The Wroblewski parameter from lattice QCD

Workshop on Field Theories Near Equilibrium
Rajiv V. Gavai
The Wroblewski parameter from lattice QCD

Introduction

$\lambda_s$ from Quark Number Susceptibility

Pressure for small baryon density

Summary
Introduction

- Quark-Gluon Plasma in Heavy Ion Collisions.

- Reliable signals needed to establish it.

- Enhancement of strangeness production as a promising signal of QGP (Rafelski-Müller, Phys. Rev. Lett ’82, Phys. Rept ’86...).

- A variety of aspects studied and many different variations proposed.

- Most signal considerations based on Simple Models.
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**Strangeness Enhancement**

- **Key Idea:** $T_{QGP} \gg T_c \approx m_s \approx 150$ MeV

- **Energy Threshold**
  
  \[
  \begin{align*}
  q + \bar{q} & \rightarrow s + \bar{s} \\
  g + g & \rightarrow s + \bar{s} \\
  \pi + N & \rightarrow \Lambda + K \\
  K + \pi & \rightarrow \bar{\Lambda} + N
  \end{align*}
  \]
  
  $E_{\text{thres}} \approx 2m_s \approx 300$ MeV
  $E_{\text{thres}} \approx 530$ MeV
  $E_{\text{thres}} \approx 1420$ MeV

- **Production Rate**
  
  $\sigma_{QGP}(s\bar{s}) > \sigma_{HG}(s\bar{s})$

- **Pauli Blocking**
  
  \[
  \begin{array}{c}
  \mu \\
  T = 0 \\
  m_s
  \end{array}
  \]

  Expect an enhancement especially for multi-strange anti-baryons.

- **Measure:**
  
  $\Lambda = (uds) \rightarrow p\pi^- \quad 64\%$
  
  $\Xi^- = (dss) \rightarrow \Lambda\pi^- \quad 100\%$
  
  $\Omega^- = (sss) \rightarrow \Lambda K^- \quad 68\%$

  and their anti-particles.
Ratio of newly created strange quarks to light quarks:

\[ \lambda_s = \frac{2\langle s\bar{s}\rangle}{\langle u\bar{u} + d\bar{d}\rangle} \quad (1) \]
Wroblewski Parameter

- Hadron gas fireball model (Becattini-Heinz '97).

- 3 Free parameters: $T$, $V$, and $N_{ss}$.

- Fit many hadron abundances.

- Obtain $\lambda_s$ from data.

- Find $\lambda_s \sim 0.4 \ (0.2)$ for $AA \ (pp)$. 


\[ pp \ \sqrt{s} = 27.4 \text{ GeV} \]
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We have argued that

\[ \lambda_s = \frac{2\chi_s}{\chi_u + \chi_d}. \]  

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(Gavai & Gupta, PR D '02)
Quark Number Susceptibility

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♠ Theoretical Checks: Resummed Perturbation expansions, Dimensional Reduction..
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Theoretical Checks: Resummed Perturbation expansions, Dimensional Reduction..

Our improvement: Fixed $m_q/T_c$, Continuum limit...
Assuming three flavours, \( u, d, \) and \( s \) quarks, and denoting by \( \mu_f \) the corresponding chemical potentials, the QCD partition function is
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Higher order susceptibilities are defined by

$$\chi_{fg\ldots} = \frac{T}{V} \frac{\partial^n \log Z}{\partial \mu_f \partial \mu_g \ldots} = \frac{\partial^n P}{\partial \mu_f \partial \mu_g \ldots}.$$  \hspace{1cm} (4)
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\begin{align*}
\eta_i &= \frac{T}{V} \frac{\partial \ln Z}{\partial \mu_i}, \\
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\end{align*}

Higher order susceptibilities are defined by

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(4)

These are Taylor coefficients of the pressure $P$ in its expansion in $\mu$. 

All of these can be written as traces of products of $M^{-1}$ and various derivatives of $M$. 
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\[
\chi_0 = \frac{T}{2V} [\langle O_2(m_u) + \frac{1}{2} O_{11}(m_u) \rangle] \quad (5)
\]

\[
\chi_3 = \frac{T}{2V} \langle O_2(m_u) \rangle \quad (6)
\]

\[
\chi_s = \frac{T}{4V} [\langle O_2(m_s) + \frac{1}{4} O_{11}(m_s) \rangle] \quad (7)
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Here $O_2 = \text{Tr} \ M_u^{-1} M_u'' - \text{Tr} \ M_u^{-1} M'_u M_u^{-1} M'_u$, and $O_{11}(m_u) = (\text{Tr} \ M_u^{-1} M'_u)^2$, and the traces are estimated by a stochastic method:

$$
\text{Tr} \ A = \sum_{i=1}^{N_v} R_i^\dagger A R_i / 2N_v 
$$

$$
(\text{Tr} \ A)^2 = 2 \sum_{i>j}^L (\text{Tr} \ A)_i (\text{Tr} \ A)_j / L(L-1) 
$$

where $R_i$ is a complex vector from a set of $N_v$ subdivided in $L$ independent sets.
$\chi_{FFT}$ — Ideal gas results for same Lattice.
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![Graph showing data points and curves for $\chi/\chi_{FFT}$ vs. $T/T_c$ for a 312 x 4 Lattice. The graph includes symbols for $N_f = 2; m_s/T_c = 0.1$ and $N_f = 0; Quenched$.](image)
$\chi_{FFT} \quad \text{Ideal gas results for same Lattice.}$

Note that PDG values for strange quark mass

\[ m_{v}^{\text{strange}} / T_c \]
\[ \approx 0.3-0.7 \ (N_f=0); \]
\[ 0.45-1.0 (N_f=2). \]
Perturbation Theory
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Weak coupling expansion gives:

\[
\frac{\chi}{\chi_{FFT}} = 1 - 2\left(\frac{\alpha_s}{\pi}\right) + 8\sqrt{1 + 0.167N_f}\left(\frac{\alpha_s}{\pi}\right)^{\frac{3}{2}}
\]

(Kapusta 1989).
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(Kapusta 1989).

♣ Minm 0.981 (0.986) at 0.03 (0.02) for \(N_f = 0\) (2).
♣ For \(1.5 \leq T/T_c \leq 3\) pert. theory \(\rightarrow 0.99-0.98\) (1.08=1.03) for \(N_f = 0\) (2).
Resummed Perturbation Theory
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Hard Thermal Loop & Self-consistent resummation give:

(Blaizot, Iancu & Rebhan, PLB ’01; Chakraborty, Mustafa & Thoma, EPJC ’02).
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\[ N_f = 0, \quad \bar{\mu} = \pi T \ldots 4\pi T \]

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Our results for $N_t = 4 \leadsto$ Lattice artifacts?
Check for larger $N_t$ and improved actions.
\[ Xud \]
Off-diagonal Susceptibility: $\chi_{ud} = \langle \frac{T}{V} \text{Tr} \, M_u^{-1} M_u' \text{Tr} \, M_d^{-1} M_d' \rangle$
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\( \heartsuit \) Identically zero for Ideal gas but \( O(\alpha_s^3) \) in P.T.

Using the same scale and \( \alpha_s \) as for \( \chi_3 \longrightarrow \chi_{ud} \sim O(10^{-4}) \)!!
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123 × 4 Lattice; Quenched.

$T = 0.75T_c$

Gavai, Gupta & Majumdar, PR D 2002
Taking Continuum Limit

(Gavai & Gupta, PR D '02 and PR D '03)
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♠ Investigate larger $N_t : 6, 8, 10, 12$ and 14.
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♠ Naik action : Improved by $O(a)$ compared to Staggered. Introduction of $\mu$ nontrivial but straightforward.

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Does improve the $N_t$-dependence of the free fermions.
Results at $2T_c$:

![Graph showing data points and trend lines]
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$N_t^{-2} \sim a^2$ extrapolation works and leads to same results within errors for both staggered and Naik fermions.
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$\chi^3/T^2$ vs $1/N_t^2$

- $N_t^{-2} \sim a^2$ extrapolation works and leads to same results within errors for both staggered and Naik fermions.

- Milder $N_t^{-2} \sim a^2$-dependence for Naik fermions.
The continuum susceptibility vs. $T$ therefore is:
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Naik action (Squares) and Staggered action (circles)
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♡ Note that $\chi_{ud}$ behaves the same way for ALL $N_t$ and both fermions, leading to the same $O(10^{-6})$ values in continuum too.
Wroblewski Parameter

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\[\begin{array}{cccccc}
T/T_c & 0.38 & 0.4 & 0.42 & 0.44 & 0.46 & 0.48 \\
\lambda_s & 0.8 & 0.85 & 0.9 & 0.95 & 1 & 1.05 \\
\end{array}\]

$N_t=4$, 2 flavour QCD, $\mu/T_c=0.1$, $m_s/T_c=1.0$
Caveats

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- Extrapolation to $T_c$ – Straightforward but better to do it for full QCD.
- Preliminary results for Full 2-flavour QCD (Gavai & Gupta):
  - Large finite volume effects below $T_c$
  - Up to $12^3$ Lattices used.
  - Strong dependence on $m_s$ expected.
  - Large finite $a$ effects.
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• Assumed: characteristic time scale of plasma are far from the energy scales of strange or light quark production.
  – Observation of spikes in photon production may falsify this.

• Assumed: Chemical equilibration in the plasma.
EoS for nonzero baryon density

Recall,
\[
\chi_{fg\cdots} = \frac{T}{V} \frac{\partial^n \log Z}{\partial \mu_f \partial \mu_g \cdots} = \frac{\partial^n P}{\partial \mu_f \partial \mu_g \cdots}.
\] (8)

Thus \(\chi_{uuuu}\) involves terms having fourth derivative w. r. to \(\mu\) while \(\chi_{uudd}\) only second derivatives.

In continuum, \(f(a\mu) = 1 + a\mu \rightarrow f'''(0) = 0\).

On lattice, in general, all derivatives exist and depend on the nature of function: prescription dependence!

Fodor-Katz used \(f_{HK}\) and got \(\mu_E = 725\) MeV for \(N_t = 4\). If they were to use \(f_{BG}\), then \(\mu_E = 692\) MeV.

Easy to show that \(f'''(0) = 1\) always but all higher derivatives depend on choice of
Thus, one can write

\[ \chi_{uuuu} = \chi_{uuuu}^{HK} + \Delta f^{(3)} \left( \frac{\chi_{uu}}{T^2} \right) \left( \frac{4}{N_t^2} \right), \]  

(9)

where \( \Delta f^{(3)} = f^{(3)} - 1 \) is 2 for \( f_{BG} \).

Prescription dependence must go away for small \( a \) or large enough \( N_t \).

How large an \( N_t \) needed? \( N_t \geq 10 \), see below.

Defining

\[ \frac{\mu_*}{T} = \sqrt{\frac{12\chi_{uu}/T^2}{|\chi_{uuuu}|}}, \]  

(10)

and \( \Delta P = P(\mu) - P(\mu = 0) \), the Taylor series expansion for Pressure \( P \) for 2 flavours can be re-organized as,

\[ \frac{\Delta P}{T^4} = \left( \frac{\chi_{uu}}{T^2} \right) \left( \frac{\mu}{T} \right)^2 \left[ 1 + \left( \frac{\mu/T}{\mu_*/T} \right)^2 + \mathcal{O} \left( \frac{\mu^4}{\mu_*^4} \right) \right]. \]  

(11)
Note that

- Each term in $\Delta P$ is prescription dependent, except the 1st. Physical $\Delta P$ may be best obtained by evaluating each in continuum limit, as we do below. More important for larger $\mu$. 
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- $\mu \ll \mu^*$ for prescription independence, provided still higher susceptibilities $\leq \chi_{uuuu}$.

- $(T_E, \mu_E)$ may be identified from the radius of convergence using many higher susceptibilities obtained in continuum limit term by term. What about series on finite lattice and estimate of $(T_E, \mu_E)$ as done presently?
Our Results

Our results for $\chi_{uuuu}$ and $\Delta P$:

Gavai and Gupta, PR D68, '03

\[ \frac{\chi_{uuuu}}{T/T_c} \]

\[ \frac{\Delta P}{T/T_c} \]

\[ N_t = 8 \]
\[ N_t = 10 \]
\[ N_t = 12 \]
\[ N_t = 14 \]

Continuum limit

Ideal Gas
Our Results

Our results for $\chi_{uuuu}$ and $\Delta P$: Gavai and Gupta, PR D68, ’03

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♥ Our results for $P$ agree with Fodor-Katz (PL B568, '03) and the recent Bielefeld results (PR D68, '03).
Defining $\mu^*_i$ to extend the definition of $\mu^*_2$ ($i^{th}$ term = $(i + 2)^{th}$ term), the Taylor series expansion for Pressure $\Delta P$ for 2 flavours can be re-organized as,

$$\frac{\Delta P}{T^4} = \left(\frac{\chi_{uu}}{T^2}\right) \left(\frac{\mu}{T}\right)^2 \left[1 + \left(\frac{\mu}{\mu^*_2}\right)^2 \left[1 + \left(\frac{\mu}{\mu^*_4}\right)^2 \left[1 + \ldots \right]\right]\right].$$  \hspace{1cm} (12)
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- Phase diagram in $T - \mu$ on small $N_t = 4$ has begun to emerge: Different methods, $\sim$ same $(T_E, \mu_E)$. Beware of prescription dependence and look forward to larger $N_t$. 