The electrical conductivity of the QCD plasma

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1. The setup: linear response theory and the need for a lattice

2. Lattice (Euclidean) correlators: Bayesian methods and an example

3. Functional Bayesian analysis: large ω and small ω

4. Parametrised Bayesian analysis: σ and phenomenology; Landau damping and lattice evidence for it.

Linear Response Theory

The response, $\mathbf{A}(t)$, of a system to a force $\mathbf{F}(t)$ if non-linear terms are neglected—

$$\mathbf{A}(t) = \int_{-\infty}^{\infty} dt' \chi(t-t') \mathbf{F}(t') \qquad \text{hence} \qquad \mathbf{A}(\omega) = \chi(\omega) \mathbf{F}(\omega).$$

Causality implies $\chi(t) = 0$ for $t < 0$. As a result $\chi(\omega)$ is regular in the upper half plane and dispersion relations follow. The spectral density is the imaginary part of $\chi(\omega)$ as ω approaches the real axis from above. A microscopic computation explicitly relates $\chi(\omega)$ to the retarded propagator. From this follow the Kubo formulæ relating the transport coefficient and the zero energy limit of the spectral density—

$$\chi \propto \lim_{\epsilon \to 0} \int d^3x' \int_{-\infty}^{t} dt'' \mathrm{e}^{\epsilon(t''-t)} \int_{-\infty}^{t''} dt' \langle \mathbf{A}(\mathbf{x}, t) \mathbf{A}(\mathbf{x}', t') \rangle.$$

J. Hilgevoord, *Dispersion Relations and Causal Description*, North-Holland, 1960

Electrical conductivity and photon emissivity

The differential photon emissivity is given by—

$$\omega \frac{d\Omega}{d^3p} = \frac{C_{EM}}{8\pi^3} n_B(\omega; T) \rho^\mu_\mu(\omega, \mathbf{p}; T) \qquad \text{where} \qquad C_{EM} = 4\pi\alpha \sum_f e_f^2 \approx \frac{1}{21}.$$

In terms of the DC electrical conductivity ($\mathbf{j} = \sigma \mathbf{E}$)

$$\sigma(T) = \frac{C_{EM}}{6} \; \frac{\partial}{\partial\omega} \rho^i_i(\omega, \mathbf{0}; T)\Big|_{\omega=0}, \qquad \frac{8\pi^3\omega}{C_{EM}T^2} \frac{d\Omega}{d^3p} = 6\,\frac{\sigma}{T}.$$

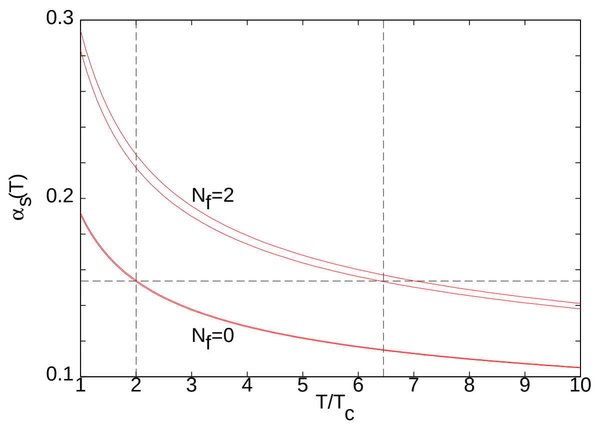
Since $k^\mu \rho_{\mu\nu} = 0$, we have $\rho_{00} = 0$ along the line $\mathbf{p} = 0$. Formally,

$$\rho_{00}(\omega, \mathbf{0}; T) = 2\pi \chi_Q \omega \delta(\omega),$$

where χ_Q is the charge susceptibility.

Lattice: Gauge theories at any coupling

For long distance physics the coupling g can become large. In QCD at experimentally accessible heavy-ion collider energies $g \geq 1$.



Effects— pressure cannot be computed quantitatively in perturbation theory, many quark number susceptibilities depart strongly from perturbative results.

Euclidean Correlators

In the Euclidean theory one constructs equilibrium correlation functions which are related to the spectral function by the relation—

$$G(\tau, \mathbf{p}; T) = \int_0^\infty \frac{\omega}{2\pi} K(\omega, \tau; T) \rho(\omega, \mathbf{p}; T).$$

In a lattice theory there are N_t points in the τ direction, but there is a continuous infinity of ω.

Replace integral by sum, the linear relation above becomes a set of linear equations: more variables than equations. Inverse of K is ill defined. Convert to a minimisation/Bayesian problem.

(Opposite of least-squares fit: more equations than unknowns)

Regularisation

Maximize the Bayesian probability—

$$P(\rho|G) \propto P(G|\rho)P(\rho) = \exp[-F(\rho)],$$

$$F(\rho) = (G - K\rho)^T \Sigma^{-1} (G - K\rho) + \beta U(\rho)$$

β is a regularisation parameter, and $U(\rho)$ is a function which we are free to choose. This encodes our prior knowledge of the system.

A. N. Tikhonov and V. Y. Arsenin, *Solutions of Ill-posed Problems*, Wiley, New York (1977)

1. Maximum Entropy Method: $U = \sum \rho \log(\rho/\rho_0) - \rho$. Y. Nakahara, M. Asakawa and T. Hatsuda, *Phys. Rev.*, D 60 (1999) 091503
2. Linear: $U = \rho^T L^T L \rho$, $L = 1$, D, D^2, *etc.*. S. Gupta, hep-lat/0301006
3. Include known information into the Bayesian probability. G. P. Lepage *et al.*, *Nucl. Phys.*, B (Proc. Suppl.) 106 (2002) 12.

An example (1/3)

Determine the parameters of the line $a + bx$ passing through (1,1)

Method 1: General linear regulator $U = \rho^T L^T L\rho$

$$F(a,b) = (1-a-b)^2 + \beta(l_{11}a^2 + l_{22}b^2 + 2l_{12}ab)$$

U is positive definite. The minimum occurs at

$$M\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{where} \quad M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \beta \begin{pmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \end{pmatrix}$$

$$\text{Most probable } \beta = 0: \quad \begin{pmatrix} a \\ b \end{pmatrix} = \overbrace{\frac{1}{1+x}\begin{pmatrix} x \\ 1 \end{pmatrix}}^{l_{11}\neq 0} \quad \text{or} \quad \overbrace{\frac{1}{1+x}\begin{pmatrix} 1 \\ x \end{pmatrix}}^{l_{22}\neq 0} \qquad (l_{12}=0).$$

For all L the best fit passes through the data

Example (2/3)

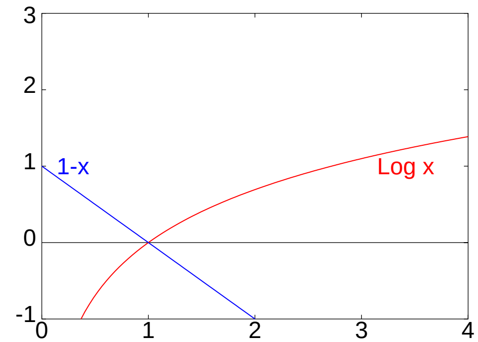
Method 2: MEM

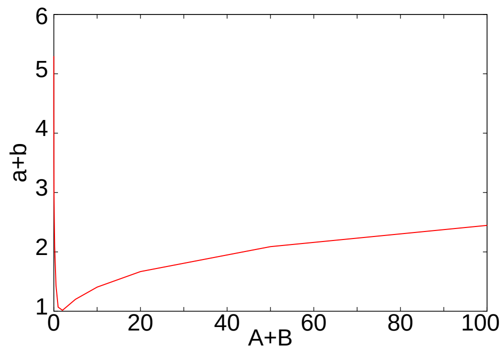
$$F(a,b) = (1-a-b)^2 + \beta \left(a \log \frac{a}{A} + b \log \frac{b}{B} - a - b \right)$$

The minimum is at

$$\frac{a}{A} = \frac{b}{B} = x \quad \text{where} \quad 1 - Sx = \frac{\beta}{2} \log x,$$

where $S = A + B$. Solutions exist only for $S > 0$. If $S < 1$ then $x > 1$ and vice versa. Also $F = 1 - S^2 x^2 - \beta S x$.





The best fit does not pass through the data except when $A + B = 1$

Example (3/3)

Method 3: Partial knowledge

Prior: the line passes through the origin. We can choose the Bayesian probability distribution to be a Gaussian of width σ around the origin—

$$F = (1 - a - b)^2 + \frac{a^2}{2\sigma} \quad \text{giving} \quad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If only information on b is needed then a can be integrated out of the Bayesian probability to give the marginal distribution

$$P(b)db \propto db \exp\left[-(1-b)^2\right].$$

In this case marginalisation gives b=1 which is the same result as minimisation. This is true if the distribution is unimodal.

Lattice gauge theory with functional Bayesian methods

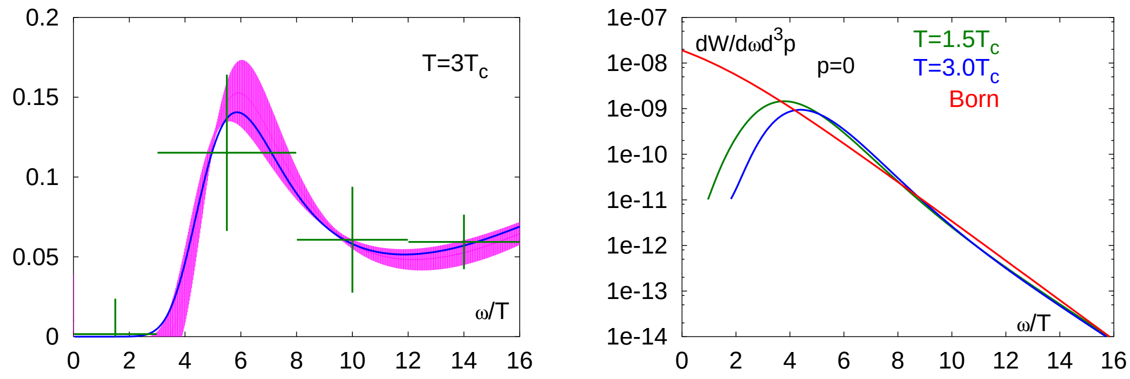
The lattice problem is to use—

$$G(\tau, \mathbf{p}; T) = \int_0^\infty \frac{\omega}{2\pi} K(\omega, \tau; T) \rho(\omega, \mathbf{p}; T),$$

and the data on G to extract ρ.

One has only limited data from the lattice. This can be treated in different ways to extract different kinds of physics. One method of analysis will not give us all the information needed. We must tune the method to the problem.

Large ω using MEM



F. Karsch et al, Phys.Lett.B530:147,2002

Full agreement with Born for $\omega/T \geq 4$.

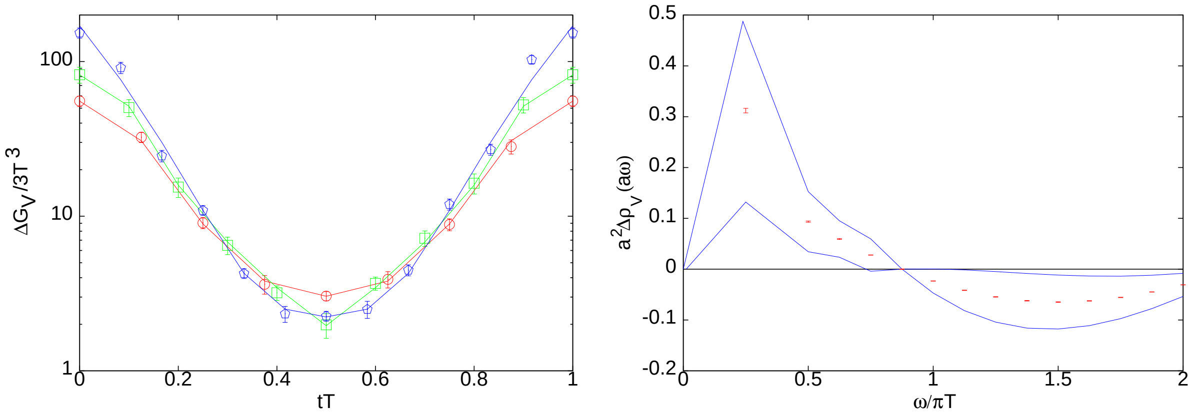
Default model: ideal gas behaviour. Output: $\rho(\omega)$ grows as ω^2 at large ω. Extracted value vanishes as $\omega \to 0$. Need to examine low ω region by another method in more detail.

Small ω using linear regulator

Since the problem is linear, work with

$$\Delta G(\omega, \mathbf{p}; T) = G_{full}(\omega, \mathbf{p}; T) - G_{ideal}(\omega, \mathbf{p}; T).$$

This gets rid of the ω^2 divergence at infinity, at the cost of the positivity of $\Delta\rho$. Use a linear regulator. This shows a bump at small ω.



S. Gupta, hep-lat/0301006

Lattice gauge theory with parametrised Bayesian methods

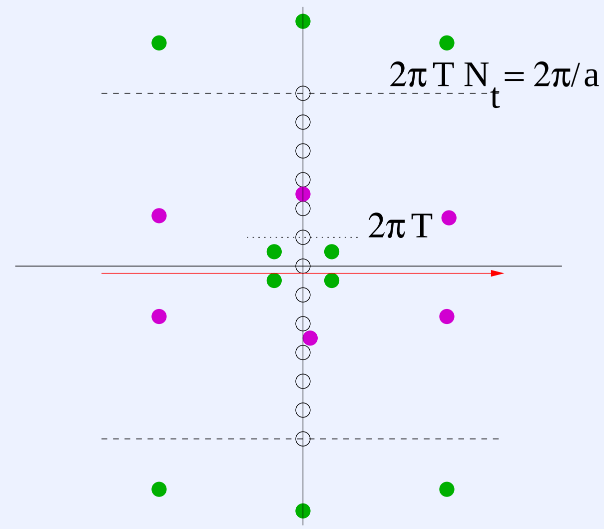
Use a sequence of parametrisations for the spectral density

$$\frac{\Delta\rho}{T^2} = \frac{z\sum_{n=0}^{N}\gamma_n z^{2n}}{1+\sum_{m=1}^{M}\delta_m z^{2m}}.$$

Use with Fourier space correlators—

$$\Delta G(\omega_n, \mathbf{p}; T) = \oint \frac{d\omega}{2i\pi}\,\frac{\Delta\rho(\omega, \mathbf{p}; T)}{\omega-\omega_n}$$

where $\omega_n = 2i\pi nT$.



Two types of poles: S_2 is relaxation time and S_4 is Landau damping.

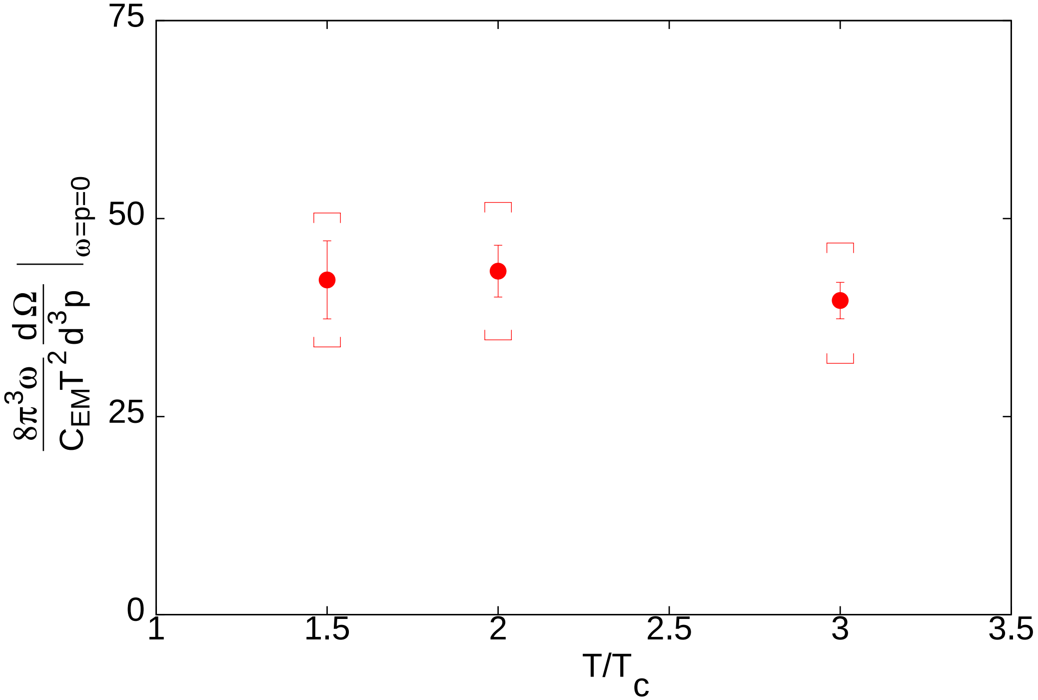
Use χ^2 parameter fitting if $N+M+1 \le N_t$, Bayesian otherwise.

F. Karsch and H. W. Wyld, *Phys. Rev.*, D 35 (1987) 2518; S. Sakai *et al.*, hep-lat/9810031

Electrical conductivity: continuum limit

Electrical conductivity depends only on the parameter γ. Obtain this by marginalising over the remaining parameters. S. Gupta, hep-lat/0301006

$$\frac{\sigma}{T} \approx 7C_{EM} \text{ for } 1.5 \le T/T_c \le 3$$



Dynamical scales and phenomenology

1. A photon emitted in the plasma is reabsorbed if its path length is

$$\ell = \frac{1}{\sigma} \approx \frac{1}{7C_{EM}T} \approx 3 \text{ fm}.$$

   Typical fireball dimensions at RHIC are a few fm, so the fireball is marginally transparent to photons. This may no longer be so at LHC.

2. Typical hadronic length/time scales in the plasma are

$$\tau \approx \frac{1}{7T} \approx 0.15 \text{ fm}.$$

   Hydrodynamic description of the final state in the plasma work if its thermalisation time is less than 1 fm. Hydrodynamics may work at both RHIC and LHC.

3. Spontaneous thermal fluctuations of flavour even out by diffusion. The diffusion constants, D, are related to the electrical conductivity as

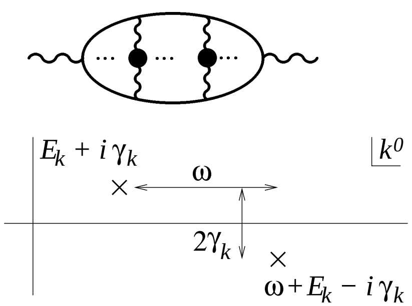
$$\sigma = \sum_f e_f^2 D_f \chi_f,$$

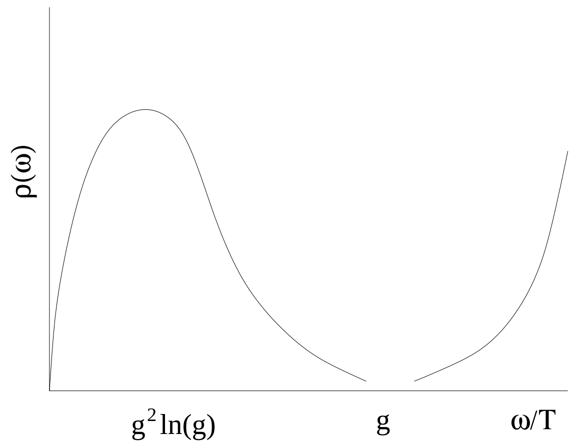
where χ_f is the thermal susceptibility for particle number. We find $D \approx \frac{1}{7T} \approx 0.15 \text{ fm}$, and hence the only visible chemical fluctuations are those at freeze out. Strong implications for strangeness production.
R.V. Gavai and S. Gupta, Phys.Rev.D65:094515,2002

4. Jets traversing the plasma are quenched. This calls for high shear viscosity, η. No reason, now, not to expect this. Try a computation of η in near future.

Pinch singularities and transport

There are pinch singularities at small external energy, ω, from ladder diagrams. These ladder diagrams correspond to multiple scatterings off particles in the plasma.





Transport: G. Aarts and J.M.M. Resco JHEP 0204:053,2002;

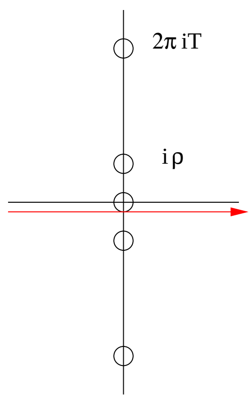
Gluon damping rate: E. Braaten and R. Pisarski, Phys.Rev.D42:2156-2160,1990

Parametric behaviour

$$\Delta G(\omega_n, \mathbf{p}; T) = \oint \frac{d\omega}{2i\pi} \frac{\Delta\rho(\omega, \mathbf{p}; T)}{\omega - \omega_n}$$

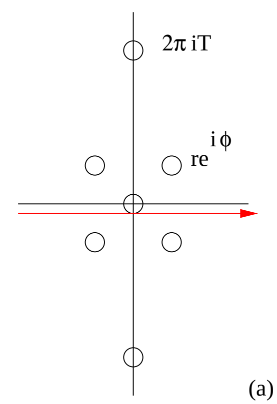
dominated by poles closest to the origin. If these are poles of $\Delta\rho$ then

Relaxation time



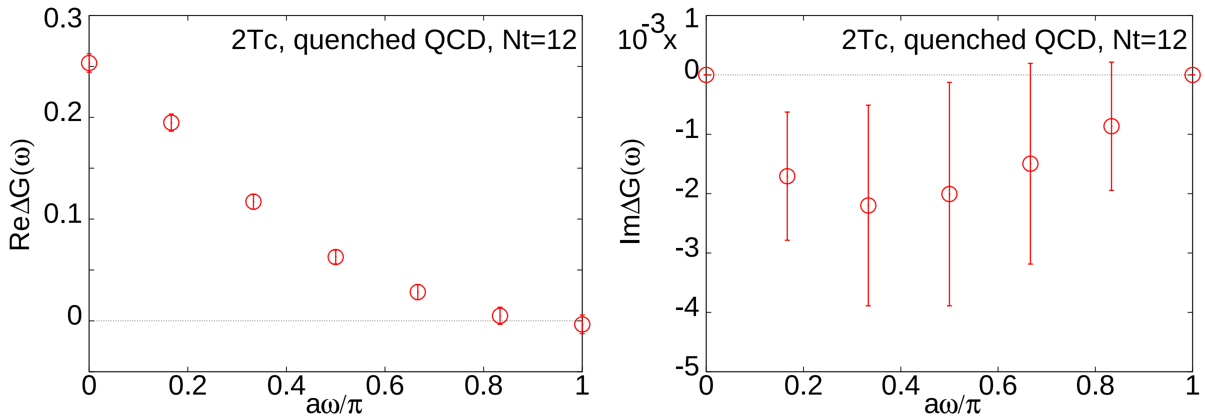
- sign $\Re G(\omega_n) \gg |\Im G(\omega_n)|$ ($\star$)
- sign $\Re G(\omega_n) =$ sign $\Im G(\omega_n)$ ($\star$)
- $\Re G(\omega_n) < 0$.

Landau damping



- sign $\Re G(\omega_n) \gg |\Im G(\omega_n)|$ ($\star$)
- sign $\Re G(\omega_n) \neq$ sign $\Im G(\omega_n)$ ($\star$)
- $\Re G(\omega_n) > 0$.

Lattice results



- Relaxation time approach ruled out. Landau damping possible.
- For $N_t = 12$ fits possible with $|n| \leq 3$.
- Smaller lattices give qualitative results but a fit is not possible.
- Landau poles at $\rho/2\pi T \approx 0.15$ and $\phi \approx 0.2$

Summary

1. Lattice computations needed for comparison with experiments, but it contains too little information to parametrize near-equilibrium physics completely. Prior assumptions necessary.

2. Analysis methods must try to identify or isolate important physical behaviour. MEM has done well in the large energy region. Linear Bayesian methods do well in the small energy region.

3. First computation of transport coefficient: the electrical conductivity. Small time scale for transport seen.

4. Parametrised forms check specific models of interactions. Relaxation time approximations ruled out. Evidence supports physics of Landau damping.