Field Theories Near Equilibrium

TIFR, Mumbay, Dec. 2004

THERMODYNAMICS OF THE HIGH TEMPERATURE QGP

Weak coupling calculations

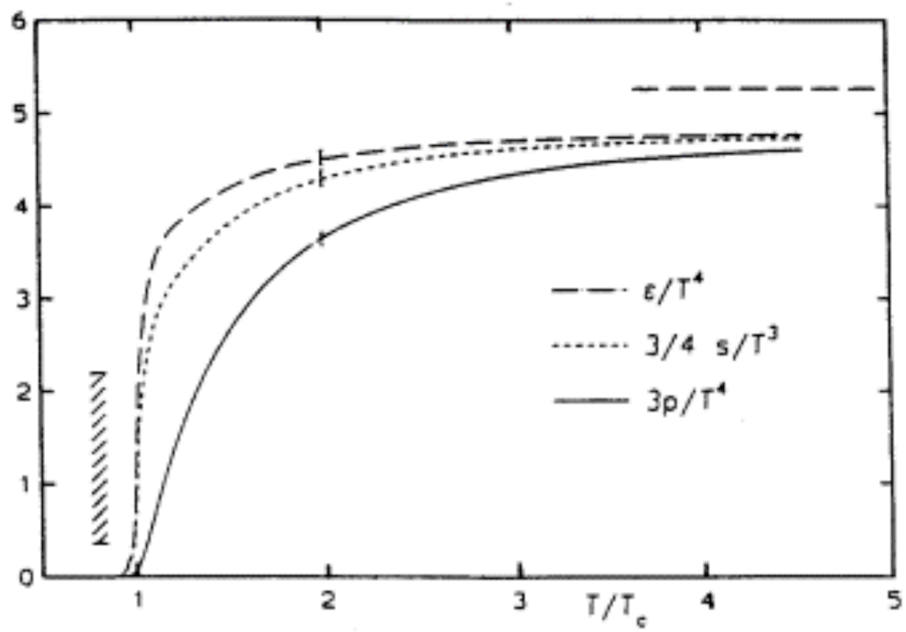
$$\alpha(\mu) \sim \frac{1}{\ln(\mu/\Lambda_{QCD})}$$

Based on work done with E. Iancu and A. Rebhan:
- hep-ph/0303185
- hep-ph/0303045

J.-P. Blaizot SPhT-Saclay

SU(3) EQUATION OF STATE

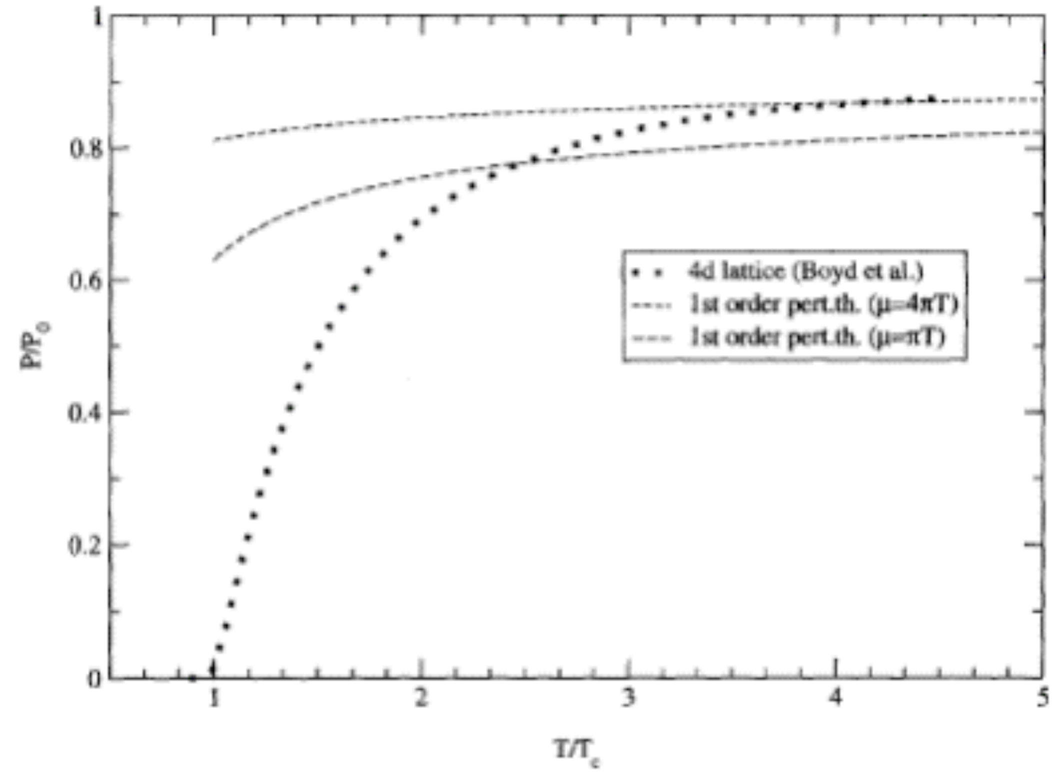
From E. Laermann, Nucl.Phys. **A610**(1996)1c



- Thermodynamical functions approach (slowly) the free gas limit at high T

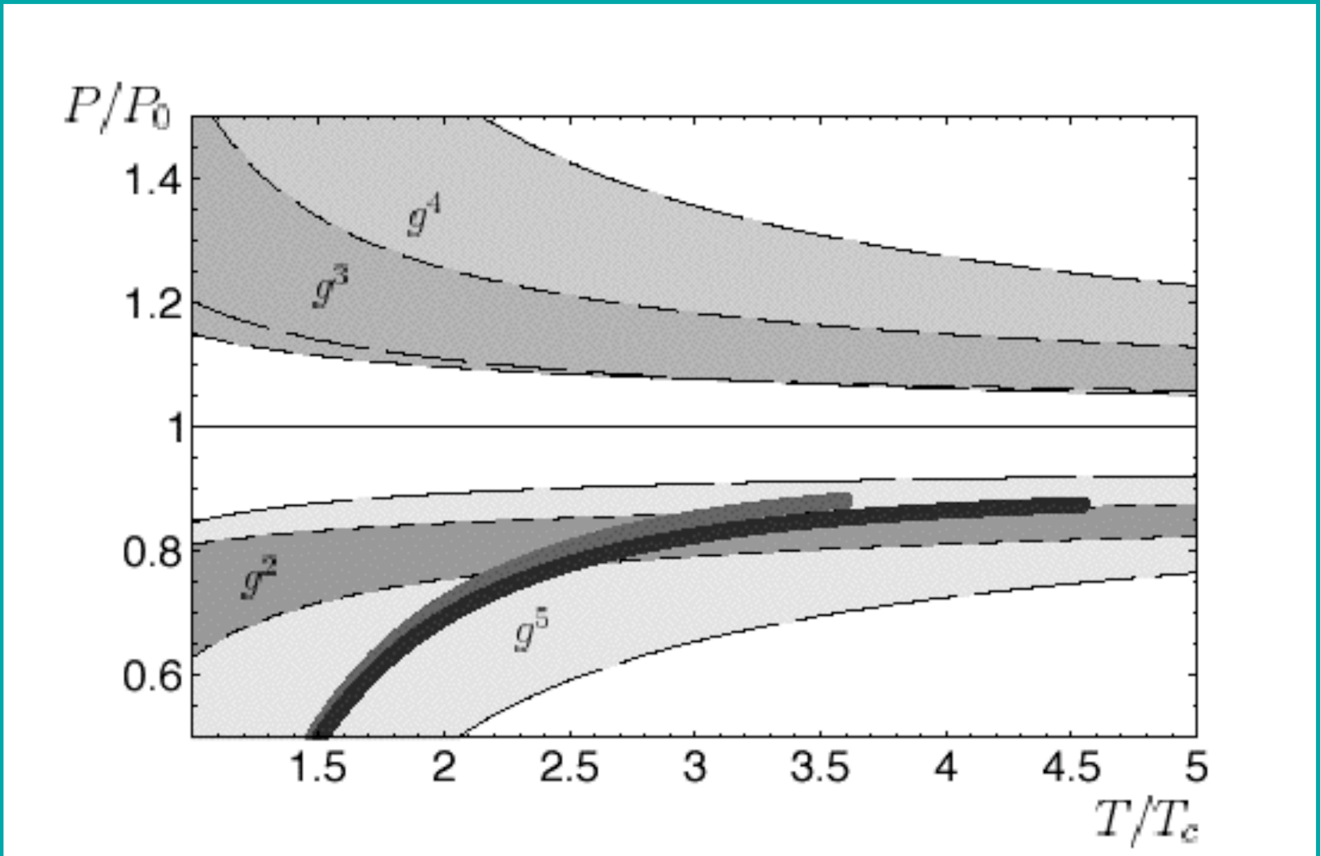
Perturbation theory at high temperature

PERTURBATION THEORY — LOWEST ORDER



$$\frac{P}{P_0} = 1 - \frac{15g^2}{16\pi^2} \qquad P_0 = (N_c^2 - 1)\,\frac{\pi^2 T^4}{45}$$

Perturbation theory up to order g^5

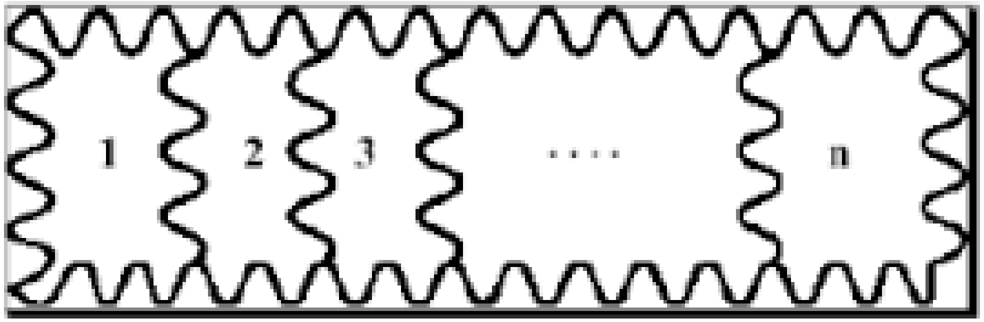


Lattice:

G. Boyd *et al.*, Nucl. Phys. **B469**, 419 (1996).
M. Okamoto *et al.*, Phys. Rev. **D60**, 094510 (1999).

E. V. Shuryak, Sov. Phys. JETP **47**, 212 (1978).
J. I. Kapusta, Nucl. Phys. **B148**, 461 (1979).
T. Toimela, Phys. Lett. **B124**, 407 (1983).
P. Arnold and C.-X. Zhai, Phys. Rev. **D51**, 1906 (1995).
C.-X. Zhai and B. Kastening, Phys. Rev. **D52**, 7232 (1995).

Breakdown of perturbation theory (Linde 79)



Contribution to free energy

$$g^{2(n-1)}\left(T\int \mathrm{d}^3k\right)^n \frac{k^{2(n-1)}}{(k^2+\mu^2)^{3(n-1)}}$$

$$n=4 \qquad \sim g^6T^4\ln(T/\mu)$$

$$n>4 \qquad \sim g^6T^4\left(g^2T/\mu\right)^{n-4}$$

[n loop, $2(n-1)$ 3-gluon vertices, $3(n-1)$ propagators]

If $\mu\sim g^2T$, *all* the diagrams with $n\geq 4$ loops contribute to the same order $\mathcal{O}(g^6)$.

Quantum ChromoDynamics

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \sum_f \bar{\psi}_f(i\not{D} - m_f)\psi_f$$

$$D_\mu = \partial_\mu - igA_\mu$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$$

Non linear effects become important when

$$\langle(\partial A)^2\rangle \sim g^2\langle A^4\rangle \sim g^2\langle A^2\rangle^2$$

Long wavelength thermal fluctuations

$$\langle A^2\rangle_\kappa \approx \int^\kappa \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{k}\frac{1}{\mathrm{e}^{k/T}-1} \approx \int^\kappa \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{T}{k^2} \approx \kappa T$$

Breakdown of thermal perturbation theory

$$\langle(\partial A)^2\rangle \sim g^2\langle A^2\rangle^2$$

$$\kappa^2 \sim g^2\langle A^2\rangle$$

$$\kappa \approx g^2 T$$

$$\langle F_{\mu\nu}^2\rangle \sim \langle(\partial A)^2\rangle \sim g^6 T^4$$

SCALES, DEGREES OF FREEDOM and FLUCTUATIONS

$$\langle A^2 \rangle \approx \int \frac{d^3k}{(2\pi)^3} \frac{N_k}{\varepsilon_k} \qquad N_k = \frac{1}{e^{\varepsilon_k/T} - 1} \qquad D_j = \partial_j - igA_j$$

- **Hard degrees of freedom: the plasma particles**

$$k \sim T \qquad \langle A^2 \rangle_T \sim T^2 \qquad \langle (\partial A)^2 \rangle_T \sim T^4 \qquad g^2 \langle A^2 \rangle_T^2 \sim g^2 T^4$$

- **Soft degrees of freedom, collective modes**

$$k \sim gT \qquad \langle A^2 \rangle_{gT} \sim gT^2 \qquad \langle (\partial A)^2 \rangle_{gT} \sim g^3 T^4 \qquad g^2 \langle A^2 \rangle_{gT}^2 \sim g^4 T^4$$

- **Ultrasoft degrees of freedom, unscreened magnetic fluctuations**

$$k \sim g^2 T \qquad \langle A^2 \rangle_{g^2T} \sim g^2 T^2 \qquad \langle (\partial A)^2 \rangle_{g^2T} \sim g^6 T^4 \qquad g^2 \langle A^2 \rangle_{g^2T}^2 \sim g^6 T^4$$

Effective theories

Classical field approximation
Dimensional reduction

Classical field approximation (1)

In the high temperature limit, $\beta \to 0$, and

$$Z \approx \mathcal{N} \int \mathcal{D}(\phi) \exp\left\{-\beta \int \mathrm{d}^3x\, \mathcal{H}(\mathbf{x})\right\},$$

$$\mathcal{H} = \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + V(\phi)$$

Equivalently, zero Matsubara frequency

$$G_0(\mathbf{k}) = \frac{T}{\varepsilon_k^2}, \qquad N(\varepsilon_k) = \frac{1}{\mathrm{e}^{\beta\varepsilon_k} - 1} \approx \frac{T}{\varepsilon_k}.$$

Valid only for $\varepsilon_k \ll T$, implying $N(\varepsilon_k) \gg 1$.

$$\boxed{k_{soft} \lesssim \Lambda \lesssim k_{hard}}$$

Classical field approximation (2)

(Dimensional reduction)

Classical field approximation = leading term in a systematic expansion

$$\phi(\tau) = \frac{1}{\beta}\sum_{\nu} \mathrm{e}^{-i\omega_\nu \tau}\phi(i\omega_\nu), \qquad \phi_0 \equiv \phi(\omega_\nu = 0)$$

Effective action for the "zero mode": $S[\phi_0]$

$$Z = \mathcal{N}_1 \int \mathcal{D}(\phi_0) \exp\left\{-S[\phi_0]\right\}, \qquad \phi_0 \equiv \phi(\omega_\nu = 0)$$

where

$$\exp\left\{-S[\phi_0]\right\} = \mathcal{N}_2 \int \mathcal{D}(\phi_{\nu\neq 0}) \exp\left\{-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\, \mathcal{L}_E(x)\right\}$$

Dimensional reduction

Integration over the hard modes $(gT \square\square \square T)$

$$p(T) = p_T(T) + \frac{T}{V} \ln \left[\int \mathcal{D}A_i^a \, \mathcal{D}A_0^a \, \exp\left(-S_E\right) \right]$$

$$\mathcal{L}_E \;=\; \frac{1}{2} \mathrm{Tr}\, \mathcal{F}_{ij}^2 + \mathrm{Tr}\left[D_i, \mathcal{A}_0\right]^2 + m_E^2 \,\mathrm{Tr}\, \mathcal{A}_0^2 + \lambda_E^{(1)} \left(\mathrm{Tr}\, \mathcal{A}_0^2\right)^2 + \lambda_E^{(2)} \,\mathrm{Tr}\, \mathcal{A}_0^4 + \ldots$$

$$D_i = \partial_i - i g_E A_i \qquad g_E \sim g\sqrt{T} \qquad m_E \sim gT \qquad \lambda_E \sim g^4 T$$

Integration over the soft modes $(g^2T \square\square \square gT)$

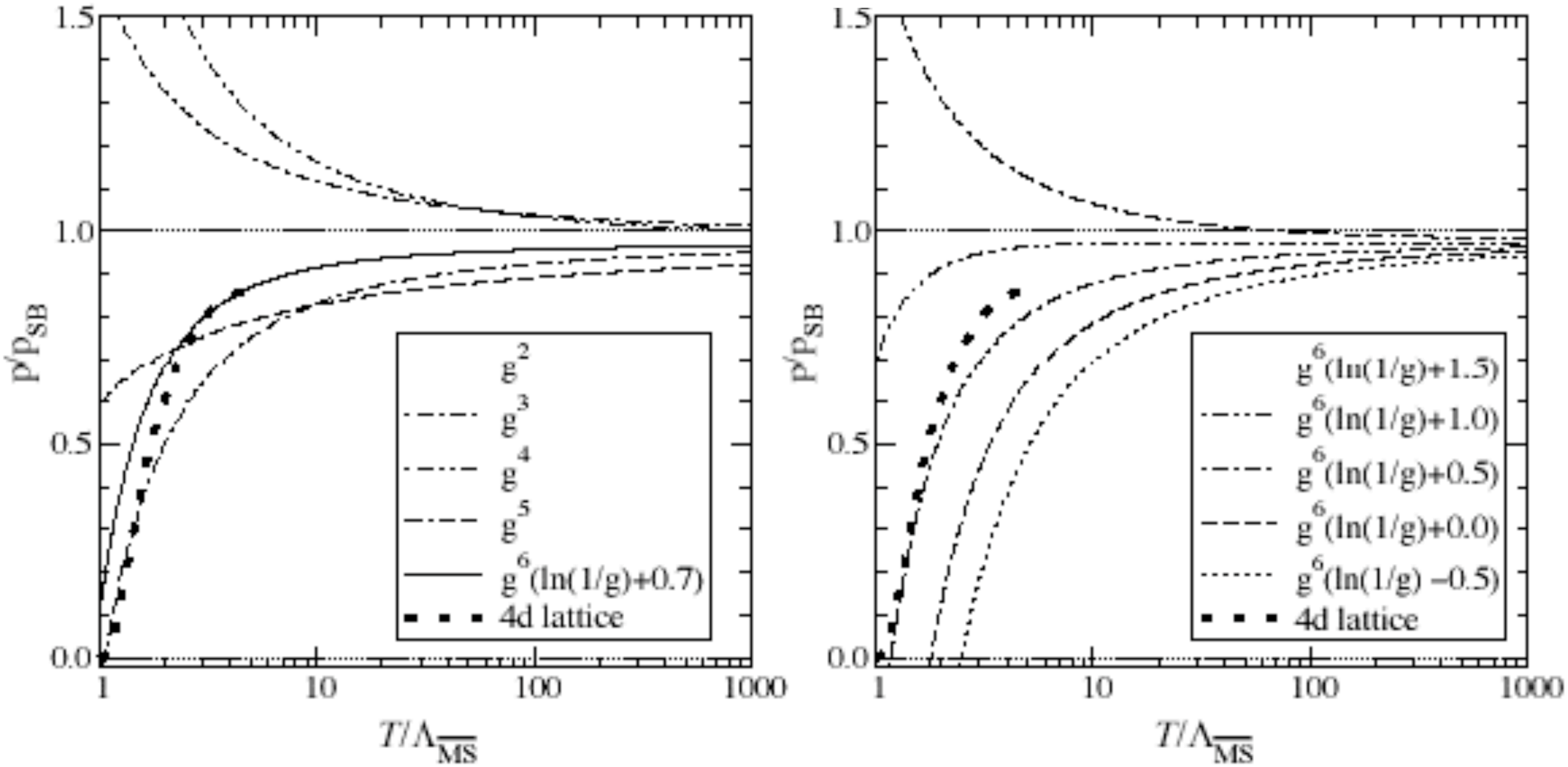
$$\frac{T}{V} \ln \left[\int \mathcal{D}A_i^a \, \mathcal{D}A_0^a \, \exp\left(-S_E\right) \right] = p_E(T) + \frac{T}{V} \ln \left[\int \mathcal{D}A_i^a \, \exp\left(-S_M\right) \right]$$

$$\mathcal{L}_M = \frac{1}{2} \mathrm{Tr}\, \mathcal{F}_{ij}^2 + \ldots$$

E. Braaten and A. Nieto, *Phys. Rev.* D 51 (1995) 6990 , *Phys. Rev.* D 53 (1996) 3421

$$p = T^4 \left[c_0 + c_2\, g^2 + c_3\, g^3 + (c_4' \ln g + c_4)\; g^4 + c_5\, g^5 + c_6\, g^6 \right]$$

$$c_6 = N_c^3 \, \frac{N_c^2 - 1}{(4\pi^4)} \left[\left(\frac{215}{12} - \frac{805}{768}\,\pi^2 \right) \ln \frac{1}{g} + 8\,\delta \right]$$



K. Kajantie, M. Laine, K. Rummukainen, and Y. Schröder, *Phys. Rev. Lett.* 86 (2001) 10 , *Phys. Rev.* D 65 (2002) 045008 , *Phys. Rev.* D 67 (2003) 105008 , *JHEP* 0304 (2003) 036

Effective theories

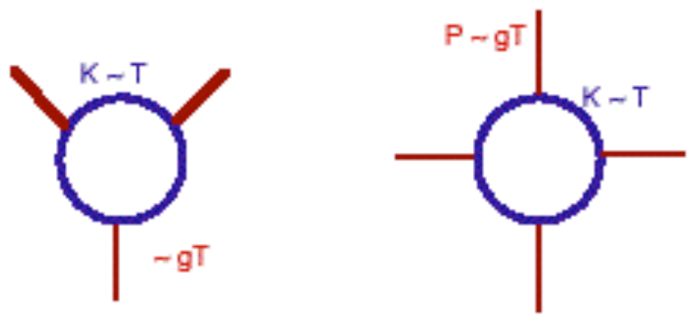
Collective phenomena
Hard thermal loops

HARD THERMAL LOOPS (1)

[Braaten and Pisarski (90), Frenkel and Taylor (90)]

- **Large thermal contributions to the dynamics of the soft fields.**

$$\Pi(P) \sim g^2T^2 \sim P^2 \qquad \text{for} \qquad P \sim gT$$



$$g^3\frac{T^2}{P} \sim gP \qquad\qquad g^4\frac{T^2}{P^2} \sim g^2$$

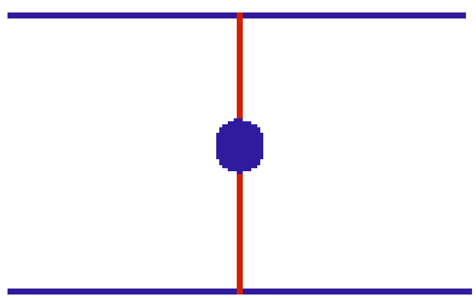
HARD THERMAL LOOPS (2)

- **Debye screening:**

$$\Pi_{el}(\omega \ll p) \simeq m_D^2 \Longrightarrow D_{el}(\omega \ll p) \simeq \frac{1}{p^2 + m_D^2}$$

- **Dynamical , screening:**

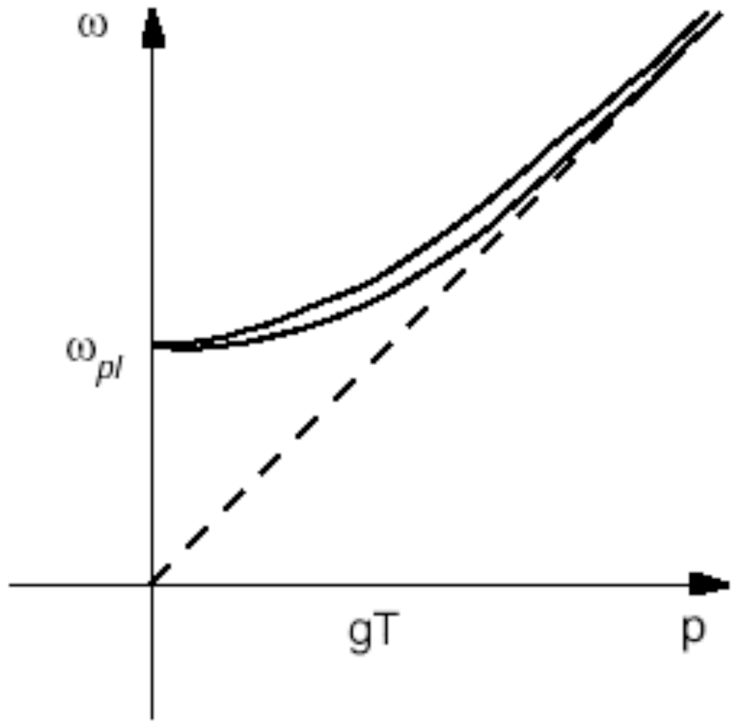
$$D_{mag}(\omega \ll p) \simeq \frac{1}{p^2 - i\frac{\omega}{p}\, m_D^2}$$



- Applications: Transport coefficients, quasiparticle damping rates, baryon number violation at high T, colour superconductivity, etc.

- Quasiparticle poles

Gluonic collective modes

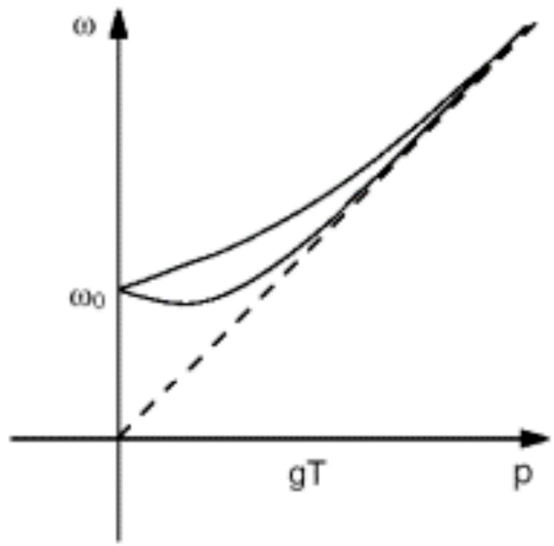


Dispersion relations for the modes $\omega_L(p)$ and $\omega_T(p)$

$$p^2 + \Pi_L(\omega_L, p) = 0, \qquad \omega_T^2 = p^2 + \Pi_T(\omega_T, p), \qquad \omega_{pl} \equiv m_D/\sqrt{3}$$

"asymptotic mass": $$m_\infty^2 \equiv \Pi_T^{1-loop}(\omega^2 = p^2) = \frac{m_D^2}{2}$$

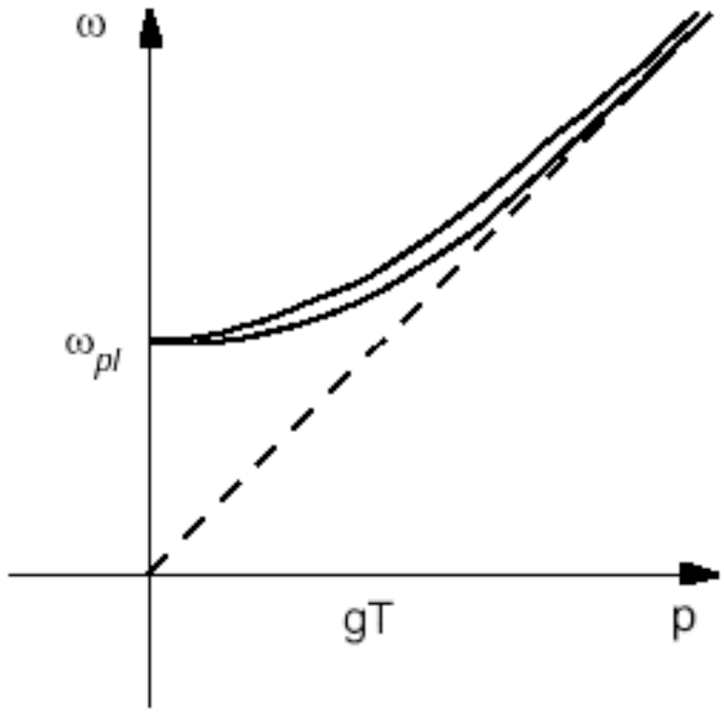
Collective fermionic excitations



$$p \gg \omega_0 \qquad \omega_+^2(p) \simeq p^2 + M_\infty^2 \,, \qquad M_\infty^2 \equiv 2\omega_0^2$$

$$p \ll \omega_0 \qquad \omega_+(p) \simeq \omega_0 + \frac{p}{3} + \cdots, \qquad \omega_-(p) \simeq \omega_0 - \frac{p}{3} + \cdots,$$

COLLECTIVE MODES AND THERMODYNAMICS



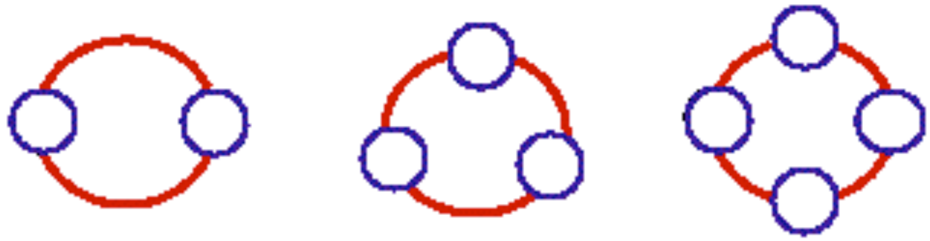
• **Question: how can one include this information on the modes in the calculations of thermodynamical quantities?**

Resummations

RESUMMATIONS (1)

- **Plasma particles,** $k \sim T$ **("hard")**

- **Collective excitations,** $k \sim gT$ **("soft")**



- **Hard thermal loop** $\Pi(\omega, q)$

RESUMMATIONS (2)

- **Resumed propagator** $D(\omega, q)$

$$D_0 = \frac{1}{\omega^2 - p^2} \longrightarrow D = \frac{1}{\omega^2 - p^2 - \Pi(\omega, p)}, \quad p^2 \sim \Pi$$

- **Corrections to hard particles due to their coupling to soft modes**

"SCREENED PERTURBATION THEORY

- **Scalar field** [Karsch, Patkos, Petreczky, PLB401]

$$\mathcal{L} = \mathcal{L}_0 - \frac{1}{2}m^2\phi^2 + \frac{1}{2}m^2\phi^2 + \mathcal{L}_{int}$$
$$= \mathcal{L}_0' + \mathcal{L}_{int}'$$

Convergence is improved.

But artificial UV pbs and scheme dependence.

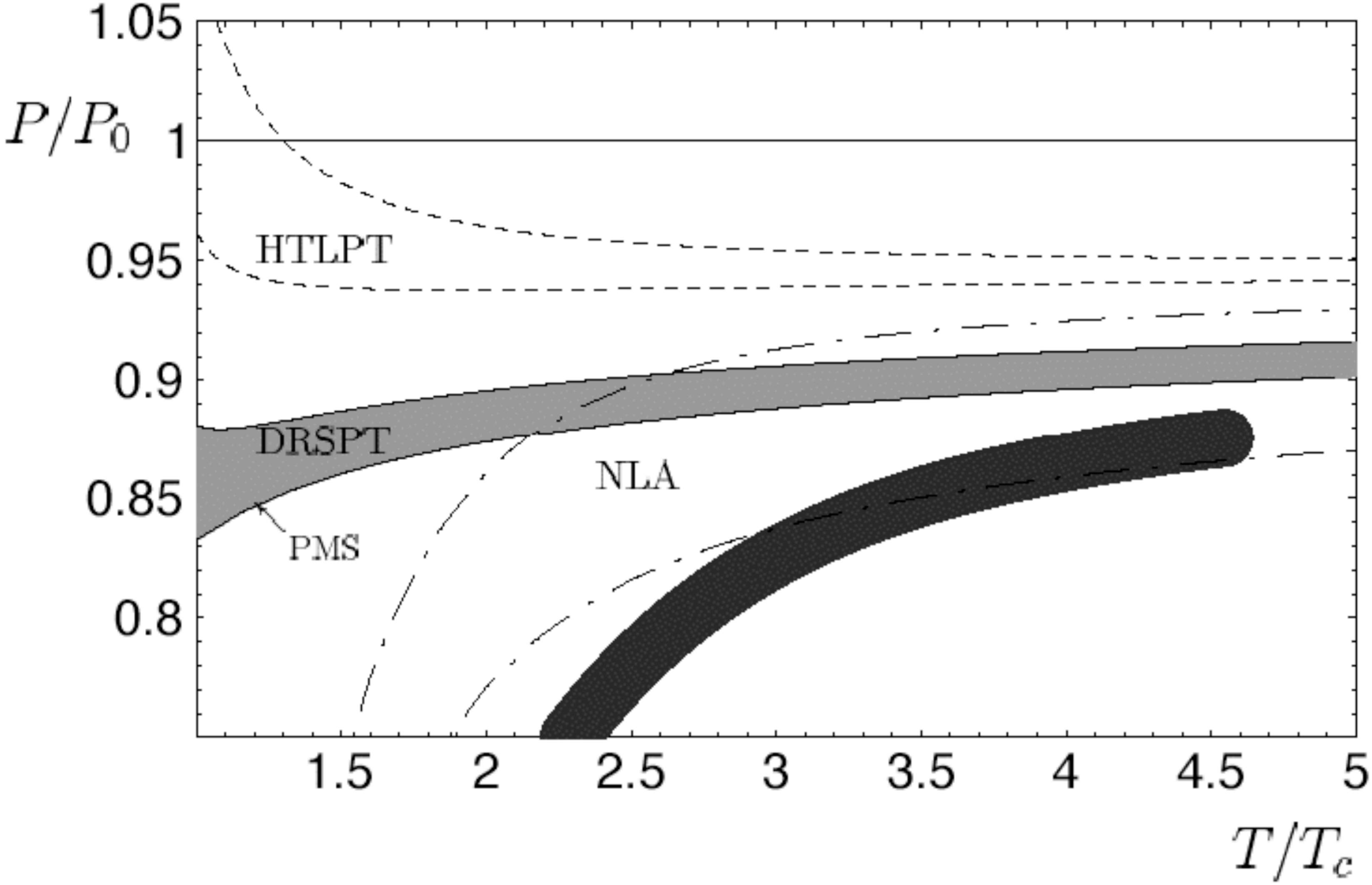
- **QCD: HTL perturbation theory**

[Andersen, Braaten, Strickland]

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{HTL} - \mathcal{L}_{HTL} + \mathcal{L}_{int}$$
$$= \mathcal{L}_0' + \mathcal{L}_{int}'$$

$\mathcal{L}_{HTL}$ not accurate for $\omega, p \gtrsim T$

Artificial T-dependent UV divergences



(From hep-ph/0303045)

Skeleton expansion and the entropy

• Based on work by J.-P. Blaizot, **E. Iancu** and **A. Rebhan**, Phys.Rev.Lett. **83**(1999)2906; Phys.Lett.**B470**(1999)181; hep-ph/0005003.

See also hep-ph/0303185

SKELETON EXPANSION

[Luttinger and Ward (60)]

- Expression of the free energy in terms of the **full propagator** D

$$\mathcal{F}[D] = \frac{1}{2}\mathrm{Tr}\ln D^{-1} - \frac{1}{2}\mathrm{Tr}\Pi D + \Phi[D],$$

Example of the scalar field

- **Stationarity property**

$$\Pi[D] \, = \, 2\frac{\delta\Phi}{\delta D} \Longrightarrow \frac{\delta\mathcal{F}[D]}{\delta D} \, = \, 0.$$

- **Self-consistent approximations** [Baym (62)]

Select some particular skeletons in Φ and solve:

$$D^{-1} = D_0^{-1} + \Pi[D]$$

(self-consistent Dyson equation, or «gap equation»)

- **The entropy $\mathcal{S}[D]$:**

$$\mathcal{S} = -\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}T} = -\left.\frac{\partial \mathcal{F}}{\partial T}\right|_D$$

Entropy is simple!

Simple model with a scalar field

- Use the simple two-loop diagram for Φ.
- Ansatz for the spectral function:

$$\rho(k_0, k) = 2\pi\epsilon(k_0)\delta(k_0^2 - k^2 - m^2)$$

- Net results for the pressure (after renormalization):

$$P = -T \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \ln\left(1 - \mathrm{e}^{-\beta\varepsilon_k}\right) + \frac{m^2}{2} I_T(m) + \frac{m^4}{128\pi^2},$$

where $\epsilon_k = (k^2 + m^2)^{1/2}$ and

$$I_T(m) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{n_k}{\varepsilon_k}.$$

- **The entropy is simpler**. Formally it is that of **non interacting massive particles**:

$$S = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left\{(1 + n_k)\ln(1 + n_k) - n_k \ln n_k\right\},$$

with $n_k = 1/(\mathrm{e}^{\varepsilon_k/T} - 1)$.

THE 2-LOOP ENTROPY

$$\boxed{S = -\int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\partial N}{\partial T} \left\{ \mathrm{Im} \ln D^{-1} - \mathrm{Im}\Pi \, \mathrm{Re}D \right\}}$$

- Effectively **one-loop** expression
- Manifestly ultraviolet-finite
- Perturbatively correct up to order g^3

- Why entropy ?
  - $\mathcal{S}$ is most directly related to the quasiparticle spectrum
  - Residual interactions start contributing order 3-loop
- Reconstructing the pressure $\mathcal{P} = -\mathcal{F}$:

$$\mathcal{P}(T) = \int_{T_0}^{T} \mathrm{d}T' \, \mathcal{S}(T') + \mathcal{P}(T_0)$$

with $\mathcal{P}(T_0)$ taken from the lattice data.

Approximately Self-Consistent Entropy (1)

By itself, the self-consistent truncation is not a gauge invariant approximation.

- Replace self-consistency by gauge-invariant approximations to Π, correct up to order g^3
- Compute the entropy exactly with these approximations for Π

$\omega, p \sim gT : \Pi_{soft} \approx \Pi_{HTL}$

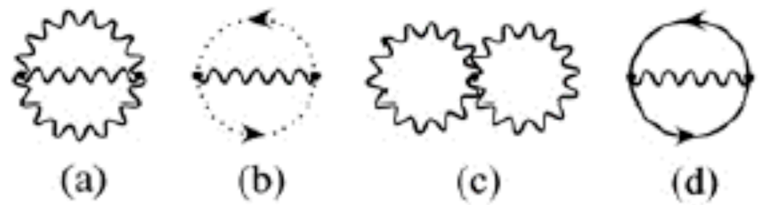
$\omega, p \sim T : \Pi_{hard}(\omega^2 \sim p^2)$

Remarks

- Up to order g^3, the self-energy of the hard particles is needed only on the light-cone: $\mathrm{Re}\,\Pi_{hard}(\omega^2 = p^2) \quad \rightarrow$ "thermal masses" for the plasma particles $\omega = p \longrightarrow \omega = \sqrt{p^2 + m_\infty^2}, \;\; m_\infty \sim gT$
- All these approximations <u>are</u> gauge-invariant!

QCD

- **2-loop skeletons**



- **Approximate self-consistency** In "leading order" $\Pi \sim \Pi_{HTL}$ and $\Sigma \sim \Sigma_{HTL}$
- **Corrections to hard particles**

Approximately Self-Consistent Entropy

- **The HTL, or leading, approximation:**

$\Pi = \Pi_{HTL}$ at all momenta $\quad \Longrightarrow \quad \boxed{\mathcal{S} = \mathcal{S}_{HTL}}$

Perturbative content: $\mathcal{O}(g^2) \ + \ \frac{1}{4}\,\mathcal{O}(g^3)$

- **The next-to-leading approximation:**

$\Pi_{soft} = \Pi_{HTL}, \quad \Pi_{hard} = \Pi_{HTL} + \delta\Pi \quad \Longrightarrow \quad \boxed{\mathcal{S} = \mathcal{S}_{NLA}}$

$$\delta\Pi(\omega = p) \sim \mathcal{O}(g^3 T^2)$$

Perturbative content: $\mathcal{O}(g^2) \ + \ \mathcal{O}(g^3)$

The entropy for SU(3) Yang-Mills

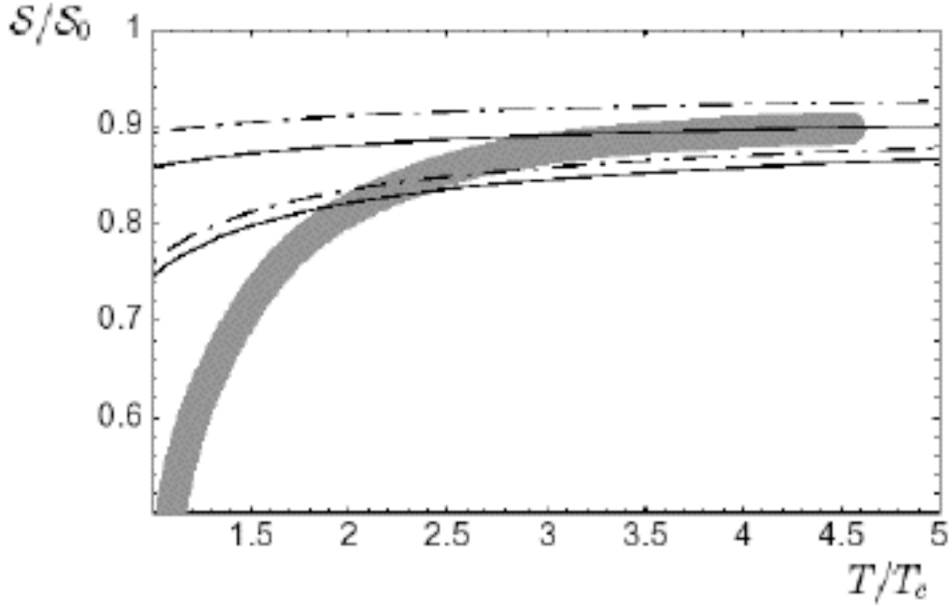


Figure 1: The entropy of pure SU(3) gauge theory normalized to the ideal gas entropy $\mathcal{S}_0$.

Full lines: $\mathcal{S}_{HTL}$. Dashed-dotted lines: $\mathcal{S}_{NLA}$.

2-loop β-function $\rightarrow$ the running coupling constant $\alpha_s(\bar{\mu})$.

The $\overline{\mathrm{MS}}$ renormalisation scale: $\bar{\mu} = \pi T \cdots 4\pi T$.

The dark grey band: lattice result by Boyd et al (1996).

Quark-Gluon Plasma with two massless flavours

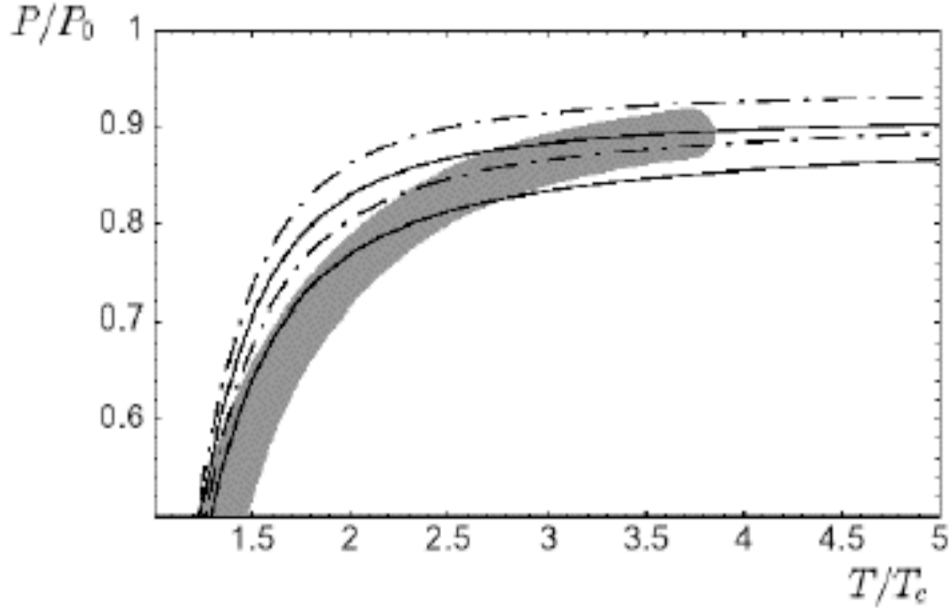


Figure 2: The pressure for a quark-gluon plasma with $N_f = 2$ versus the lattice results by Karsch et al. (2000).
Full lines: $\mathcal{P}_{HTL}$. Dashed-dotted lines: $\mathcal{P}_{NLA}$.

Summary

Accuracy of perturbation theory depends on momentum scale (fluctuations)

Weak coupling calculations provide a coherent picture of the quark-gluon plasma in terms of quasiparticles for $T \geq 3T_c$

THE ENTROPY
of the QUARK-GLUON PLASMA

"First principle" calculation of the **ENTROPY** of the Quark-Gluon Plasma using analytical techniques

- Based on work by J.-P. Blaizot, **E. Iancu** and **A. Rebhan**, Phys.Rev.Lett. **83**(1999)2906; Phys.Lett.**B470**(1999)181; hep-ph/0005003.

- Related work: J.O.Andersen, E. Braaten and M. Strickland, Phys.Rev.Lett. **83**(1999)2139; Phys.Rev**D61**(2000)014017,174016.