the fact that the metric g_{ab} is proportional to η_{ab}.

For such a reparametrisation, $t'(\sigma,t)$ satisfies

$$\partial_{+}\partial_{-}t'(\sigma,t) = 0$$

This is the same equation as the one satisfied by the X^{μ}. So we can use it to choose t proportional to one of the X^{μ}, in particular to X^{+}.

More precisely, we set:

$$X^{+}(\sigma,t) = x^{+} + 2\alpha' p^{+} t$$

where x^{+}, p^{+} are constants. This amounts to eliminating all the oscillators in X^{+}, and keeping only the zero modes.

Using $\partial_- X^+ = \alpha' p^+$, we can solve the constraints for X^-:

$$\partial_- X^- = \frac{1}{2\alpha' p^+} \partial_- X^i \, \partial_- X^i$$

Inserting the mode expansion of X^-, we find that

$$\alpha_n^- = \frac{1}{2\sqrt{2\alpha'} p^+} \sum_{m \in \mathbb{Z}} \alpha^i_{n-m} \, \alpha^i_m$$

This determines all modes of X^- in terms of those of X^i. In particular,

$$p^- = \frac{1}{\sqrt{2\alpha'}} \, \alpha_0^- = \frac{1}{2\alpha' p^+} \sum_{m=1}^{\infty} \alpha^i_{-m} \, \alpha^i_m + \frac{1}{2p^+} p^i p^i$$

This can be rewritten as

$$2p^+ p^- - p^i p^i = \frac{1}{\alpha'} \sum_{m=1}^{\infty} \alpha^i_{-m} \, \alpha^i_m$$

This leaves the independent oscillators $\alpha_n^i,\ i = 1, 2, \ldots, D-2$, as well as the zero modes x^i, p^i. These are canonically quantised:

$$[\alpha_m^i, \alpha_n^j] = m\,\delta_{m+n,0}\,\delta^{ij}, \quad [x^i, p^j] = i\delta^{ij}, \quad [\alpha_m^i, x^j] = [\alpha_m^i, p^j] = 0$$

Physical states of the string are now constructed by defining a ground state $|0\rangle$ satisfying

$$\alpha_n^i|0\rangle = 0,\ n > 0$$

We also define the state

$$|k\rangle = e^{ik^i x^i}|0\rangle$$

satisfying

$$p^i|k\rangle = k^i|k\rangle$$

which represents a string in its ground state with transverse momentum k^i.

The excited states of the string will then be of the form:

$$\alpha^{i_1}_{-n_1}\,\alpha^{i_2}_{-n_2}\ldots\alpha^{i_\mathrm{N}}_{-n_\mathrm{N}}\,|k\rangle$$

These transform as tensors under the transverse rotations, $SO(D-2)$. Full Lorentz symmetry, $SO(D-1,1)$, is not manifest.

What are the masses of these states? We have the standard relativistic formula:

$$M^2 = -p^\mu p_\mu = 2p^+p^- - p^ip^i$$

From a previous equation, we see that:

$$M^2 = \frac{1}{\alpha'}\sum_{m=1}^{\infty}\alpha^i_{-m}\,\alpha^i_m = \frac{1}{\alpha'}\sum_{m=1}^{\infty} m\,N_m$$

where

$$N_m = \frac{1}{m}\alpha^i_{-m}\,\alpha^i_m$$

is the number operator that counts the number of excitations of mode number m.

Thus, we have:

$$M^2|0\rangle = 0, \quad M^2(\alpha^i_{-1}|0\rangle) = \frac{1}{\alpha'}(\alpha^i_{-1}|0\rangle)$$

so apparently the ground state is massless and the first excited state has $(\text{mass})^2 = 1/\alpha'$.

We have just encountered a serious paradox.

The first excited state is massive and has $D-2$ physical components. In fact, it is a vector of $SO(D-2)$.

But according to the representation theory of the Lorentz group, only a massless state in D spacetime dimensions can have $D-2$ physical components. A massive state necessarily has $D-1$ components!

Thus the first excited state of the string really ought to be massless. If we cannot ensure this, then we have lost/misplaced our Lorentz invariance, and that is the end of string theory...

Fortunately, string theory does manage to evade this paradox.

To see this, we must go back to the equation:

$$M^2 = \frac{1}{\alpha'} \sum_{m=1}^{\infty} \alpha^i_{-m}\, \alpha^i_m = \frac{1}{\alpha'} \sum_{m=1}^{\infty} m\, N_m$$

As a classical formula it is perfectly correct. But after quantising the oscillators α^i_m, the formula develops an ordering ambiguity.

The operators α^i_m and α^i_{-m} do not commute. Indeed,

$$\alpha^i_{-m}\, \alpha^i_m = \alpha^i_m\, \alpha^i_{-m} - m$$

Since we allowed the interchange of orderings at the classical level, we must allow for an additive constant in the formula for M^2.

Thus, the corrected formula is:

$$M^2 = \frac{1}{\alpha'}\left(\sum_{m=1}^{\infty} \alpha^i_{-m}\,\alpha^i_m - a\right) = \frac{1}{\alpha'}\left(\sum_{m=1}^{\infty} m\,N_m - a\right)$$

where a is a constant.

We can determine a by a heuristic argument. Originally, the classical expression arose from the term:

$$\frac{1}{2}\sum_{m=-\infty}^{\infty} \alpha^i_{-m}\,\alpha^i_m$$

If we assume that the oscillators in this expression are replaced by quantum operators in the same ordering, then we must re-order precisely half of the terms to get the normal-ordered expression:

$$\frac{1}{2}\sum_{m=-\infty}^{\infty} :\alpha^i_{-m}\,\alpha^i_m: \; = \sum_{m=1}^{\infty} \alpha^i_{-m}\,\alpha^i_m$$

It is easy to see that

$$\frac{1}{2}\sum_{m=-\infty}^{\infty}\alpha_{-m}^{i}\,\alpha_{m}^{i}=\frac{1}{2}\sum_{m=-\infty}^{\infty}:\alpha_{-m}^{i}\,\alpha_{m}^{i}:\;+\;\frac{D-2}{2}\sum_{m=1}^{\infty}m$$

With zeta-function regularisation,

$$\sum_{m=1}^{\infty}m=-\frac{1}{12}$$

so, if we believe this, then we have determined

$$a=\frac{D-2}{24}$$

But we have already determined a by consistency with the representation theory of the Lorentz group. It must be such as to render the first excited state massless. This happens only if $a = 1$.

It follows that

$$D = 26$$

so the bosonic string is consistent only in 26 spacetime dimensions!

There is another interesting consequence. The mass formula has become:

$$M^2 = \frac{1}{\alpha'}\left(\sum_{m=1}^{\infty} m\, N_m - 1\right)$$

This implies that the ground state is tachyonic:

$$M^2|0\rangle = -\frac{1}{\alpha'}|0\rangle$$

Have we solved one problem only to encounter another, worse one?

No. Once we accept 26 dimensions and a tachyon, everything else actually falls into place very nicely.

Indeed, one can show that the degeneracies of all the excited states

$$\alpha^{i_1}_{-n_1}\,\alpha^{i_2}_{-n_2}\ldots\alpha^{i_\mathsf{N}}_{-n_\mathsf{N}}\,|k\rangle$$

match the dimensions of irreps of the little group $SO(D-1)$.

For example, at the second excited level we have:

$$\alpha^{i}_{-2}|0\rangle, \quad \alpha^{i}_{-1}\alpha^{j}_{-1}|0\rangle$$

Each of these separately cannot be a representation of $SO(D-1)$. But together, they can. The total number of states is

$$(D-2)+\frac{1}{2}(D-2)(D-1)=\frac{1}{2}(D-2)(D+1)$$

which is precisely the dimension of the symmetric, traceless representation of the little group $SO(D-1)$.

(iii) Open strings: covariant quantisation and gauge invariance

We have seen that the NN open bosonic string has a tachyonic ground state and a massless vector as its first excited state.

At low energies, one may expect that these are described by a scalar field $\phi(x)$ and a vector field $A_\mu(x)$, with a field-theory Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2, \qquad m^2 = -1/\alpha'$$

The gauge field action has the well-known Abelian gauge invariance:

$$\delta A_\mu = \partial_\mu \Lambda(x)$$

If the free action was not gauge invariant, the field theory would have negative-norm states, as we learn in electrodynamics.

Can we find direct evidence for gauge invariance from string theory? This would be strong confirmation that at low energies, string theory is described by familiar field theories.

In light-cone gauge, we worked only with physical degrees of freedom, so we could not have seen gauge invariance.

Instead, we now briefly consider covariant quantisation of the string. In this formalism, we quantise all the oscillators.

Consider a general linear combination of the first excited states:

$$\zeta_\mu(k)\,\alpha^\mu_{-1}|k\rangle$$

It is natural to identify $\zeta_\mu(k)$ with the Fourier transform of the gauge field $A_\mu(x)$.

Now we impose the constraints:

$$L_n|\mathsf{phys}\rangle = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot\alpha_{n+m}|\mathsf{phys}\rangle = 0$$

to define states $|\mathsf{phys}\rangle$ that are in the physical Hilbert space.

It is easily verified that L_n, L_{-n} do not commute with each other in general. Their commutator includes a c-number term.

This means we cannot require both

$$L_n|\text{phys}\rangle = 0 \;\; \text{and} \;\; L_{-n}|\text{phys}\rangle = 0$$

so in covariant quantisation we only impose

$$L_n|\text{phys}\rangle = 0, \;\; n \geq 0$$

Applying this requirement on

$$|\text{phys}\rangle = \zeta_\mu(k)\, \alpha^\mu_{-1}|k\rangle$$

we find that only L_0, L_1 give nontrivial constraints, which turn out to be:

$$k_\mu k^\mu\, \zeta_\nu(k) = 0, \qquad k^\mu\, \zeta_\mu(k) = 0$$

The first condition says the field is massless, which we know. Taken together, the two conditions give us the free-field equation of motion:

$$k^\mu\,(k_\mu\zeta_\nu - k_\nu\zeta_\mu) = 0 \quad \leftrightarrow \quad \partial^\mu F_{\mu\nu} = 0$$

Now consider the state

$$L_{-1}|k\rangle \sim k_\mu\, \alpha^\mu_{-1}|k\rangle$$

This is clearly a massless state, but it is unphysical in the following sense. We have

$$\langle \text{phys}|L_{-1}|k\rangle = 0$$

for every physical state $\langle \text{phys}|$, because

$$L_1|\text{phys}\rangle = 0 \;\Rightarrow\; \langle \text{phys}|L_{-1} = 0$$

So the state is orthogonal to the physical Hilbert space, and is equivalent to zero. Thus for arbitrary $\Lambda(k)$, we have the equivalence of polarisation vectors:

$$\zeta_\mu(k) \sim \zeta_\mu(k) + k_\mu\, \Lambda(k)$$

This is just the momentum space version of the gauge equivalence:

$$A_\mu \sim A_\mu + \partial_\mu \Lambda$$

Thus, at least at the non-interacting level, string theory has gauge invariance. We did not require it, rather it emerged upon quantising the theory.

This is impressive confirmation that string theory has a profound degree of internal consistency. It embodies symmetry principles that are fundamental to particle physics.

To summarise, the open bosonic string has a tachyon and a massless photon in its spectrum.

There is also an infinite tower of higher excited states, whose mass^2 are integrally spaced in units of $1/\alpha'$.

(iv) Closed strings: spectrum and gauge invariance

Without much effort, we can repeat the same steps for the closed bosonic string. As we have seen, there are twice as many oscillators, because of the independent left- and right-moving sectors.

However, the zero modes x^μ, p^μ are not doubled.

Both $L_0, \widetilde{L}_0$ have normal ordering ambiguities and must have a constant $-1/\alpha'$ subtracted from them. We find:

$$M^2 = L_0 + \widetilde{L}_0 = \frac{2}{\alpha'}\left(\sum_{m=1}^{\infty} \alpha^i_{-m}\,\alpha^i_m + \sum_{m=1}^{\infty} \widetilde{\alpha}^i_{-m}\,\widetilde{\alpha}^i_m - 2\right)$$

It follows that the ground state $|k\rangle$ is a tachyon of

$$M^2 = -\frac{4}{\alpha'}$$

What about excited states?

In principle we could act with any number of oscillators $\alpha^i_{-n}, \tilde{\alpha}^i_{-n}$ on the ground state.

But there is a subtlety. The zero mode p^- is common for both left- and right-movers. So the procedure we followed for the open string now gives:

$$p^- = \frac{1}{\alpha' p^+}\left(\sum_{m=1}^{\infty} \alpha^i_{-m}\,\alpha^i_m - 1\right) + \frac{1}{2p^+}p^i p^i$$
$$p^- = \frac{1}{\alpha' p^+}\left(\sum_{m=1}^{\infty} \tilde{\alpha}^i_{-m}\,\tilde{\alpha}^i_m - 1\right) + \frac{1}{2p^+}p^i p^i$$

and hence:

$$\sum_{m=1}^{\infty} \alpha^i_{-m}\,\alpha^i_m = \sum_{m=1}^{\infty} \tilde{\alpha}^i_{-m}\,\tilde{\alpha}^i_m$$

Thus the total level $\sum nN_n$ must be equal for left- and right-movers. This rules out the states

$$\alpha^i_{-1}|k\rangle, \quad \widetilde{\alpha}^i_{-1}|k\rangle$$

and therefore the first excited state of the closed bosonic string is:

$$\alpha^i_{-1}\,\widetilde{\alpha}^j_{-1}|k\rangle$$

which is massless. A general linear combination is

$$\zeta_{ij}(k)\,\alpha^i_{-1}\,\widetilde{\alpha}^j_{-1}|k\rangle$$

As a representation of $SO(D-2)$, $\zeta_{ij}(k)$ decomposes into three irreducible representations: symmetric traceless, antisymmetric, and a trace part which is a singlet.

Each one can be identified with the transverse components of a field:

$$\begin{aligned} \zeta_{(ij)}(k) - \frac{1}{D-2}\,\delta^{ij}\,\zeta_{ij}(k) &\;\rightarrow\; G_{ij}(x) \\ \zeta_{[ij]}(k) &\;\rightarrow\; B_{ij}(x) \\ \delta^{ij}\,\zeta_{ij}(k) &\;\rightarrow\; \Phi(x) \end{aligned}$$

These fields, in turn, are the transverse components of the massless fields $G_{\mu\nu}, B_{\mu\nu}, \Phi$ of the Lorentz group $SO(25,1)$.

Thus we have shown that the massless first excited state of the bosonic string consists of these three fields. They should be described at low energies by a suitable field theory action.

But it is a theorem that the only consistent action for a massless symmetric rank-2 tensor field is that of Einstein's gravity.

Therefore, closed string theory, if consistent, is a theory of gravity!.

We therefore expect the low-energy effective action to be:

$$\mathcal{L} = \sqrt{-\|G\|}\Big(-\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + R \\ - \partial_{[\mu}B_{\nu\lambda]}\,\partial^{[\mu}B^{\nu\lambda]} - \frac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi\Big),\ m^2 = -4/\alpha'$$

which is to be taken seriously only to quadratic order in the fields. For the metric this means we make the linearised approximation:

$$G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

and keep terms quadratic in $h_{\mu\nu}$.

If this is true, the symmetries that should be visible (at the level of inhomogeneous transformations) are:

(I) linearised reparametrisation invariance:

$$\delta h_{\mu\nu}(x) = \partial_\mu \Lambda_\nu(x) + \partial_\nu \Lambda_\mu(x)$$

(II) tensor gauge invariance:

$$\delta B_{\mu\nu}(x) = \partial_\mu \Lambda_\nu(x) - \partial_\nu \Lambda_\mu(x)$$

In each case, $\Lambda_\mu(x)$ is a vector gauge parameter.

In the first symmetry transformation, $\Lambda_\mu(x)$ labels an infinitesimal reparametrisation in spacetime.

The second transformation is less familiar, but it is known to remove negative-norm states in the field theory of a second-rank antisymmetric tensor field (which we call a 2-form field for short).

By now we know how to check this.

Carry out covariant quantisation of the closed string, and require that the constraints annihilate the state:

$$\zeta_{\mu\nu}(k)\,\alpha^{\mu}_{-1}\,\widetilde{\alpha}^{\nu}_{-1}|k\rangle$$

This leads to the linearised equations of motion .

Next, allow addition of the unphysical states:

$$\Lambda_{\mu}(k)\,\left(L_{-1}\widetilde{\alpha}^{\mu}_{-1}\pm\alpha^{\mu}_{-1}\widetilde{L}_{-1}\right)|k\rangle$$

and obtain the gauge equivalences:

$$\zeta_{\mu\nu}\sim\zeta_{\mu\nu}+\left(k_{\mu}\,\Lambda_{\nu}\pm k_{\nu}\,\Lambda_{\mu}\right)$$

as desired.

The trace part of the state obtained in light-cone quantisation corresponds to a scalar field $\Phi(x)$ called the dilaton. This field plays a very important role in string theory, and we will have more to say about it later.

To summarise, the closed bosonic string has a tachyon and a massless graviton, a 2-form field and a dilaton.

There is also an infinite tower of higher excited states whose mass^2 are integrally spaced in units of $4/\alpha'$.

Between open and closed strings, the twin principles of gauge invariance and gravity are embodied. The price we have paid is 26 dimensions and a tachyon.

3. Classical Superstrings

(i) The closed fermionic string

In analogy with the bosonic string, it is fairly straightforward to define the spinning or fermionic string.

This is an object with fermionic coordinates on the worldsheet. These will be related to the bosonic coordinates X^μ by worldsheet supersymmetry.

We therefore introduce Majorana worldsheet fermions $\psi^\mu_\alpha(\sigma, t)$ where $\alpha = 1, 2$ is a spinor index on the worldsheet.

It may seem a little odd for these fermions to be vectors in spacetime, as the μ index indicates. However, μ does not transform under worldsheet reparametrisations, so there is no obvious conflict.

All it means is that the worldsheet fermions are not spacetime fermions.

We would like to have worldsheet supersymmetry between X^μ and ψ^μ_α. But since there is also an auxiliary variable g_{ab} on the worldsheet, we introduce its fermionic superpartner $\chi_{\alpha a}$, rather like a worldsheet gravitino.

The action will have local worldsheet supersymmetry as well as worldsheet reparametrisation invariance. There is no kinetic term for the graviton and gravitino.

Thus, the action is that of non-dynamical worldsheet supergravity coupled to supersymmetric matter:

$$S = -\frac{T}{2}\int d\sigma\, dt\,\sqrt{-\|g\|}\Big(g^{ab}\partial_a X^\mu \partial_b X_\mu - i\overline{\psi}^\mu \rho^a \partial_a \psi_\mu \\ + 2\,\overline{\chi}_a\,\rho^b \rho^a \psi^\mu \partial_b X_\mu + \frac{1}{2}\,\overline{\psi}_\mu \psi^\mu\,\overline{\chi}_a\,\rho^b \rho^a \chi_b\Big)$$

Here, ρ^a are 2×2 gamma-matrices, and all spinor indices have been suppressed.

As promised, this action has local worldsheet supersymmetry as well as reparametrisation invariance. As before, we can use reparametrisations (and Weyl invariance) to effectively fix the metric $g_{ab} = \eta_{ab}$.

Less obvious, but equally true, is that we can use local supersymmetry as well as a super version of Weyl invariance, to effectively set $\chi_{\alpha a} = 0$.

The result is an action in superconformal gauge, which is rather simple:

$$S = -\frac{T}{2} \int d\sigma \, dt \left(\partial_a X^\mu \partial_a X_\mu - i \overline{\psi}^\mu \rho^a \partial_a \psi_\mu \right)$$

The equations of motion of the bosonic and fermionic coordinates are the familiar Klein-Gordon and Dirac equations in two dimensions:

$$\partial_a \partial^a X^\mu = 0, \qquad \rho^a \partial_a \psi^\mu = 0$$

In light-cone coordinates, and with a decomposition of ψ^μ_α into its chiral components $\psi_\pm$ in a suitable basis of gamma-matrices, these become:

$$\partial_-\partial_+X^\mu = 0, \qquad \partial_-\psi^\mu_+ = 0, \quad \partial_+\psi^\mu_- = 0$$

The superconformal gauge action has global worldsheet supersymmetry:

$$\delta X^\mu = \bar{\epsilon}\psi^\mu, \qquad \delta\psi^\mu = -i\rho^a\partial_a X^\mu$$

Correspondingly, there is a conserved supercurrent:

$$J_{\alpha a} = \frac{1}{2}\,\rho^b\rho_a\,\psi^\mu_\alpha\,\partial_b X_\mu, \qquad \partial^a J_{\alpha a} = 0$$

The gravitino equations of motion, derived from the original action, reduce in superconformal gauge to the new constraint $J_{\alpha a} = 0$.

Thus the constraints that must be imposed are the vanishing of the energy-momentum tensor T_{ab} and the supercurrent $J_{\alpha a}$.

In superconformal gauge and light-cone coordinates, these "Super-Virasoro" constraints are:

$$T_{--} = \partial_- X^\mu \partial_- X_\mu + \frac{i}{2}\psi_-^\mu\, \partial_- \psi_{-\mu} = 0$$

$$J_{--} = \psi_-^\mu\, \partial_- X_\mu = 0$$

along with their left-moving $(+)$ counterparts.

Next we would like to make a mode expansion of the fermionic coordinates ψ^μ. As we are doing closed strings, they would naively be periodic in σ, but it is equally consistent to take them antiperiodic in σ.

For historical reasons, we assign the names:

$$\begin{aligned} \text{periodic:} \quad & \psi^\mu(\sigma+\pi,t) = \psi^\mu(\sigma,t) \quad \rightarrow \text{ Ramond (R)} \\ \text{antiperiodic:} \quad & \psi^\mu(\sigma+\pi,t) = -\psi^\mu(\sigma,t) \rightarrow \text{ Neveu-Schwarz (NS)} \end{aligned}$$

and we will see that both are important.

The mode expansions are then:

$$\text{R sector}: \psi_-^\mu(\sigma,t) = \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} d_n^\mu \, e^{-2in(t-\sigma)}$$

$$\text{NS sector}: \psi_-^\mu(\sigma,t) = \sqrt{2\alpha'} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu \, e^{-2ir(t-\sigma)}$$

along with their left-moving counterparts.

Now we can write the constraints in terms of bosonic and fermionic modes:

R sector:

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot\alpha_{n+m} + \frac{1}{2}\sum_{m\in\mathbb{Z}}\left(\frac{n}{2}+m\right) d_{-m}\cdot d_{n+m} = 0$$

$$G_n = \sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot d_{m+n} = 0$$

NS sector:

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot\alpha_{n+m} + \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}}\left(\frac{n}{2}+r\right) b_{-r}\cdot b_{n+r} = 0$$

$$G_s = \sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot b_{m+s} = 0$$

with similar expressions for the left-movers.

Note that at this stage the NS/R boundary conditions can be chosen independently for right and left movers.

(ii) The open fermionic string

For the open string, the boundary conditions in the variation of the fermionic part of the action are easily seen to be:

$$\int dt \left[\delta\psi_+^\mu\, \psi_{+\mu} - \delta\psi_-^\mu\, \psi_{-\mu}\right]_0^\pi = 0$$

The solution of these conditions is:

$$\psi_-^\mu(0,t) = \eta_1\, \psi_+^\mu(0,t)$$
$$\psi_-^\mu(\pi,t) = \eta_2\, \psi_+^\mu(\pi,t)$$

where $\eta_1, \eta_2 = \pm 1$.

While the sign can be chosen independently at each end, the terminology depends on the relative sign:

$$\eta_1 = \eta_2 : \ \text{Ramond}, \qquad \eta_1 = -\eta_2 : \ \text{Neveu-Schwarz}$$

The mode expansion for the worldsheet fermion of an open string is in terms of integer modes d_n in the R sector and half-integer modes b_r in the NS sector.

As long as we consider NN or DD open strings, the Super-Virasoro constraints for the open string are the same as for one sector (left- or right-movers) of the closed string.

For DN, ND strings, however, the bosonic oscillators also become half-integer moded and we must modify the sums in the constraints accordingly.

4. Quantising the Superstring

(i) Open string: Neveu-Schwarz sector

Quantisation of the fermionic string proceeds along very similar lines as the bosonic string. So we can be less explicit about the familiar details.

But there will be some new subtleties to take care of, mainly having to do with boundary conditions on the fermions.

Let us perform light-cone quantisation of the action:

$$S = -2T \int d\sigma\, dt \left(\partial_+ X^\mu \partial_- X_\mu + i\psi_+^\mu \partial_- \psi_{+\mu} + i\psi_-^\mu \partial_+ \psi_{-\mu} \right)$$

together with the constraints:

$$\begin{aligned} T_{--} &= \partial_- X^\mu \partial_- X_\mu + \frac{i}{2} \psi_-^\mu \, \partial_- \psi_{-\mu} = 0 \\ J_{--} &= \psi_-^\mu \, \partial_- X_\mu = 0 \end{aligned}$$

As before, we make the choice:

$$X^+ = x^+ + 2\alpha' p^+ t$$

Recall that this choice is made using the local reparametrisations that preserve conformal gauge, which satisfy

$$\partial_+ \partial_- t'(\sigma, t) = 0$$

It seems natural that we should fix ψ^+ to be similarly free of oscillators. For this, we need to consider the local supersymmetry transformations that preserve superconformal gauge.

The analysis, which we will skip, reveals that these are given by supersymmetry parameters $\epsilon_\pm$ satisfying

$$\partial_+\epsilon_- = \partial_-\epsilon_+ = 0$$

Again these are the same equations as those satisfied by $\psi^\mu_\pm$. We can therefore set

$$\psi^+_\pm = 0$$

Imposing these gauge choices, we find that

$$\partial_- X^- = \frac{1}{2\alpha' p^+}\left(\partial_- X^i\,\partial_- X^i + +\frac{i}{2}\psi_-\partial_-\psi_-\right)$$
$$\psi^-_- = \frac{1}{2\alpha' p^+}\psi^i_-\partial_- X^i$$

Let us now choose Neveu-Schwarz boundary conditions. Recall that in this sector, the open string satisfies:

$$\psi_-^\mu(0,t) = \psi_+^\mu(0,t)$$
$$\psi_-^\mu(\pi,t) = -\psi_+^\mu(\pi,t)$$

Therefore it has a half-integer mode expansion:

$$\psi_-^\mu(\sigma,t) = \sqrt{\alpha'} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu \, e^{-ir(t-\sigma)}$$
$$\psi_+^\mu(\sigma,t) = \sqrt{\alpha'} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu \, e^{-ir(t+\sigma)}$$

Note the absence of any zero-frequency mode.

In terms of modes, the light-cone gauge condition leads to:

$$\alpha_n^- = \frac{1}{2\sqrt{2\alpha' p^+}}\left(\sum_{m\in\mathbb{Z}} :\alpha^i_{n-m}\,\alpha^i_m: + \sum_{r\in\mathbb{Z}+\frac{1}{2}}\left(r-\frac{n}{2}\right):b^i_{n-r}b^i_r: - a\,\delta_{n,0}\right)$$

$$b_r^- = \frac{1}{\sqrt{2\alpha' p^+}}\sum_{s\in\mathbb{Z}+\frac{1}{2}} \alpha^i_{r-s}b^i_s$$

As before, we have included a possible normal-ordering constant a in α_0^-.

Canonical quantisation leads to the anticommutators:

$$\{b^i_r,\, b^j_s\} = \delta_{r+s,0}\,\delta^{ij}$$

What is the normal-ordering constant this time? If we calculate it as before by evaluating a divergent sum, we find:

$$a = \frac{D-2}{16}$$

Now consider the ground state $|k\rangle$ annihilated by all positive modes:

$$\alpha^i_n|k\rangle = b^i_r|k\rangle = 0, \quad n > 0,\ r > 0$$

At the first excited level, we get a vector state:

$$b^i_{-\frac{1}{2}}|k\rangle$$

This state again has only $D-2$ components and therefore has to be massless. This fixes

$$a = \frac{1}{2}$$

and therefore also

$$D = 10$$

Thus we see that the fermionic string is consistent in 10 spacetime dimensions.

The mass-shell condition is now:

$$M^2 = \frac{1}{\alpha'}\left(\sum_{m=1}^{\infty} \alpha^i_{-m}\,\alpha^i_m + \sum_{r=\frac{1}{2}}^{\infty} r\, b^i_{-r}\, b^i_r - \frac{1}{2}\right)$$

So this sector appears to have a tachyon, with

$$M^2 = -\frac{1}{2\alpha'}$$

We will soon demonstrate that this is not quite true. First we have to consider the Ramond sector.

(ii) Open string: Ramond sector

In this sector, the fermions have integer modes:

$$\psi_-^\mu(\sigma,t) = \sqrt{\alpha'} \sum_{n\in\mathbb{Z}} d_n^\mu \, e^{-in(t-\sigma)}$$

$$\psi_+^\mu(\sigma,t) = \sqrt{\alpha'} \sum_{n\in\mathbb{Z}} d_n^\mu \, e^{-in(t+\sigma)}$$

and the canonical anticommutators are:

$$\{d_m^i, \, d_n^j\} = \delta_{m+n,0} \, \delta^{ij}$$

In this sector, it turns out that the normal-ordering constant a is zero. This is a consequence of worldsheet supersymmetry (which was broken by boundary conditions in the NS sector).

Thus the mass-shell condition is:

$$M^2 = \frac{1}{\alpha'}\left(\sum_{m=1}^{\infty} \alpha_{-m}^i \, \alpha_m^i + \sum_{n=1}^{\infty} n \, d_{-n}^i \, d_n^i \right)$$

Because of the zero-frequency mode d^i_0, the quantisation of this sector is somewhat novel.

The ground state is defined, as usual, by

$$\alpha^i_n|k\rangle = d^i_n|k\rangle = 0, \quad n > 0$$

This state clearly has $M^2 = 0$.

The operators $\alpha^i_{-n}, d^i_{-n},\ n > 0$ are creation operators for the excited states:

$$\alpha^{i_1}_{-n_1}\,\alpha^{i_2}_{-n_2}\ldots\alpha^{i_N}_{-n_N}\,d^{j_1}_{-r_1}\,d^{j_2}_{-r_2}\ldots d^{j_M}_{-r_M}\,|k\rangle$$

But what about the zero modes d_0^i ? They satisfy the Clifford algebra:

$$\{d_0^i,\, d_0^j\} = \delta^{ij}$$

which in particular tells us that

$$\left(d_0^i\right)^2 = 1, \quad \text{all } i$$

Now consider the state $d_0^i|k\rangle$. Because of the above relation, this state cannot be zero. On the other hand, since

$$[M^2, d_0^i] = 0$$

we see that $d_0^i|k\rangle$ has the same value of M^2 as $|k\rangle$, namely $M^2 = 0$.

Thus we have the phenomenon of ground-state degeneracy.

Given a ground state in the R sector, we can generate a degenerate set of ground states:

$$|k\rangle, \quad d_0^i|k\rangle, \quad d_0^i\, d_0^j|k\rangle, \cdots$$

The smallest such set is the irreducible representation of the Clifford algebra, obtained by taking the combinations:

$$D^1 = d_0^1 + i d_0^2, \quad D^2 = d_0^3 + i d_0^4, \quad D^3 = d_0^5 + i d_0^6, \quad D^4 = d_0^7 + i d_0^8$$

which obey

$$\{D^I, D^{J\dagger}\} = 2\delta^{IJ}, \qquad \{D^I, D^J\} = 0$$

Then we can consistently require $|k\rangle$ to satisfy:

$$D^I |k\rangle = 0, \quad I = 1, 2, 3, 4$$

and the only nonzero states are

$$|k\rangle, \quad D^{I\dagger}|k\rangle, \quad D^{I\dagger}D^{J\dagger}|k\rangle, \quad D^{I\dagger}D^{J\dagger}D^{K\dagger}|k\rangle, \quad D^{I\dagger}D^{J\dagger}D^{K\dagger}D^{L\dagger}|k\rangle$$

Because of antisymmetry, there are $2^4 = 16$ such states.

This is understandable, since the d_0^i satisfy the anticommutation relations of gamma-matrices in 8 Euclidean dimensions, whose dimension is $2^{\frac{8}{2}} = 16$. This implies that the 16 ground states transform as the spinor representation of $SO(8)$ (and, less obviously, of $SO(9,1)$).

By the spin-statistics theorem, these must therefore be fermions in spacetime!

These 16-component spinors are reducible into two 8-component spinors, one of each chirality. Thus the Ramond sector of the open fermionic string has two massless spacetime fermions, one of each chirality.

We label these as s (spinor) and c (conjugate spinor):

$$|\alpha, k\rangle_s, \quad |\alpha, k\rangle_c, \quad \alpha = 1, 2, \dots 8$$

The excited states in this sector are obtained by acting with α_{-n}^i, d_{-n}^i on this spinorial ground state. That means they are all spacetime fermions.

(iii) Open string: The GSO projection

The open string spectrum that we have found in the NS and R sectors will be displayed, for the lowest levels, in a table on the following page.

In the NS sector there are states at every half-integer level, while the R sector only has states at integer levels.

We will see an interesting pattern. At the integer levels, the degeneracy of the R states is exactly double that of the NS states.

Note also that all the NS states are spacetime bosons, while all the R states are spacetime fermions.

$\alpha' M^2$	Neveu-Schwarz	Degen.	Ramond	Degen.
$-\frac{1}{2}$	$\|k\rangle$	1	—	—
0	$b^i_{-\frac{1}{2}}\|k\rangle$	8	$\|\alpha; k\rangle_{s,c}$	16
$\frac{1}{2}$	$b^i_{-\frac{1}{2}}\, b^j_{-\frac{1}{2}}\|k\rangle$	28	—	—
	$\alpha^i_{-1}\|k\rangle$	8		
1	$b^i_{-\frac{1}{2}}\, b^j_{-\frac{1}{2}}\, b^k_{-\frac{1}{2}}\|k\rangle$	56	$d^i_{-1}\|\alpha; k\rangle_{s,c}$	128
	$b^j_{-\frac{3}{2}}\|k\rangle,\ \alpha^i_{-1} b^j_{-\frac{1}{2}}\|k\rangle$	8+64	$\alpha^i_{-1}\|\alpha; k\rangle_{s,c}$	128

We see that the NS sector by itself is unsatisfactory for a physical theory, because it has only bosons. Moreover, it has a tachyon.

The R sector by itself is also unsatisfactory for a physical theory. It has only fermions (though fortunately no tachyon). Moreover, the fermions occur in both chiralities, which makes it harder to obtain a parity violating theory.

If we could somehow combine the two sectors, and also project out the NS states at half-integer levels as well as half the R states at the integer levels, we would have solved all these problems.

This is exactly what we will now do! And the resulting theory will have supersymmetry in spacetime. It will be the open superstring in 10 dimensions.

Projecting out some levels of a string theory is not generally consistent. It can be done consistently only if we take the quotient of the theory by a symmetry operator.

This we consider a simple example of an orbifold. In this case, the symmetry is not a spacetime or geometrical symmetry.

In the NS sector, define F to be the worldsheet fermion number. This is an operator that should satisfy:

$$[F, b^i_r] = b^i_r, \ \ r < 0$$
$$[F, b^i_r] = -b^i_r, \ \ r > 0$$

It is easy to see that

$$F = \sum_{r=\frac{1}{2}}^{\infty} b^i_{-r}\, b^i_r$$

Consider the symmetry operator $(-1)^F$. One can check that

$$\{(-1)^F, b_r^i\} = 0, \text{ all } r$$
$$\left((-1)^F\right)^2 = 1$$

In particular the eigenvalues of $(-1)^F$ are ± 1.

We now take the orbifold of the entire NS sector by $(-1)^F$. This means we project in all the states with eigenvalue $+1$, and project out those with eigenvalue -1.

It is up to us to assign the eigenvalue of the NS ground state $|k\rangle$. We choose

$$(-1)^F|k\rangle = -|k\rangle$$

which gets rid of the tachyon.

Moreover, all half-integer levels get projected out, as desired.

In the R sector, the corresponding operator is:

$$(-1)^F = \hat{\Gamma}\ (-1)^{\sum_1^\infty d_{-n}^i d_n^i}$$

The exponential part causes this to anticommute with the d_n^i, while the factor

$$\hat{\Gamma} = \Gamma^1 \Gamma^2 \cdots \Gamma^8$$

anticommutes with the $d_0^i \sim \Gamma^i$.

On the Ramond ground state, the exponent does not contribute, but $\hat{\Gamma}$, the chirality matrix, projects out one of the two 8-component spinors. Thus at the end we are left with a single massless Majorana-Weyl spinor in 10 spacetime dimensions:

$$|\alpha, k\rangle_s \quad \text{or} \quad |\alpha, k\rangle_c$$

The procedure we have carried out is called the GSO projection. It may look a little artificial, but has a sound basis – as further study (not in this course!) will reveal.

Combining the projected NS and R sectors, we get a theory whose massless sector is a 10-dimensional photon A_μ and a 10-dimensional Majorana-Weyl (real, chiral) fermion λ_α. Both have 8 physical degrees of freedom.

Indeed, at the noninteracting level, they can be combined into a 10 dimensional gauge supermultiplet.

What is much more remarkable is that at every excited level, the bosonic states from the NS sector and the fermionic states from the R sector combine to form massive supermultiplets.

If this procedure of combining and projecting sectors is to be more than an artifice, the interactions should also respect supersymmetry. This is a very restrictive requirement! We will see that it is automatically met by string theory.

(iv) Closed string: The GSO projection

In the closed superstring, we have a very similar structure to that derived above, but independently in the left- and right-moving sectors. Thus, we first carry out the GSO projection in each sector.

The full spectrum is obtained by combining left-and right-movers subject to the constraint that the total mode number of left- and right-moving oscillators is equal.

Thus altogether we have four sectors:

NS-NS, R-R, NS-R, R-NS

In each sector we will have massless states, as well as a tower of massive states spaced in integral multiples of $\frac{4}{\alpha'}$.

In the NS-NS sector we find the massless states:

$$b^i_{-\frac{1}{2}}\widetilde{b}^j_{-\frac{1}{2}}|k\rangle$$

which, just as for the bosonic string, breaks up into a symmetric traceless, antisymmetric and trace part.

In covariant language these are represented by massless fields propagating in 10 spacetime dimensions:

$$G_{\mu\nu}(x), B_{\mu\nu}(x), \Phi(x)$$

In the R-R sector, we have to combine

$$\Big(|\alpha, k\rangle_s \quad \text{or} \quad |\alpha, k\rangle_c \Big)_L \times \Big(|\tilde{\alpha}, k\rangle_s \quad \text{or} \quad |\tilde{\alpha}, k\rangle_c \Big)_R$$

The choice of spinor chirality (s or c) in each sector is a pure convention. But the relative choice between the left- and right-movers is significant.

Thus we have two physically inequivalent choices:

$$\Big(|\alpha, k\rangle_s \Big)_L \times \Big(|\tilde{\alpha}, k\rangle_c \Big)_R \quad \text{or} \quad \Big(|\alpha, k\rangle_s \Big)_L \times \Big(|\tilde{\alpha}, k\rangle_s \Big)_R$$

The product of two spinorial representations of the Lorentz group is a tensorial representation. Thus in both cases, the R-R sector contains only bosons.

Introduce the notation:

$$C^{(r)}_{\mu_1,\mu_2,\ldots,\mu_r}$$

for an rth rank totally antisymmetric tensor field.

A bit of group theory tells us that

$$\Big(|\alpha,k\rangle_s\Big)_L \times \Big(|\tilde{\alpha},k\rangle_c\Big)_R \;\to\; C^{(1)}_\mu(x),\; C^{(3)}_{\mu\nu\lambda}(x)$$

while

$$\Big(|\alpha,k\rangle_s\Big)_L \times \Big(|\tilde{\alpha},k\rangle_s\Big)_R \;\to\; C^{(0)}(x),\; C^{(2)}_{\mu\nu}(x),\; C^{(4)}_{\mu\nu\lambda\rho}(x)$$

These are inequivalent sets of bosonic fields in 10 dimensions. Thus there are two types of theories:

(i) opposite chiralities for left-and right-movers: type IIA

(ii) same chirality for left-and right-movers: type IIB

Finally we look at the NS-R and R-NS sectors. In each case, we are combining a tensor and spinor representation, so the result is spinorial and therefore fermionic. The massless states are as follows:

Type IIA:

$$\text{NS-R}: \quad b_{-\frac{1}{2}}|\widetilde{\alpha}, k\rangle_c$$

$$\text{R-NS}: \quad \widetilde{b}_{-\frac{1}{2}}|\alpha, k\rangle_s$$

Type IIB:

$$\text{NS-R}: \quad b_{-\frac{1}{2}}|\widetilde{\alpha}, k\rangle_s$$

$$\text{R-NS}: \quad \widetilde{b}_{-\frac{1}{2}}|\alpha, k\rangle_s$$

In every case, we get a product of a vector and a spinor. The result includes a Rarita-Schwinger fermion (in 4 dimensions it would have "spin $\frac{3}{2}$"). This is a gravitino, the supersymmetric partner of the graviton.

It has long been known that there are precisely two types of massless supermultiplets in 10 dimensions. These are the building blocks of two classical field theories called type IIA and type IIB supergravity.

Each one has $\mathcal{N}=2$ local supersymmetry in spacetime, and therefore two gravitinos. The two theories are distinguished by the relative chirality of the two gravitinos:

$$\text{opposite} \;\longrightarrow\; \text{IIA}, \qquad \text{same} \;\longrightarrow\; \text{IIB}$$

The field contents are as follows:

$$\begin{array}{lll}
\text{IIA} & \text{bosons}: & G_{\mu\nu},\; B_{\mu\nu},\; \Phi,\; C^{(1)}_{\mu},\; C^{(3)}_{\mu\nu\lambda} \\
& \text{fermions}: & \chi^{\text{left}}_{\mu,\alpha},\; \chi^{\text{right}}_{\mu,\alpha},\; \lambda^{\text{left}}_{\alpha},\; \lambda^{\text{right}}_{\alpha} \\
\text{IIB} & \text{bosons}: & G_{\mu\nu},\; B_{\mu\nu},\; \Phi,\; C^{(0)},\; C^{(2)}_{\mu\nu},\; C^{(4)}_{\mu\nu\lambda\rho} \\
& \text{fermions}: & \chi^{\text{left}}_{\mu,\alpha},\; \chi^{\text{left}}_{\mu,\alpha},\; \lambda^{\text{right}}_{\alpha},\; \lambda^{\text{right}}_{\alpha}
\end{array}$$

The massless spectra of the two closed string theories we have studied are in perfect correspondence with those of the two supergravities.

This suggests that the two types of GSO projections lead to two distinct spacetime supersymmetric string theories, type IIA and type IIB, which are related in some way to the corresponding supergravities.

These two superstring theories, and their cousins, are central to the goal of describing the real world through string theory.

5. Effective actions, Symmetries and Interactions

(i) The effective action

We have suggested that the type IIA and IIB superstring theories are somehow related to the corresponding supergravity theories.

Let us now make the relationship more precise. Supergravity is not a renormalisable quantum field theory. So its action is just a classical object. However, it can very well be the low energy effective action of some well-defined quantum theory.

It is believed that string theory is a well-defined quantum theory. It contains both massless and massive states. On integrating out the massive states, we get an effective action for the massless states, with the same symmetries as the original string theory.

We propose that this effective action is the supergravity action.

Accordingly, let us examine the symmetries of superstring theory at the level of massless states. As with the bosonic string, the covariant formalism must be used here.

We skip the details of the calculation. The results are:

(i) Local reparametrisation invariance:

$$\delta h_{\mu\nu} = \partial_\mu \Lambda_\nu(x) + \partial_\nu \Lambda_\mu(x)$$

(ii) Local supersymmetry:

$$\delta e^a_\mu = \bar{\epsilon}(x)\,\Gamma^a \chi_\mu, \ \cdots$$

(iii) Local p-form gauge invariance:

$$\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu(x) - \partial_\nu \Lambda_\mu(x)$$

$$\delta C^{(p)}_{\mu_1\mu_2\cdots\mu_p} = \partial_{[\mu_1} \Lambda_{\mu_2\cdots\mu_p]}(x)$$

These are all known to be local symmetries of the corresponding linearised supergravity!

In supergravity, these can be extended to symmetries of the interacting action too. It is reasonable to expect that superstring theory likewise incorporates these symmetries even when we introduce interactions.

Before asking how superstring interactions are computed, let us take a schematic look at a part of the supergravity action (avoiding numerical constants):

$$S_{\text{type II,NS-NS}} \sim \frac{1}{\kappa^2} \int d^{10}x \sqrt{-\|G\|}\; e^{-2\Phi} \left(R + |d\Phi|^2 - |dB|^2 \right)$$

We have restricted to the NS-NS sector, which is common to type IIA and IIB.

As in any theory of gravity, the action is highly nonlinear. Note that every term has precisely two spacetime derivatives.

A constant κ has been introduced, with dimensions of length^2. However, because of the factor $e^{-2\Phi}$ in front of the whole Lagrangian, κ is physically irrelevant. We can change it by adding a constant to Φ.

Indeed, if for any reason Φ develops a vacuum expectation value Φ_0, we can scale this out of the action to get a prefactor $e^{-2\Phi_0}$.

It follows that e^{Φ_0} acts like the coupling constant in this theory. We will see independent evidence of this from the worldsheet approach to string theory. Thus we define the string coupling:

$$g_s = e^{\Phi_0}$$

We can also deduce independently from the supergravity action, and from the worldsheet approach, that in general there are derivative corrections.

The supergravity action is unique only if we restrict to terms with two derivatives. Four-derivative terms, for example:

$$R^2, \quad R_{\mu\nu}R^{\mu\nu}, \quad \partial_\mu\partial_\nu\Phi\partial^\mu\partial^\nu\Phi$$

can appear in this action, along with appropriate fermion couplings and couplings to the RR sector. In general, all orders of derivatives are allowed by the symmetries.

However, such terms will require a dimensional constant to appear along with the derivatives. In string theory, this constant turns out to be α', the inverse string tension.

Thus the low energy action for superstring theory can contain terms like

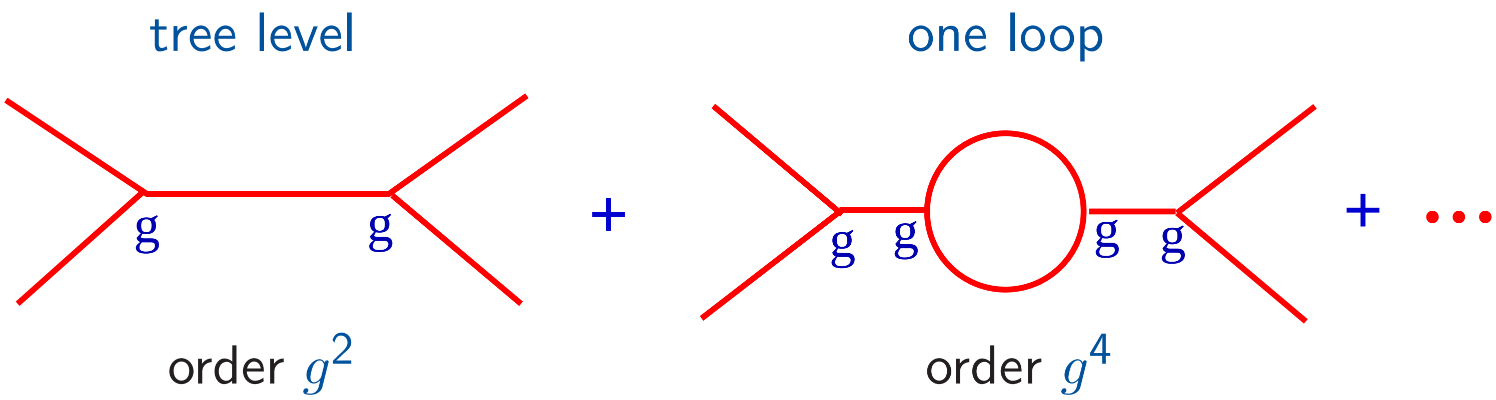
$$\frac{1}{\kappa^2}\int d^{10}x\sqrt{-\|G\|}\; e^{-2\Phi}\left(R+\alpha'(c_1R^2+c_2\,R_{\mu\nu}R^{\mu\nu}+\cdots)+\mathcal{O}(\alpha'^2)\right)$$

We will now discuss superstring amplitudes and confirm all these general arguments.

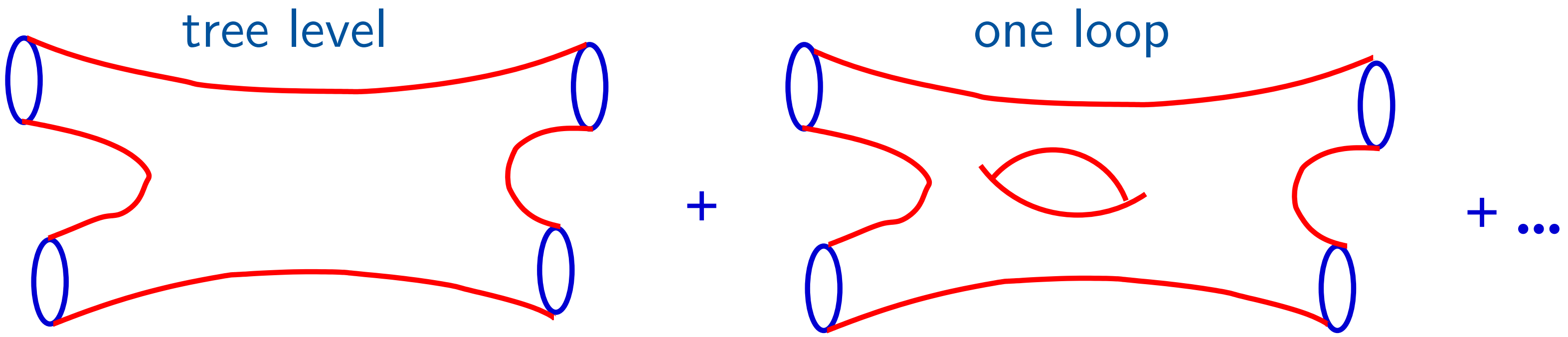
(ii) Superstring interactions

Having quantised free superstrings and obtained their spectrum and symmetries, it is natural to ask how to introduce interactions.

For particles in first-quantised language, we introduce interactions by drawing Feynman diagrams:



For strings, we know that the worldline has to be replaced by a worldsheet. So one may guess that string scattering diagrams (for closed strings) look like:

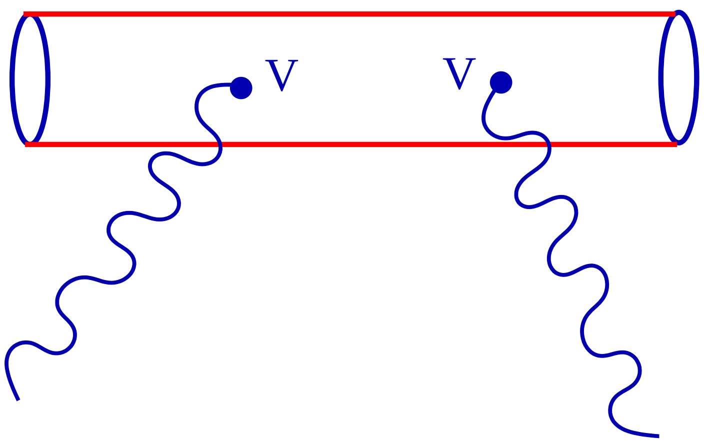


The worldsheet in the tree level diagram is topologically a sphere (with four discs cut out). The worldsheet in the one-loop diagram is a torus, and so on.

We see that the string loop expansion is related to the set of compact two-dimensional manifolds with discs cut out.

How do we quantise strings with worldsheets on such complicated worldsheets? A standard mode expansion cannot be made when there are "handles" on the worldsheet.

For tree-level amplitudes, there is no problem. We start with a cylindrical worldsheet and insert special operators V called vertex operators, to represent absorption and emission of additional strings:



For example, this 4-point amplitude, for open strings, is written as:

$$\langle 1|V_2(k_2)\,\Delta\,V_3(k_3)|4\rangle$$

where Δ is the propagator:

$$\Delta = \frac{1}{L_0} = \int_0^\infty d\tau\, e^{-\tau\, L_0}$$

and

$$L_0 = p^- = \frac{1}{2p^+}\left(\frac{1}{\alpha'}\sum_{m=1}^{\infty}\alpha^i_{-m}\,\alpha^i_m + p^i p^i - a\right)$$

is the worldsheet Hamiltonian.

For loop corrections, things are not so simple. There are two ways to proceed:

(i) make worldsheets of complicated topology by gluing cylinders together. On each cylinder the mode expansion is well-defined. To glue the cylinders together we must define a three string vertex.

(ii) give up the operator formalism we have been using so far, and work with the functional integral. In this case we can define the worldsheet to have any topology we like.

The latter formalism is extremely powerful and allows us to compute string amplitudes quite effectively. In this formalism, every string state has a vertex operator V associated to it, and string amplitudes are simply the correlation function of vertex operators on the given worldsheet.

Whichever way we do things, it turns out that in each formalism, every physical state of the string has a unique vertex operator associated to it.

The computation of the amplitude is completely specified in terms of these operators. As a result, string amplitudes are unique and cannot be postulated by hand.

Thus, for example, the graviton of the closed type II superstring has an operator

$$V_{\mathsf{G}}^{\mu\nu}(k)$$

associated to it, such that the N-graviton scattering amplitude is given symbolically by:

$$\langle V_{\mathsf{G}}^{\mu_1\nu_1}(k_1)\, V_{\mathsf{G}}^{\mu_2\nu_2}(k_2)\cdots V_{\mathsf{G}}^{\mu_{\mathsf{N}}\nu_{\mathsf{N}}}(k_{\mathsf{N}})\rangle$$

The actual computation of string amplitudes is a long story, so we will simply quote the answers and examine their physical properties.

(iii) Closed superstring tree amplitudes

Tree amplitudes are moderately simple for small values of N. For $N = 3$, we find:

$$\langle V_{\mathsf{G}}^{\mu_1\nu_1}(k_1)\, V_{\mathsf{G}}^{\mu_2\nu_2}(k_2)\, V_{\mathsf{G}}^{\mu_3\nu_3}(k_3)\rangle \sim \delta^{10}(k_1 + k_2 + k_3)\; A^{\mu_1\mu_2\mu_3}\; A^{\nu_1\nu_2\nu_3}$$

where

$$A^{\mu_1\mu_2\mu_3} = \eta^{\mu_1\mu_2}(k_1 - k_2)^{\mu_3} \;+\; \text{cyclic permutations}$$

We see that the momentum is conserved, as it should be. This is built in to the structure of vertex operators.

We also see that every term in this amplitude has exactly two momenta.

Multiplying by polarisation tensors

$$\zeta^1_{\mu_1\nu_1}\,\zeta^2_{\mu_2\nu_2}\,\zeta^3_{\mu_3\nu_3}$$

and summing over the indices, we get the three-point amplitude. The expression involves a large number of terms.

Let us try to get a feeling for a typical term:

$$k_1^\alpha\, k_2^\beta\, \zeta^1_{\mu\nu}\, \zeta^{2\mu\nu}\, \zeta^3_{\alpha\beta}$$

As we have seen previously, the polarisation tensors should be thought of as wave functions in momentum space for the corresponding field.

In this case the field is the graviton $G_{\mu\nu}$, or more precisely its fluctuation $h_{\mu\nu}$ away from flat spacetime. And the above expression is just the momentum-space version of:

$$\partial^\alpha h_{\mu\nu}\, \partial^\beta h^{\mu\nu}\, h_{\alpha\beta}$$

It turns out that this term, and all the others of its kind, are precisely the terms in the Einstein Lagrangian:

$$\sqrt{-\|G\|}\, R$$

when we make the replacement

$$G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

and expand to cubic order in $h_{\mu\nu}$.

Thus by computing a three-point amplitude following the rules of superstring theory, we are able to read off the effective coupling of the graviton, to cubic order.

The fact that it is equal to the cubic term in Einstein's action is nothing short of a miracle, though one that we expected by now.

This miracle continues at every order, but an extra feature also emerges.

Let us consider the four-graviton amplitude. In this case, it is convenient to work with the Mandelstam variables:

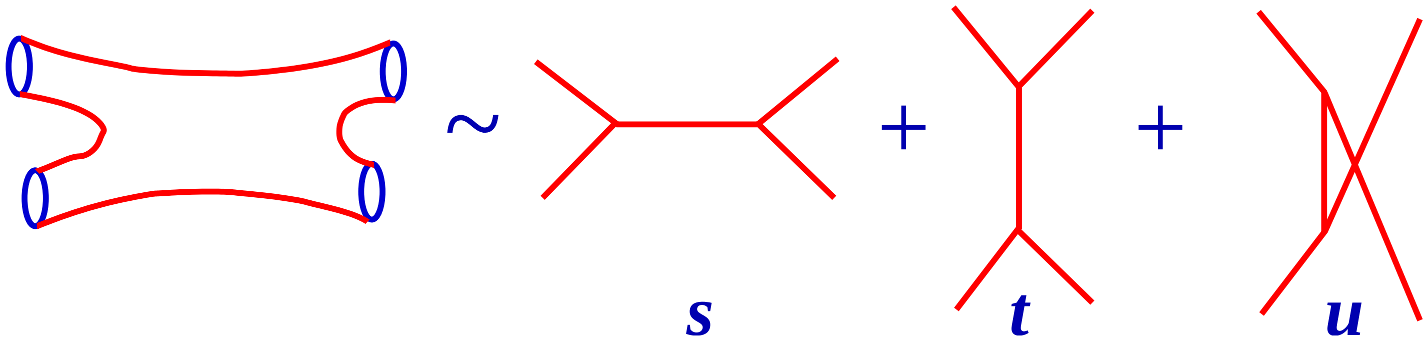
$$s = -(k_1 + k_2)^2, \qquad t = -(k_1 + k_3)^2, \qquad u = -(k_1 + k_4)^2$$

$$\langle V_{\mathsf{G}}^{\mu_1\nu_1}(k_1)\, V_{\mathsf{G}}^{\mu_2\nu_2}(k_2)\, V_{\mathsf{G}}^{\mu_3\nu_3}(k_3)\, V_{\mathsf{G}}^{\mu_4\nu_4}(k_4)\rangle \sim$$
$$\alpha'^3\, \frac{\Gamma(-\frac{1}{4}\alpha' s)\,\Gamma(-\frac{1}{4}\alpha' t)\,\Gamma_(- \frac{1}{4}\alpha' u)}{\Gamma(1+\frac{1}{4}\alpha' s)\,\Gamma(1+\frac{1}{4}\alpha' t)\,\Gamma(1+\frac{1}{4}\alpha' u)}\; K^{\mu_1\nu_1\cdots\mu_4\nu_4}(\zeta^i, k_i)$$

where $K^{\mu_1\nu_1\cdots\mu_4\nu_4}(k_i)$ is a kinematic factor containing eight powers of momenta.

This is a beautiful expression, with many important properties:

(i) It is completely symmetric in s, t, u. While it corresponds to a single "stringy Feynman diagram", it plays the role of a sum of particle diagrams:



(ii) The amplitude has poles whenever

$$s = \frac{4}{\alpha'} n \quad \text{or} \quad t = \frac{4}{\alpha'} n \quad \text{or} \quad u = \frac{4}{\alpha'} n$$

for any positive integer n.

These are precisely the values of M^2 for which the closed superstring has massive physical states.

The poles occur when any of s, t, u is equal to the mass-squared of a physical state, signalling the possibility of producing these states as resonances in the intermediate channel.

(iii) The amplitude depends on α', so we can expect to see some truly "stringy" properties. Consider the expansion of the gamma-function factor in powers of α':

$$\frac{\Gamma(-\frac{1}{4}\alpha' s)\,\Gamma(-\frac{1}{4}\alpha' t)\,\Gamma_{(}-\frac{1}{4}\alpha' u)}{\Gamma(1+\frac{1}{4}\alpha' s)\,\Gamma(1+\frac{1}{4}\alpha' t)\,\Gamma(1+\frac{1}{4}\alpha' u)} \sim -\frac{64}{\alpha'^3\, s\, t\, u} - 2\zeta(3) + \mathcal{O}(\alpha')$$

The first term cancels 6 momenta in the kinematic factor K, as well as the α'^3 coefficient in the amplitude. Thus it gives a two-derivative term of fourth order in the gravitational fluctuation $h_{\mu\nu}$.

So this must be the fourth order term in the expansion of the Einstein Lagrangian. And indeed it is.

The next term, however, gives a contribution to the effective action which has 8 derivatives and three powers of α. Dimensionally this is consistent, because $\alpha' \partial\partial$ is dimensionless.

In fact, the term in the Lagrangian required to reproduce this term in the amplitude is:

$$\sim \zeta(3)\, \alpha'^3\, t^{\mu_1\nu_1\cdots\mu_4\nu_4}\, t^{\rho_1\sigma_1\cdots\rho_4\sigma_4}\, R_{\mu_1\nu_1\rho_1\sigma_1}\, R_{\mu_2\nu_2\rho_2\sigma_2}\, R_{\mu_3\nu_3\rho_3\sigma_3}\, R_{\mu_4\nu_4\rho_4\sigma_4}$$

where t is some numerical tensor and $R_{\mu\nu\rho\sigma}$ is the Riemann curvature tensor.

As we predicted, string theory produces not only conventional (Einstein-type) actions with two derivatives, but also higher-derivative corrections to them.

The role of α' is to govern these corrections. It defines what is meant by slowly varying fields. Derivative corrections may be ignored for fields that are slowly varying on the scale of the string length $l_s \sim \sqrt{\alpha'}$.

(iv) Closed superstring loop amplitudes

The loop amplitudes illustrate several key points about superstring theory, which we will now highlight.

(i) Role of the dilaton:

We have alluded before to the role of the dilaton as the coupling constant of string theory.

Let us now see directly how this arises on the worldsheet.

The worldsheet actions we have presented were valid for strings propagating in flat spacetime and in the absence of background fields.

Suppose we allow an arbitrary dilaton field $\Phi(x)$ to be present in spacetime. It turns out that this requires a modification in the worldsheet action:

$$S \to S + \frac{1}{4\pi} \int d\sigma \, dt \, \sqrt{-\|g\|} \, R \, \Phi(X(\sigma, t))$$

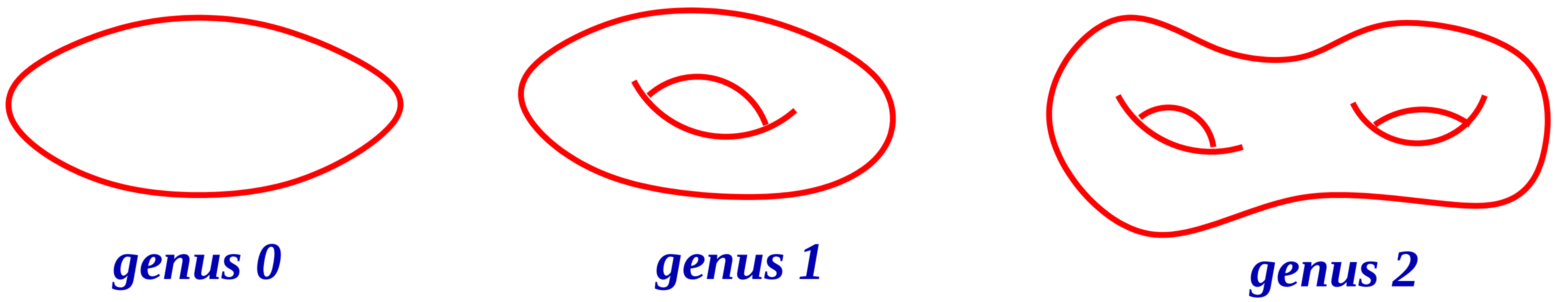
In general, this modification spoils Weyl invariance which was so important in quantising the string. But suppose the dilaton field is constant:

$$\Phi(X) = \Phi_0$$

Then the extra term becomes

$$\frac{\Phi_0}{4\pi} \int d\sigma\, dt \sqrt{-\|g\|}\, R$$

Now, worldsheets with handles are classified by their genus which is just the number of handles:



It is a theorem in differential geometry that on any two-dimensional closed surface,

$$\frac{1}{4\pi}\int d\sigma dt\,\sqrt{-\|g\|}\,R = 2-2h$$

where h is the genus.

Thus we see that the modification of the worldsheet action for the constant mode of the dilaton is:

$$S \to S + 2\Phi_0(1-h)$$

In the functional integral, we accordingly get the replacement:

$$e^{-S} \to e^{-2\Phi_0(1-h)}\,e^{-S} = \left(g_s^2\right)^{h-1} e^{-S}$$

where, as suggested earlier, we have defined:

$$g_s = e^{\Phi_0}$$

We see that every extra handle on the worldsheet comes with an extra power of g_s^2.

In Feynman diagrams each such handle is like a string loop, and we expect each extra loop to be weighted by higher powers of the string coupling.

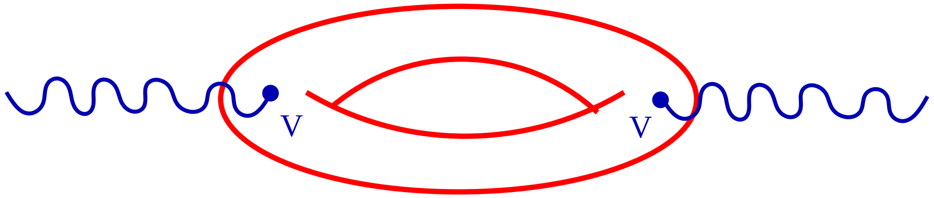
Thus the VEV of the dilaton indeed defines the string coupling constant, confirming the arguments that we earlier advanced on the basis of the spacetime action.

It is a remarkable property of string theory that it has no arbitrary dimensionless constants.

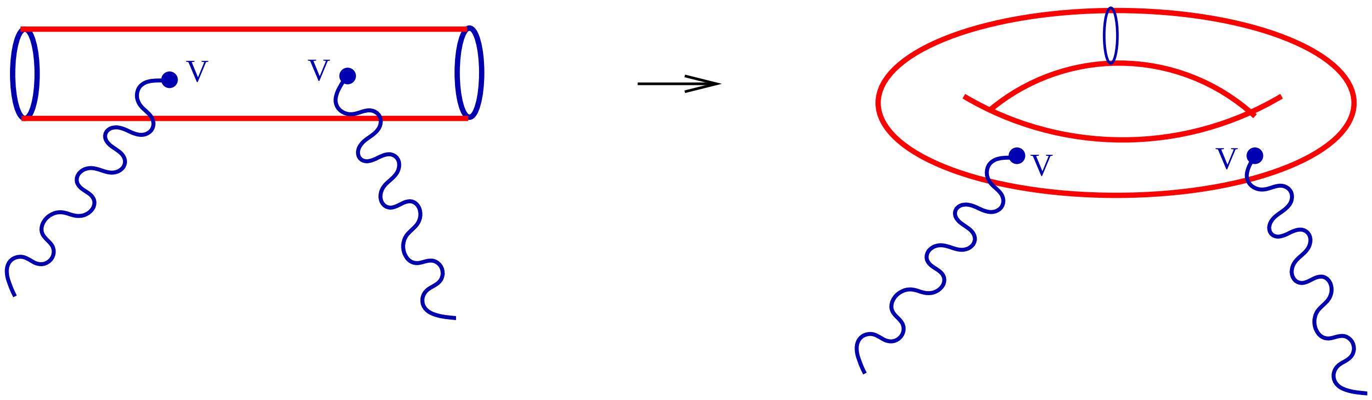
(ii) Finiteness of loop amplitudes:

It is hard to discuss general loop amplitudes without developing more formalism, but one-loop amplitudes are relatively simple.

The reason is that the one-loop diagram:



can be thought of as a cylindrical worldsheet of finite length, rolled up:



The act of rolling up the worldsheet means we are no longer computing a tree amplitude between fixed states, like:

$$\langle 1|V_2(k_2)\,\Delta\,V_3(k_3)|4\rangle$$

but are instead computing a sum like:

$$\sum_n \langle n|\Delta V_2(k_2)\,\Delta\,V_3(k_3)|n\rangle$$

where $|n\rangle$ is a complete set of string states.

This is the same as a trace in Hilbert space:

$$\mathrm{tr}\;(\Delta V_2(k_2)\,\Delta\,V_3(k_3))$$

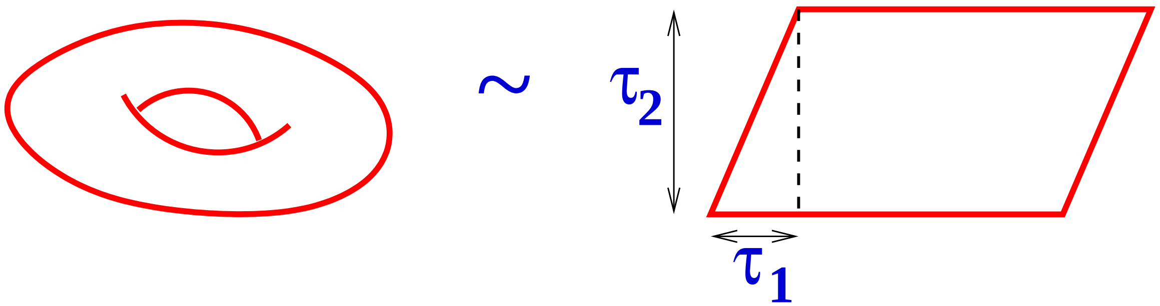
Let us specialise to closed strings. The simplest loop amplitude is the vacuum amplitude, obtained by computing

$$\int \frac{d\tau_1 d\tau_2}{(\tau_2)^2}\, Z(\tau,\overline{\tau})$$

where

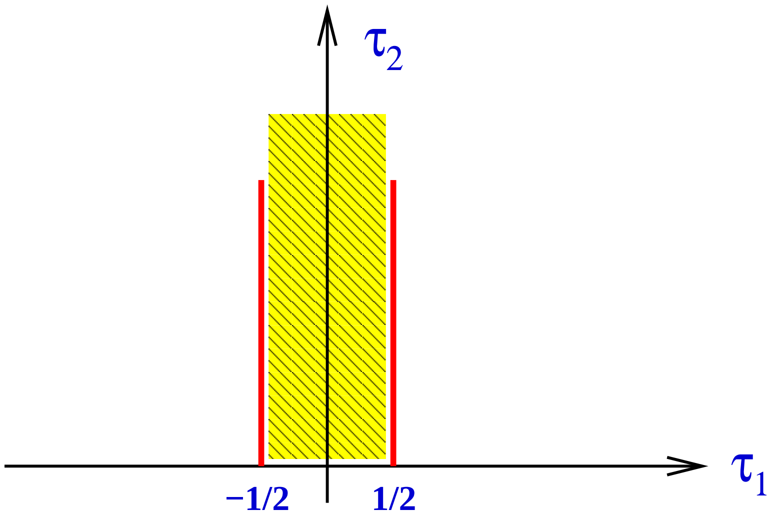
$$Z(\tau,\overline{\tau}) = \mathrm{tr}\; e^{2\pi i\tau L_0} e^{-2\pi i\overline{\tau}\widetilde{L}_0} \sim \mathrm{tr}\;\Delta$$

Here $\tau = \tau_1 + i\tau_2$ describes the shape of the toroidal worldsheet:



$Z(\tau,\overline{\tau})$ is the generating function for the number of states in the string theory, or the partition function.

We must integrate over τ. But what is the range? Naively, it looks like:



Note that the integration over τ_1 enforces the constraint:

$$L_0 = \widetilde{L}_0$$

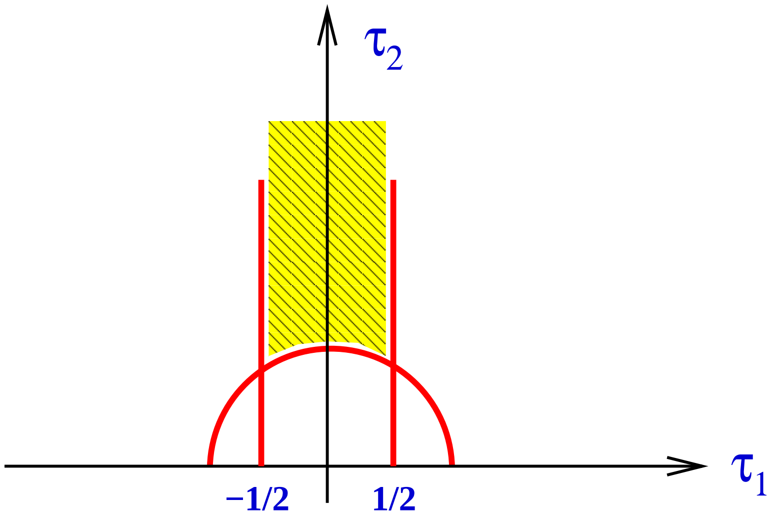
However, the partition function has an infinite discrete symmetry under:

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}$$

where a, b, c, d are integers satisfying $ad - bc = 1$. This symmetry is called modular invariance.

Under this transformation, the shaded region divides into infinitely many equivalent subregions.

Thus to avoid overcounting, we must restrict the integration over τ to a "fundamental region":



Notice that the region near $\tau \sim 0$ is now excluded.

But since $L_0 \sim p^2 + \cdots$, this means the region of large p^2 is excluded.

This cuts off the ultraviolet modes propagating in the loop. As a result, we get a finite answer.

While this example had no external string states, we can repeat the same type of calculation with, for example, external gravitons.

Again, modular invariance ensures that the one-loop amplitude is finite. Indeed, all loop amplitudes have similar invariances and they are all finite.

Thus we have evaded one of the biggest theoretical problems in gravity since the days of Einstein.

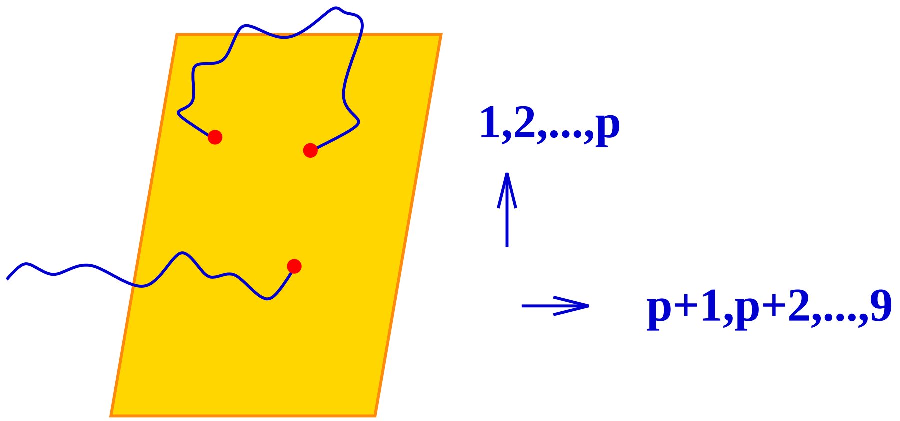
Gravity as a field theory is nonrenormalisable and gives unremovable UV divergences. But gravity in string theory has no UV divergences at all!

This is why we believe that string theory is a consistent theory of quantum gravity.

6. D-Branes and Nonabelian Gauge Symmetry

We have seen that open strings are allowed to have Neumann or Dirichlet boundary conditions independently in each direction.

Since there are altogether 9 spatial directions, we can allow p Neumann and $9-p$ Dirichlet boundary conditions:

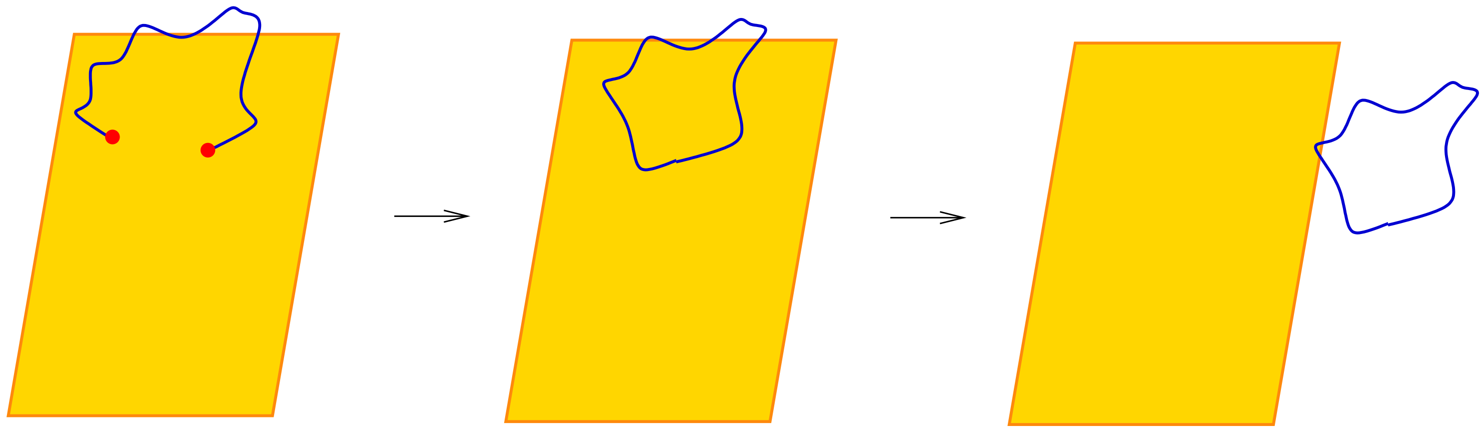


This defines a p-dimensional hypersurface in spacetime. It is called a D-brane, generalising the notion of a membrane.

What is the physical interpretation of this?

Open strings interact locally at their end points.

Consider an open string with both ends on the same brane. If the two ends meet, it can become a closed string and leave the brane.



Thus we expect open and closed strings to coexist in the same theory.

But closed strings live in 10 dimensions and have Lorentz invariance. So a D-brane, on which open strings live, must be an excited state of the closed string theory.

It is not one of the closed-string excited states that we've already seen. Indeed, it turns out to be a nonperturbative state of the theory.

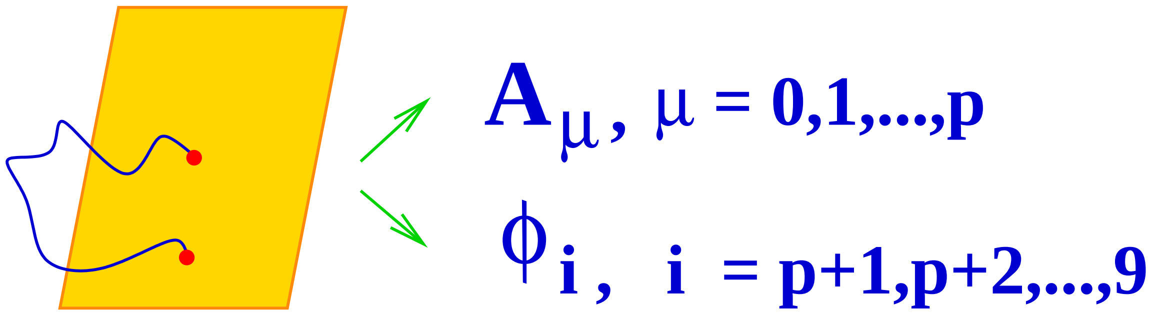
From the point of view of supergravity, it is known that there are extended solitonic excitations similar to the kinks, vortices and domain walls of ordinary quantum field theory.

A D-brane is identified as such a solitonic excitation. The new thing is that it can be explicitly studied using open strings.

Earlier we quantised open strings with NN boundary conditions. The massless states were a photon A_μ in 10 dimensions, as well as a fermion λ_α.

Now let us quantise strings with DD boundary conditions in $9-p$ spatial directions. Since the ends of the string only propagate in $p+1$ spacetime dimensions, the string states will be confined to the D-brane.

We find that the massless states are now a photon A_μ in $p+1$ dimensions, as well as $9-p$ scalar fields ϕ_i. (We ignore the fermionic partners).



In particular, there is one scalar field for each direction transverse to the brane.

It is tempting to assume that the vacuum expectation value of these scalars represent the transverse locations of the branes.

When these scalars become time-dependent, that physically represents transverse motion of the brane.

The low-energy effective field theory on a Dp-brane is a $p+1$-dimensional field theory.

It can be shown, either using supergravity or using open-string scattering amplitudes, that some Dp-branes carry a charge under Ramond-Ramond gauge fields.

As an explicit example, consider type IIA string theory, which has an RR gauge field $C^{(1)}_\mu$ with a field strength:

$$G^{(2)}_{\mu\nu} = \partial_\mu C^{(1)}_\nu - \partial_\nu C^{(1)}_\mu$$

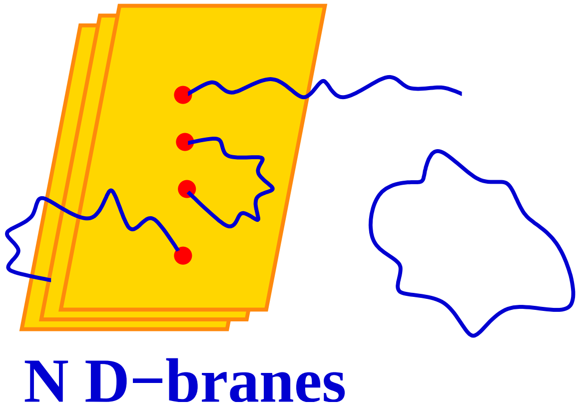
In the presence of a D0-brane, this field strength satisfies the equation

$$\partial^I G^{(2)}_{0I} \sim q\, \delta^9(x)$$

with $q = \pm 1$. Thus we have both D0 and anti-D0 branes. Because of their charge, such branes are individually stable.

There are other, uncharged, branes that are unstable. They have tachyons on their world-volume, even in superstring theory.

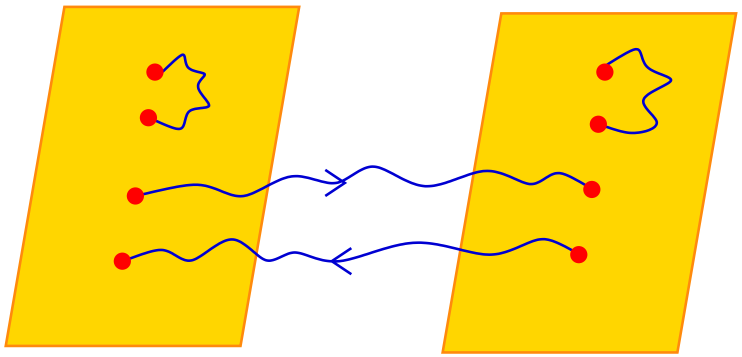
But now we can finally see our way to finding nonabelian gauge symmetry in string theory. Simply assemble a collection of N parallel D-branes:



Now an open string can start on any one of the branes and end on any other. So there are N^2 species of open strings. That means the massless gauge field is an $N \times N$ matrix A_μ^{ab}.

Is string theory clever enough to construct non-abelian gauge symmetry in this situation? Well, of course it is.

Let's consider the simplest example, $N = 2$:



We see that there are four species of strings. Of these, two are localised on individual branes, so they clearly represent the abelian gauge field of that brane. Together, they provide $U(1) \times U(1)$ gauge fields.

Now, the endpoint of a string ending on a D-brane can be shown to behave as a point charge on the brane world-volume.

So the two strings that are stretching across the branes are charged under $U(1) \times U(1)$. They provide the extra gauge fields to enhance:

$$U(1) \times U(1) \to U(2)$$

If the two D-branes are precisely coincident, then the strings stretching from one to the other can have zero length. At this point, all the four gauge fields are massless.

If we now separate the branes, two of the four strings acquire a minimum length and therefore a classical energy. So the corresponding gauge fields must be massive.

But we claimed that transverse motion of the branes is represented by giving a VEV to the transverse scalar fields.

Therefore this is string theory's realisation of the Higgs mechanism!

As usual, things now fall into place.

We can quantise the N^2 strings on a stack of D-branes and perform amplitude calculations using vertex operators.

The result is as expected. The low-energy effective theory is the Yang-Mills theory of a $U(N)$ gauge field A_μ^{ab}, $a, b = 1, 2, \dots N$, coupled to scalars and fermions in the adjoint representation of $U(N)$.

Amplitude calculations reveal the famous "three-gluon" and "four-gluon" interactions, the signature of Yang-Mills theory:

$$\mathrm{tr}\, \partial_\mu A_\nu [A^\mu, A^\nu], \qquad [A_\mu, A_\nu][A^\mu, A^\nu]$$

D-branes are extremely useful objects. Besides introducing non-Abelian gauge symmetries into string theory, they also help us to reformulate familiar notions from field theory and mathematics in a new way. This has led to several new insights about gauge theory.

7. Compactification and Physics

We have discovered that type IIA/B superstring theory is a consistent ten-dimensional theory with local $\mathcal{N} = 2$ supersymmetry.

What does this have to do with reality?

Our quantisation of the theory in 10 flat extended spacetime dimensions has perhaps been slightly misleading. We could have chosen to have the string propagate in any 10-dimensional spacetime.

All such choices need not be consistent. But there is one very simple choice that is always consistent.

Let us use new labels for the spacetime directions:

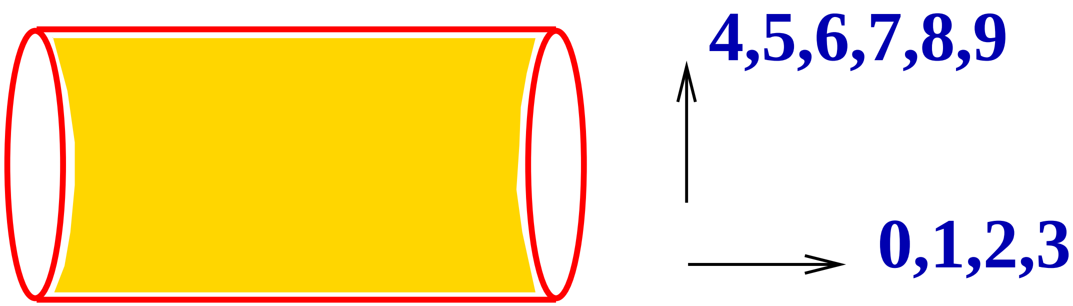
$$0, 1, 2, 3 \longrightarrow \mu, \nu \cdots$$

$$4, 5, 6, 7, 8, 9 \longrightarrow i, j, \cdots$$

Now suppose that the 6 coordinates X^i are periodic:

$$X^i \sim X^i + 2\pi R^i$$

This has nothing to do with worldsheet boundary conditions. It says that some directions of physical space are curled up:



If we probe such a world through experiments whose available energy E satisfies:

$$E << \frac{1}{R_i} \quad \text{for all } i$$

this world will not appear 10-dimensional, but rather 4-dimensional.

This is because, for its Fourier modes to fit into the compact dimension, an elementary particle needs an energy of order the inverse radius.

What would change if we formulated superstring theory in this kind of "toroidally compactified" spacetime?

(i) The periodicity of the six X^i's breaks the Lorentz group

$$SO(9,1) \to SO(3,1)$$

This is, of course, a good thing.

(ii) The mode expansion of the closed string changes and we get additional modes. Instead of

$$X^i = x^i + 2\alpha' p^i t + \text{oscillators}$$

we now have

$$X^i = x^i + 2\alpha' p^i t + 2L^i \sigma + \text{oscillators}$$

where L^i is quantised.

Originally the L^i mode was prohibited by the requirement:

$$X^i(t, \sigma + \pi) = X^i(t, \sigma)$$

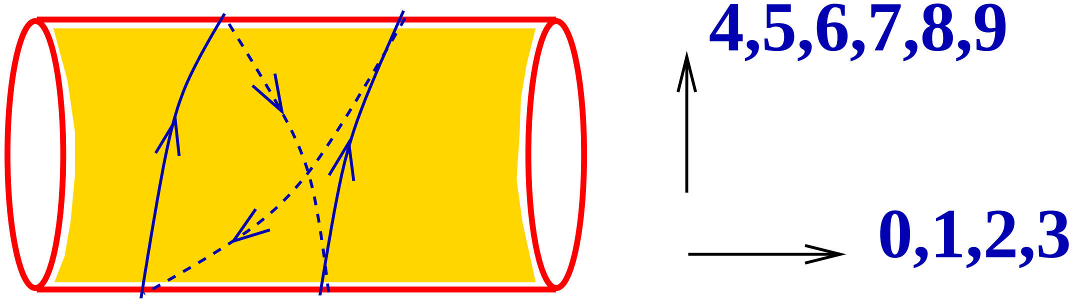
But because X^i itself is periodic, this requirement is now relaxed to:

$$X^i(t, \sigma + \pi) = X^i(t, \sigma) + 2\pi\, n_i R^i$$

from which we get

$$L^i = n_i R^i, \quad n_i \text{ integers}$$

A mode of nonzero L^i is a winding mode of the string.



In this example the string is winding twice around a compact direction.

Note that the centre-of-mass momentum is also quantised:

$$p^i = \frac{m^i}{R^i}, \quad m^i \text{ integers}$$

just as for an ordinary particle in a compact space.

The contributions to the energy from momentum and winding modes are:

$$E \sim \sum_i (p^i)^2 + \frac{(L_i)^2}{\alpha'^2} \sim \left(\frac{m^i}{R^i}\right)^2 + \frac{(n_i R^i)^2}{\alpha'^2}, \quad n_i, m^i \text{ arbitrary integers}$$

This formula has a symmetry under

$$R^i \leftrightarrow \frac{\alpha'}{R^i}, \quad n_i \leftrightarrow m^i$$

This is an exact symmetry of string theory, called T-duality or target-space duality. Physically it tells us there is a minimum length in string theory, the string length $\sqrt{\alpha'}$. Any length smaller than that can be "dualised" into a larger length.

(iii) The massless string states that we found in type IIA/B supergravity continue to exist after toroidal compactification.

However, their 10-momentum will be restricted to a 4-momentum p^μ, with the other 6 components being zero unless we supply enormous energies.

It no longer makes sense to think of them as 10-dimensional tensors. Their indices can be divided into $\mu = 0, 1, 2, 3$ and $i = 4, 5, \ldots, 9$. The latter indices are scalars under the 4-dimensional Lorentz group.

This is called Kaluza-Klein reduction.

As an example, the 10-dimensional metric or graviton field becomes:

$$G_{\mu\nu}, \quad G_{\mu i}, \quad G_{ij}$$

which means a metric, 6 gauge fields and 21 scalars in 4 dimensions.

Performing this process for all the fields of type IIA/B supergravity, we find that both theories reduce to the same supergravity theory in 4 dimensions.

The 2 gravitinos in 10 dimensions become 8 gravitinos in 4 dimensions. Thus we have 8 local supersymmetry charges in 4 dimensions.

Indeed, this is the fabled $\mathcal{N} = 8$ supergravity.

Unfortunately, $\mathcal{N} = 8$ supergravity has far too many things wrong with it to be the right low energy theory describing nature.

But the compactified string is certainly a theory in 4 spacetime dimensions, with gravity, gauge fields, fermions...

Theoretically this is just as natural a background of string theory as open 10-dimensional spacetime. And it enjoys the good properties of string theory in 10 open dimensions, such as UV finiteness of amplitudes.

Moreover, this also indicates how we should fix the scale α' of string theory. It should be chosen so that the 4-dimensional theory has the right value of the Planck mass. Then the infinite tower of string excitations would have masses of order 10^{19} GeV, and be unobservable.

As we know, to get a more realistic theory of low-energy physics, it is better to have $\mathcal{N} = 1$ supersymmetry in 4 dimensions. Hence we must search for string backgrounds with lower supersymmetry.

Besides type IIA/B superstrings, there are 3 more superstring theories in 10 dimensions, with $\mathcal{N} = 1$ supersymmetry.

One of these, called the type I superstring, arises by quotienting the closed string by orientation reversal.

This creates a kind of mirror called an orientifold plane in 10 dimensions. The latter object turns out to have -32 units of charge with respect to a D-brane. So we must put 32 D-branes in as well.

These would give the gauge group $U(32)$, but the orientifold plane breaks this to a subgroup, $SO(32)$.

This theory can also be toroidally compactified to four dimensions, but it gives us $\mathcal{N} = 4$ supergravity coupled to $\mathcal{N} = 4$ super-Yang-Mills theory. This is a slight improvement on $\mathcal{N} = 8$, but not good enough.

Another two superstring theories in 10 dimensions are called heterotic strings. They arise by fusing the left movers of the bosonic string and the right movers of the fermionic string.

For this to be possible, the bosonic string has to be compactified on a torus from 26 to 10 dimensions. Only two 16-tori are consistent (with modular invariance) and they give rise to Yang-Mills multiplets of $SO(32)$ or $E_8 \times E_8$ respectively.

The resulting theories are $\mathcal{N} = 1$ superstrings in 10 dimensions: the $SO(32)$ and $E_8 \times E_8$ heterotic strings.

Again, toroidal compactification of these theories is no use since we end up with $\mathcal{N} = 4$ supersymmetry in 4 dimensions.

Essentially this means we have to consider more complicated compactifications than just toroidal.

One can for example take an arbitrary differentiable manifold that is 6-dimensional, compact, and small.

Then one can ask whether the theory compactified on this is

(i) consistent,

(ii) $\mathcal{N} = 1$ supersymmetric,

(iii) has a zero or small cosmological constant,

(iv) parity-violating,

and so on.

Roughly, requirements (i),(ii) and (iii) rule out most 6-manifolds!

The ones that survive are a class called Calabi-Yau manifolds. But even these do not work with type IIA/B strings, because we end up with $\mathcal{N} = 2$ supersymmetry after compactifying to 4 dimensions.

With type I and heterotic $SO(32)$ superstrings, Calabi-Yau compactification gives the right supersymmetry, but no parity violation.

But with the $E_8 \times E_8$ heterotic string we can satisfy all the above basic requirements.

This is the basis of early string phenomenology. But we will not be able to touch on any of the details here.

We have been talking about the Kaluza-Klein reduction within perturbative string theory.

Many nonperturbative properties of string theory are known, though they could not be discussed here. Nonperturbative compactifications satisfying the four physical requirements listed above do exist for all the 5 superstring theories.

Typically, these involve D-branes and orientifold planes, as well as geometric singularities at which light states are pinned.

Even more remarkably, nonperturbative investigations reveal that all the 5 superstring theories are really the same. In some sense they are all deformations of a common theory. So, which one we choose as the starting point for compactification is largely a matter of convenience.