Initial conditions in heavy ion collisions

I – Gluon production by external sources

François Gelis - Tuomas Lappi

CERN and CEA/Saclay

General outline

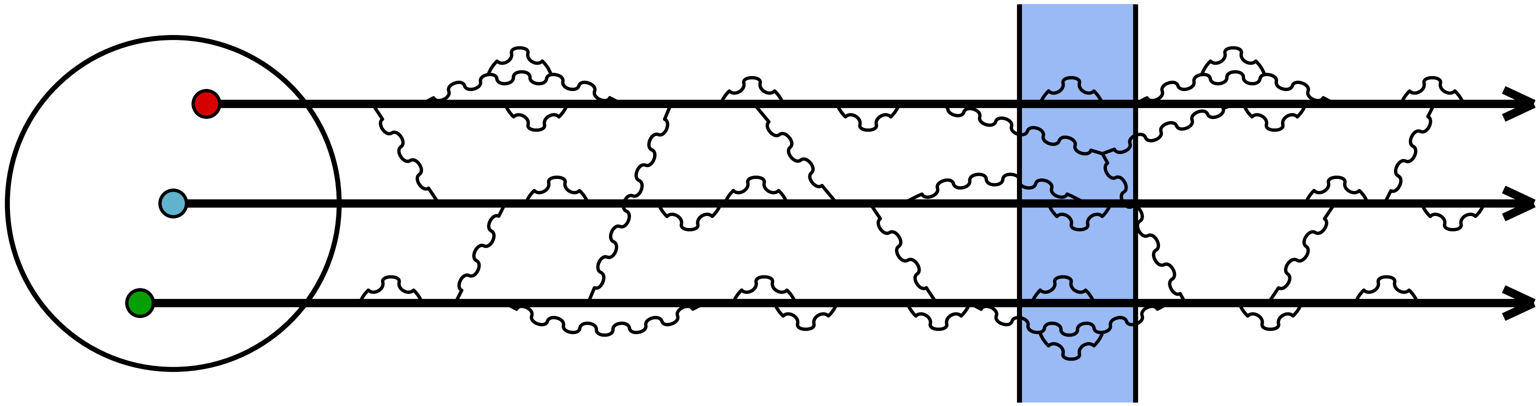
- Lecture I : Gluon production by external sources
- Lecture II : Leading Order description (T. Lappi)
- Lecture III : Next to Leading Order, Factorization (T. Lappi)
- Lecture IV : Final state evolution, Thermalization

Lecture I : Gluons from external sources

- Parton model
- Color Glass Condensate
- Power counting and bookkeeping
- Inclusive gluon spectrum
- Generating functional
- Surgery of retarded graphs

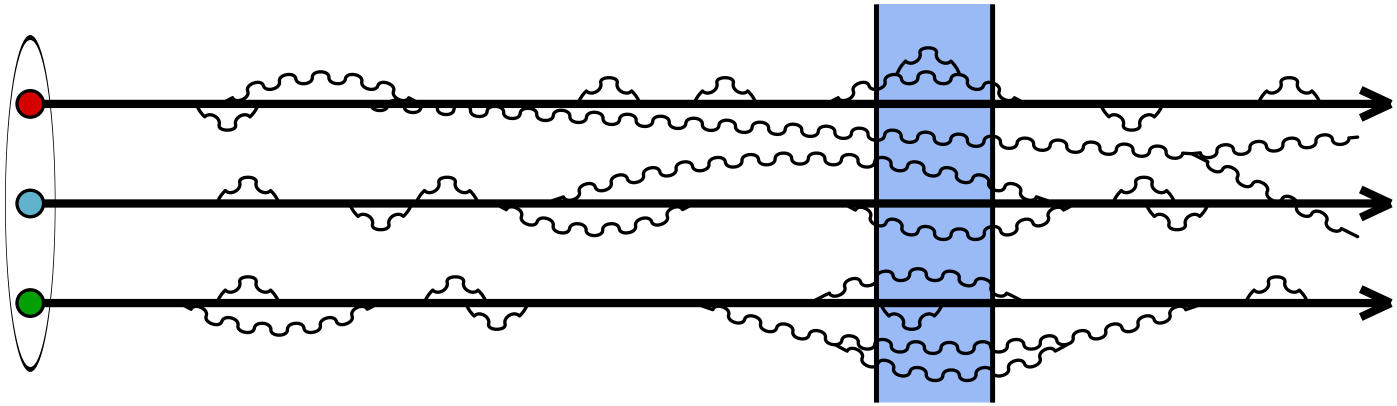
Parton model

Nucleon at rest



- A nucleon at rest is a very complicated object...
- Contains fluctuations at all space-time scales smaller than its own size
- Only the fluctuations that are longer lived than the external probe participate in the interaction process
- The only role of short lived fluctuations is to renormalize the masses and couplings
- Interactions are very complicated if the constituents of the nucleon have a non trivial dynamics over time-scales comparable to those of the probe

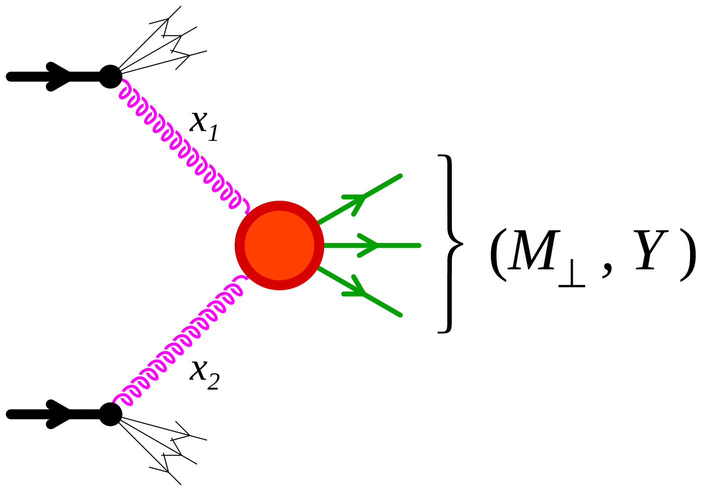
Nucleon at high energy



- Dilation of all internal time-scales for a high energy nucleon
- Interactions among constituents now take place over time-scales that are longer than the characteristic time-scale of the probe ▷ the constituents behave as if they were free
- Many fluctuations live long enough to be seen by the probe. The nucleon appears to contain more gluons at high energy
- Fast partons (fluctuations that were already visible before the boost) do not have any significant dynamics over the duration of the collision. They can be treated as static objects, that act as sources for the slower partons

Infrared and collinear divergences

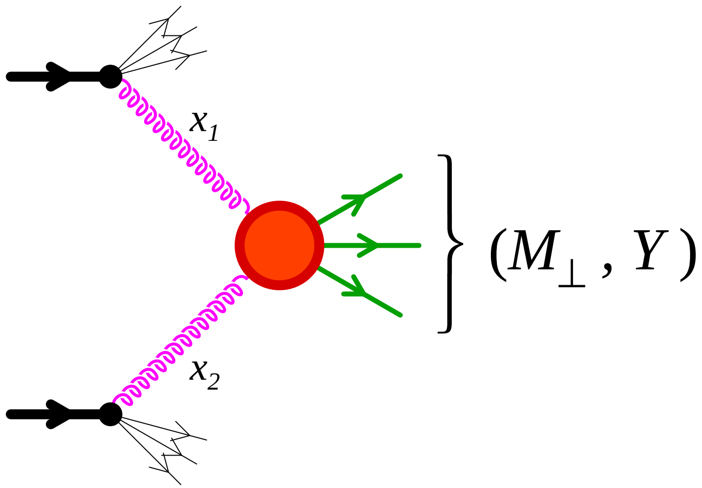
- Calculation of some process at LO :



$$\begin{cases} x_1 = M_\perp \, e^{+Y} / \sqrt{s} \\ x_2 = M_\perp \, e^{-Y} / \sqrt{s} \end{cases}$$

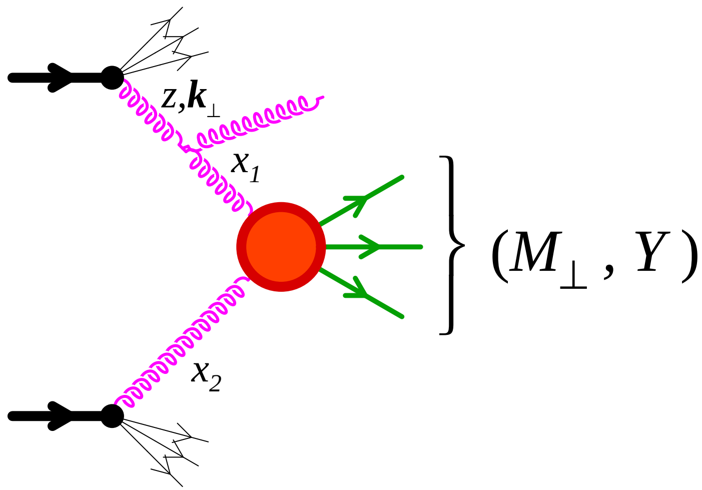
Infrared and collinear divergences

- Calculation of some process at LO :



$$\begin{cases} x_1 = M_\perp\, e^{+Y}/\sqrt{s} \\ x_2 = M_\perp\, e^{-Y}/\sqrt{s} \end{cases}$$

- Radiation of an extra gluon :



$$\Longrightarrow \quad \alpha_s \int_{x_1} \frac{dz}{z} \int^{M_\perp} \frac{d^2\vec{\boldsymbol{k}}_\perp}{\boldsymbol{k}_\perp^2}$$

Infrared and collinear divergences

- Large logs : $\log(M_\perp)$ or $\log(1/x_1)$, under certain conditions

  $\triangleright$ these logs can compensate the additional α_s, and void the naive application of perturbation theory
  $\triangleright$ resummations are necessary

- Logs of $M_\perp \Longrightarrow$ DGLAP + Collinear factorization
  - $M_\perp \gg \Lambda_{QCD}$
  - x_1, x_2 are rather large

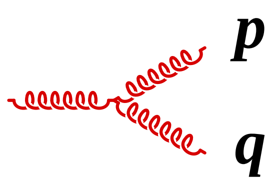
- Logs of $1/x \Longrightarrow$ BFKL + $k_\perp$-factorization
  - $M_\perp$ remains moderate
  - x_1 or x_2 (or both) are small

- Physical interpretation :
  - The physical process can resolve the gluon splitting if $M_\perp \gg k_\perp$
  - If $x_1 \ll 1$, the gluon that initiates the process is likely to result from bremsstrahlung from another parent gluon

Infrared and collinear divergences

- Factorization is the property that these logs can be absorbed in the parton distributions of the projectiles, and that these distributions are universal :
  - independent of the measured observable
  - independent of the other projectile

- Factorization is mostly a consequence of causality, because radiating a soft or collinear gluon takes a long time



$$t_f \sim \frac{1}{|\vec{\boldsymbol{p}}+\vec{\boldsymbol{q}}|-p-q} \sim \frac{1}{\sqrt{(p+q)^2+\underbrace{2pq(\cos\theta-1)}_{\text{small in the IR or collinear limits}}}-p-q}$$

$$\sim \frac{p+q}{pq(1-\cos\theta)} \to +\infty$$

▷ the processes responsible for the large logs must take place before or after –but not during– the collision

Color Glass Condensate

Criterion for gluon recombination

Gribov, Levin, Ryskin (1983)

- Number of gluons per unit area:

$$\rho \sim \frac{xG_A(x,Q^2)}{\pi R_A^2}$$

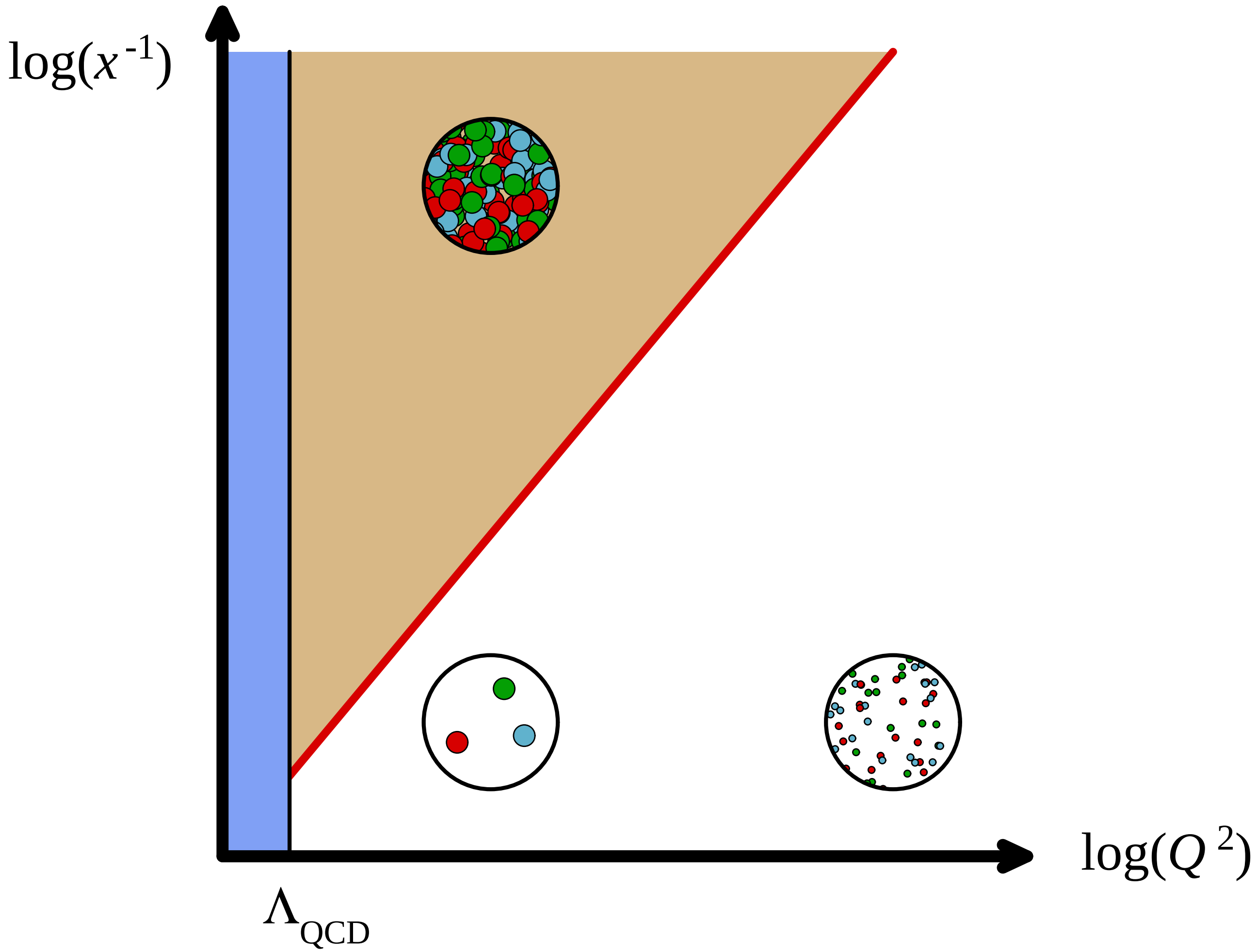
- Recombination cross-section:

$$\sigma_{gg\to g} \sim \frac{\alpha_s}{Q^2}$$

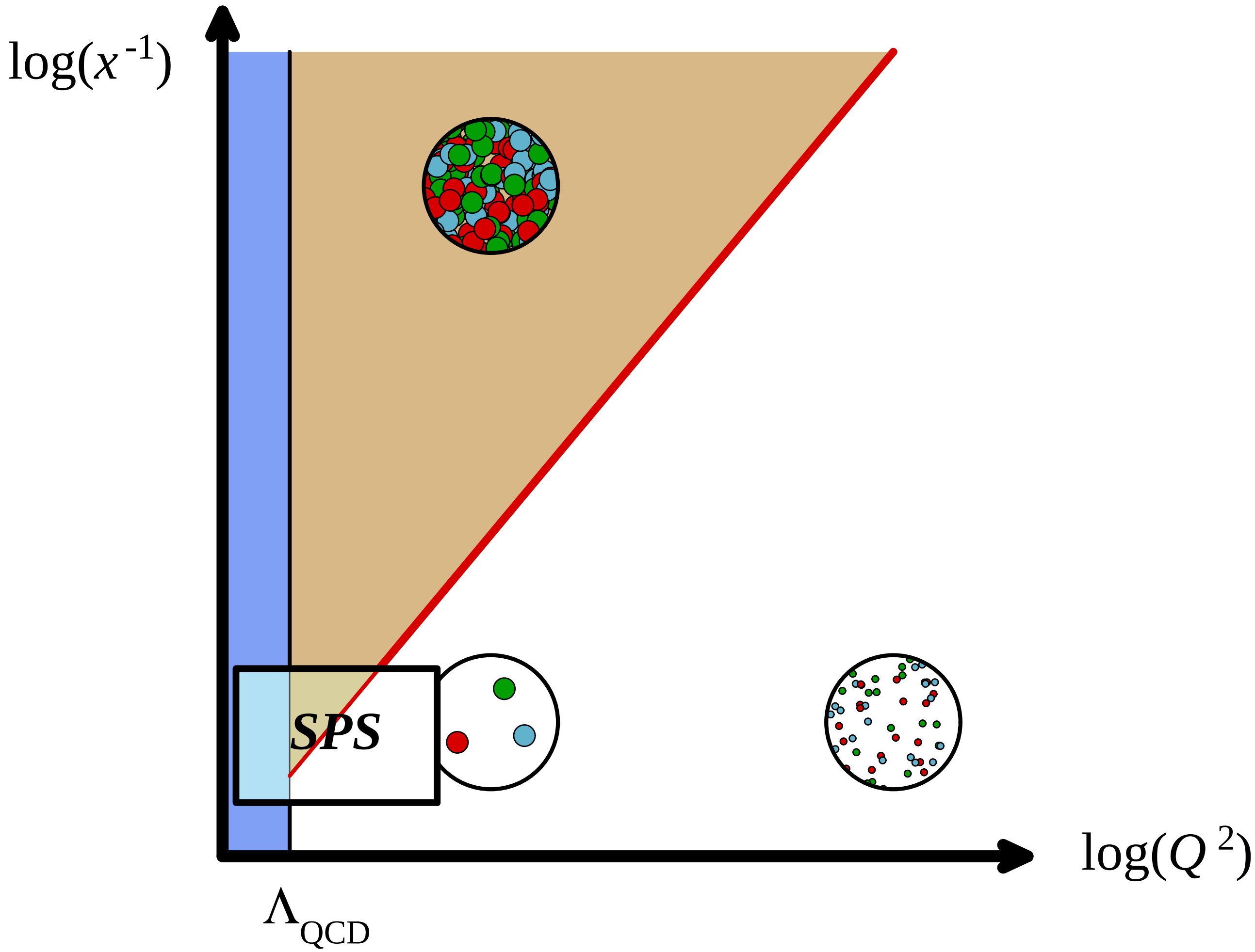
- Recombination happens if $\rho\sigma_{gg\to g} \gtrsim 1$, i.e. $Q^2 \lesssim Q_s^2$, with:

$$Q_s^2 \sim \frac{\alpha_s xG_A(x,Q_s^2)}{\pi R_A^2} \sim A^{1/3}\frac{1}{x^{0.3}}$$

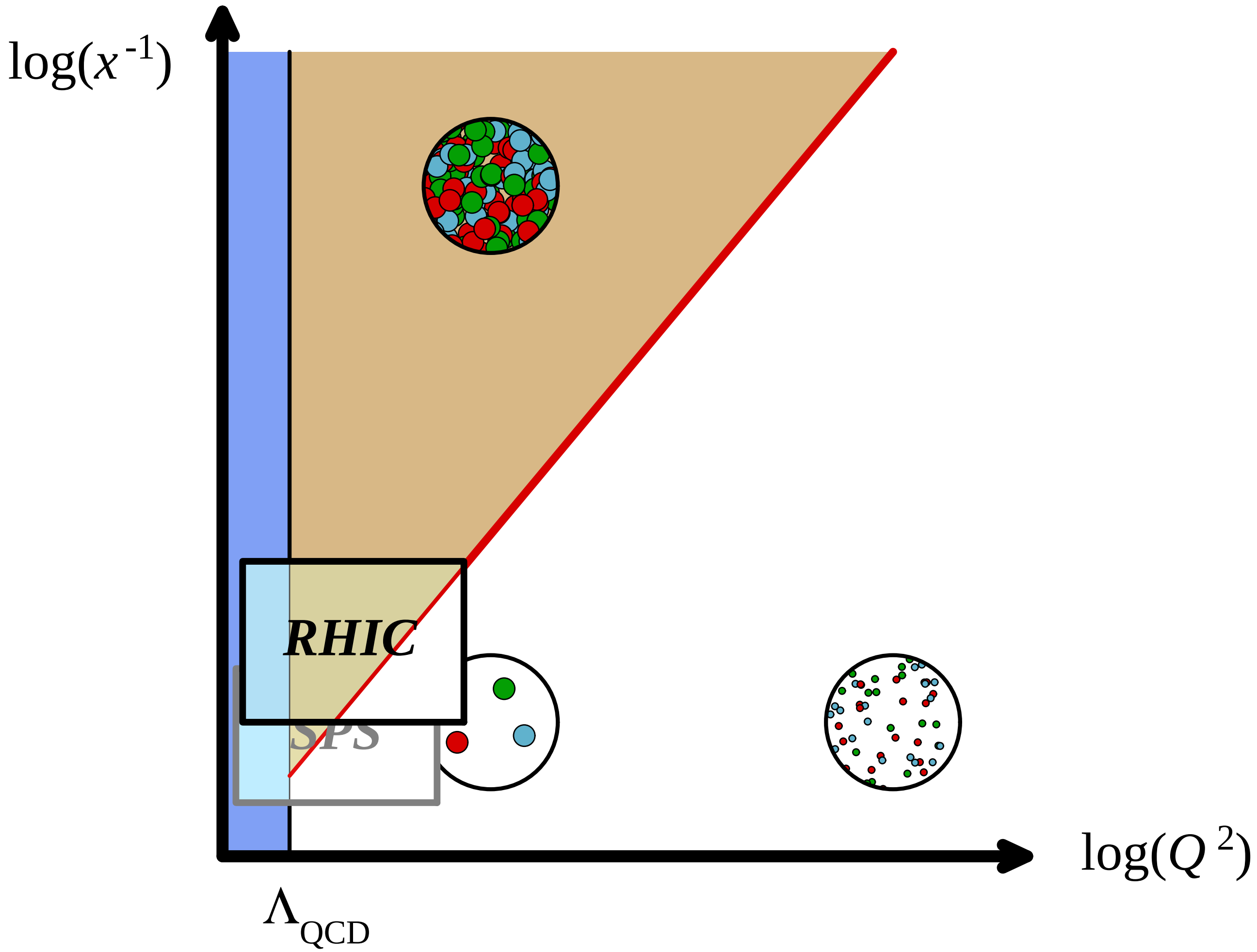
Saturation domain



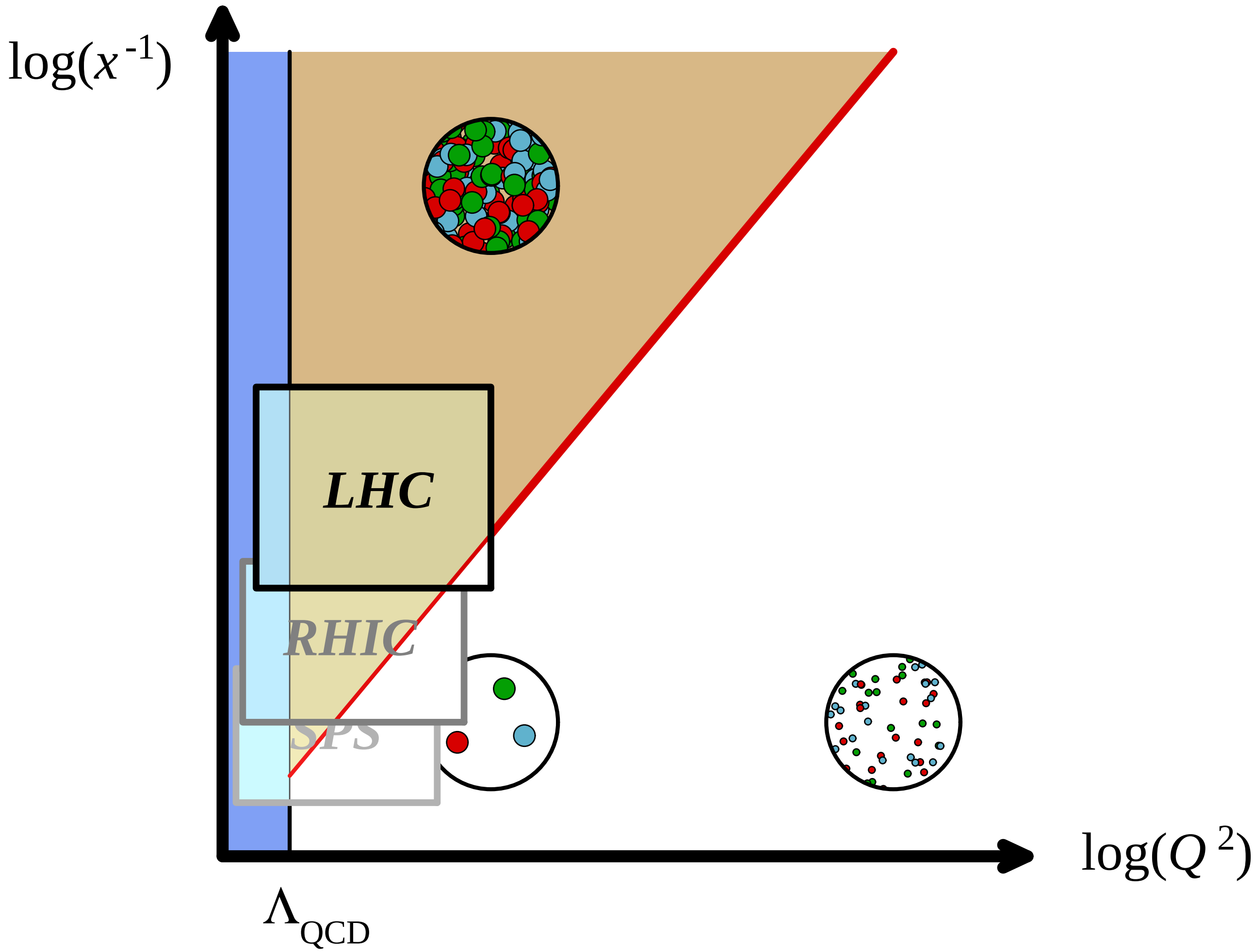
Saturation domain



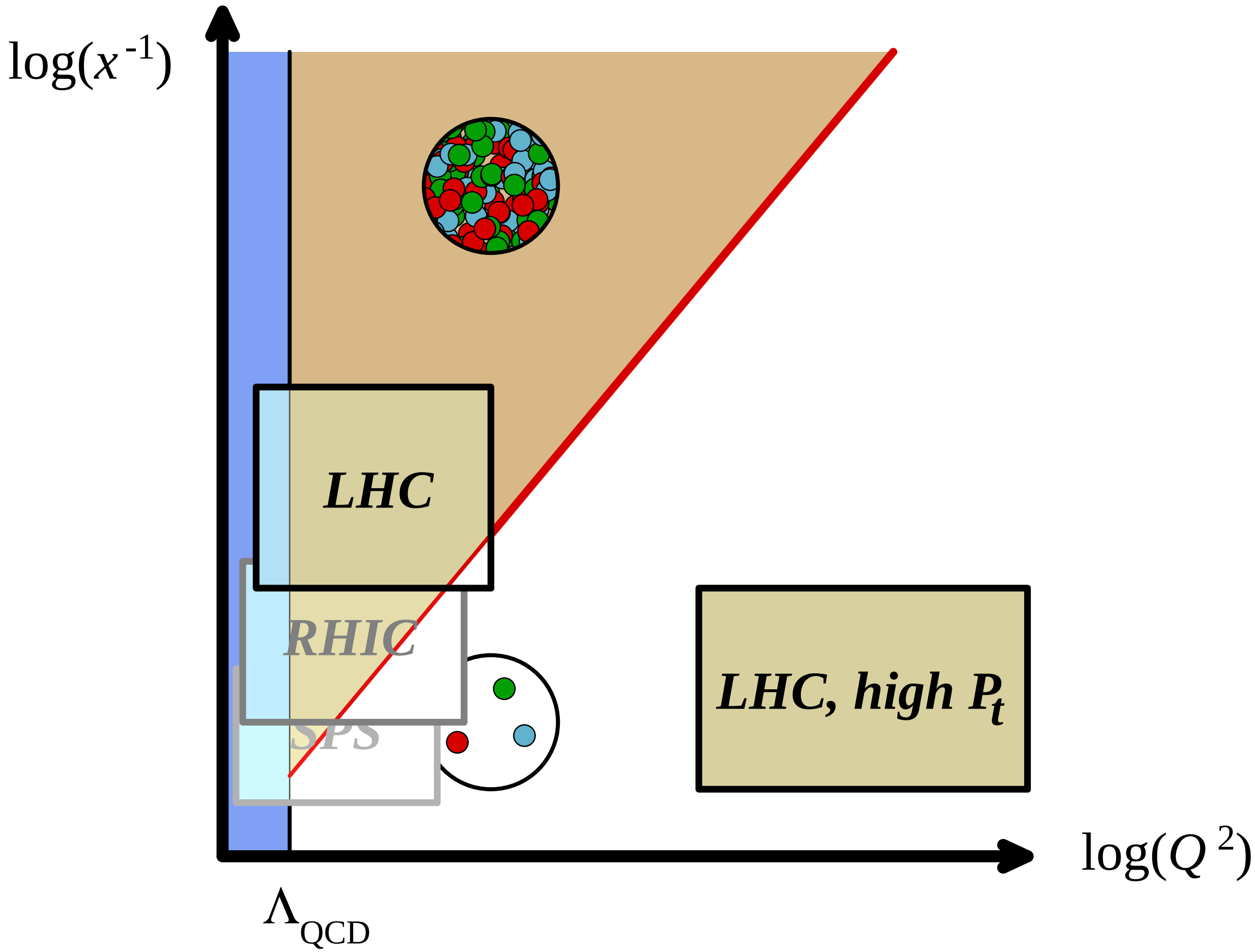
Saturation domain



Saturation domain



Saturation domain



Multiple scatterings

- The saturation criterion can also be seen as a condition for multiple scatterings

- The mean free path of a gluon in a nucleus is

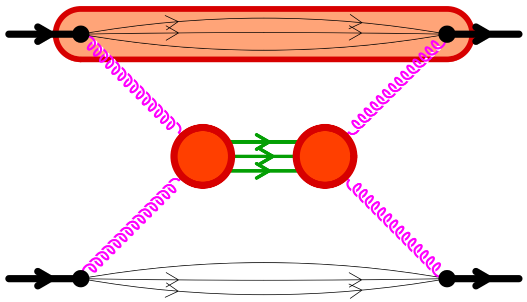
$$\lambda = \frac{1}{n\sigma_{gg\to g}} \quad , \quad n \sim \frac{xG_A(x, Q^2)}{\frac{4}{3}\pi R_A^3}$$

- Multiple scatterings are important if λ becomes smaller than the size of the nucleus, $\lambda \lesssim R_A$, i.e.

$$Q^2 \lesssim \alpha_s \frac{xG_A(x, Q^2)}{\pi R_A^2} \sim Q_s^2$$

Multiple scatterings

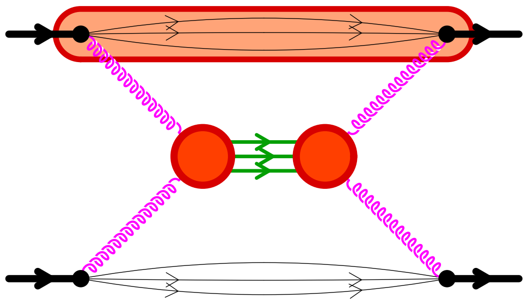
- Single scattering :



▷ 2-point function in the projectile ▷ gluon number

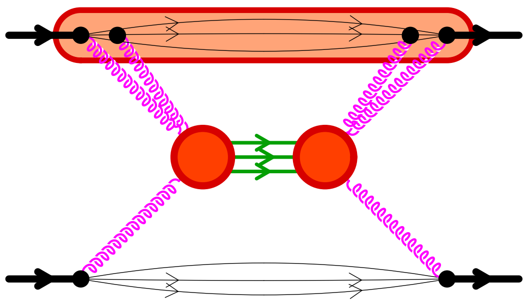
Multiple scatterings

- Single scattering :



▷ 2-point function in the projectile ▷ gluon number

- Multiple scatterings :



▷ 4-point function in the projectile ▷ higher correlations

CGC degrees of freedom

- The fast partons (large x) are frozen by time dilation
  ▷ described as static color sources on the light-cone :

$$J_a^\mu = \delta^{\mu +}\delta(x^-)\rho_a(\vec{\boldsymbol{x}}_\perp) \qquad (x^- \equiv (t-z)/\sqrt{2})$$

- Slow partons (small x) cannot be considered static over the time-scales of the collision process ▷ they must be treated as the usual gauge fields

  Since they are radiated by the fast partons, they must be coupled to the current J_a^μ by a term : $A_\mu J^\mu$

- The color sources ρ_a are random, and described by a distribution functional $W_Y[\rho]$, with Y the rapidity that separates “soft” and “hard”

  ▷ $W_Y[\rho]$ contains all the correlations needed to calculate multiple scatterings

CGC evolution

- Evolution equation (JIMWLK) :

$$\frac{\partial W_Y}{\partial Y} = \mathcal{H}\ W_Y$$

$$\mathcal{H} = \frac{1}{2}\int_{\vec{\boldsymbol{y}}_\perp} \frac{\delta}{\delta\widetilde{\mathcal{A}}_b^+(\epsilon,\vec{\boldsymbol{y}}_\perp)}\eta_{ab}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\frac{\delta}{\delta\widetilde{\mathcal{A}}_a^+(\epsilon,\vec{\boldsymbol{x}}_\perp)}$$

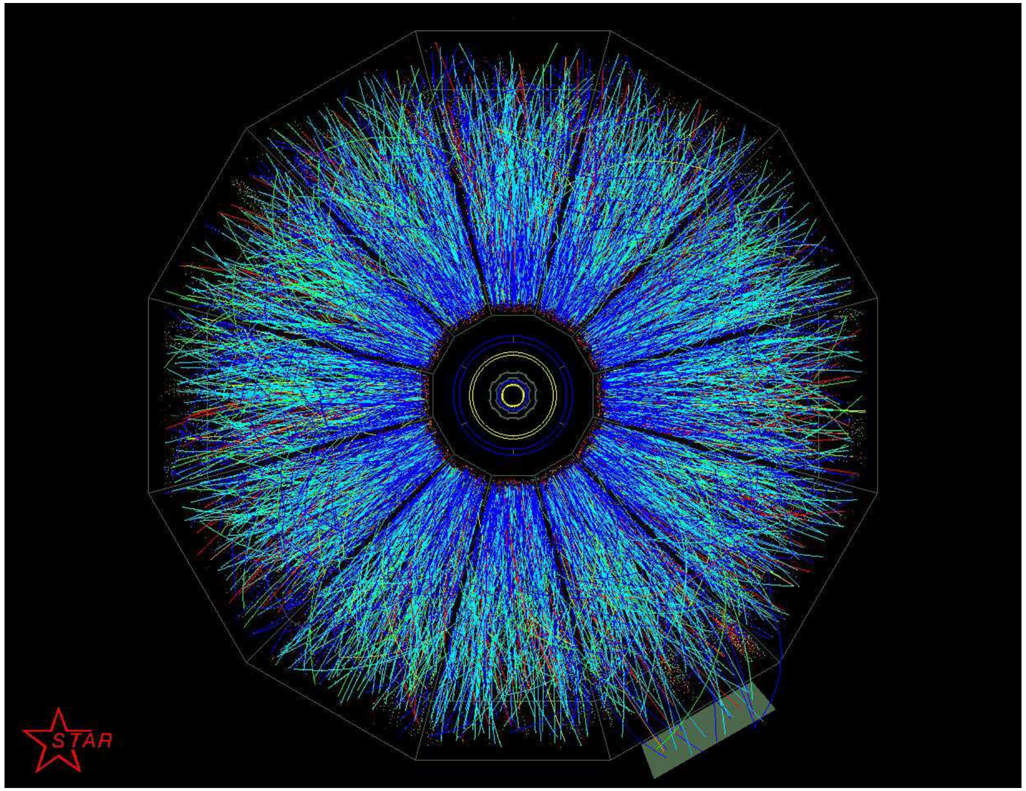
where $-\boldsymbol{\partial}_\perp^2\ \widetilde{\mathcal{A}}^+(\epsilon,\vec{\boldsymbol{x}}_\perp) = \rho(\epsilon,\vec{\boldsymbol{x}}_\perp)$

- η_{ab} is a non-linear functional of ρ

- This evolution equation resums the powers of $\alpha_s \ln(1/x)$ and of $Q_s/p_\perp$ that arise in loop corrections

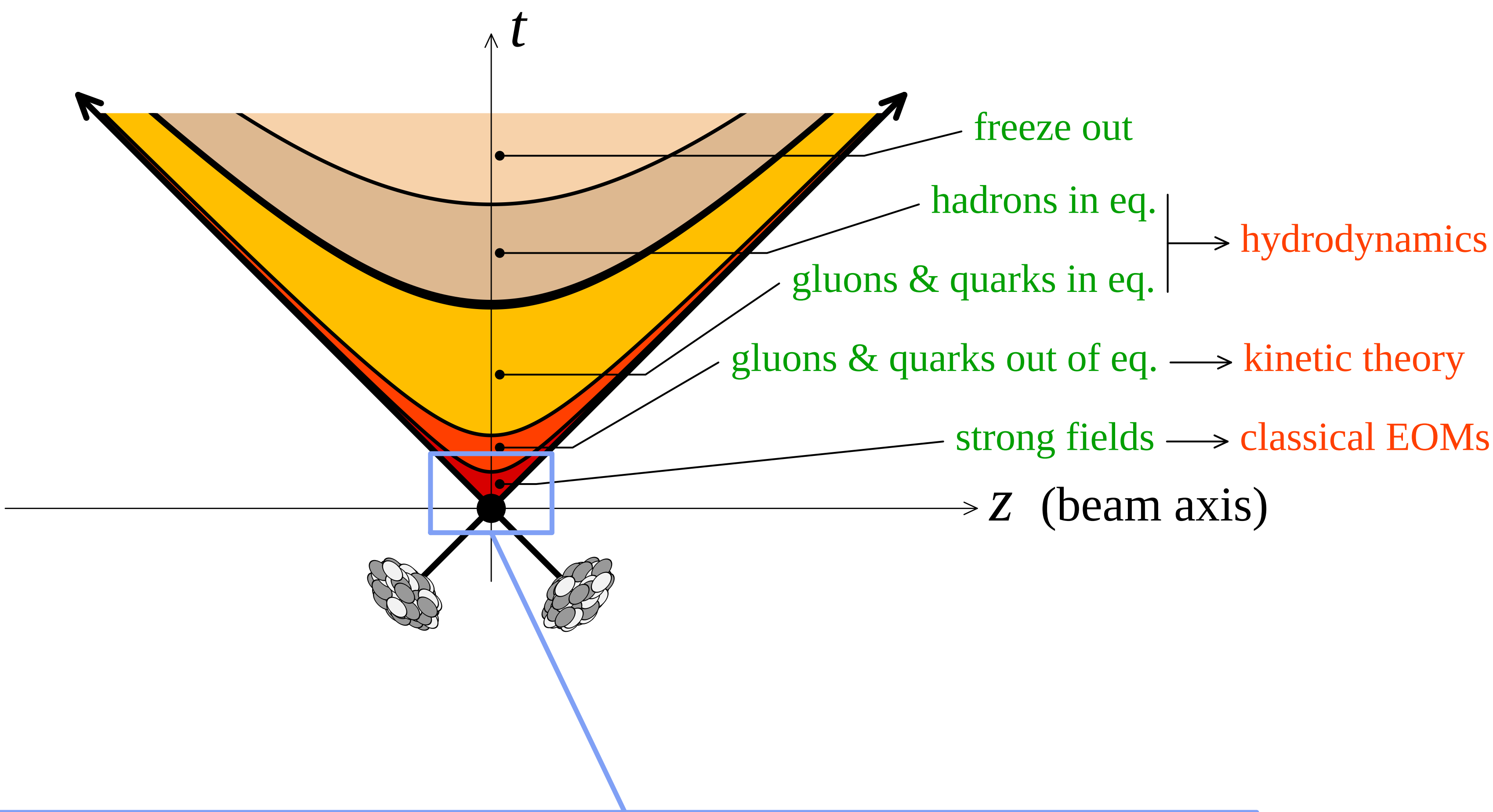
- This equation simplifies into the BFKL equation when the color density ρ is small (one can expand η_{ab} in ρ)

Heavy Ion Collisions



- 99% of the multiplicity below $p_\perp \sim 2$ GeV
- Q_s^2 might be as large as 10 GeV^2 at the LHC ($\sqrt{s} = 5.5$ TeV)
  - ▷ saturation and multiple scatterings presumably important

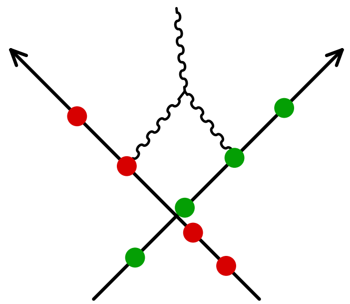
Heavy Ion Collisions



- calculate the initial production of semi-hard particles
- match to kinetic theory or hydrodynamics

CGC and Nucleus-Nucleus collisions

- For nucleus-nucleus collisions, there are two strong sources that contribute to the color current :



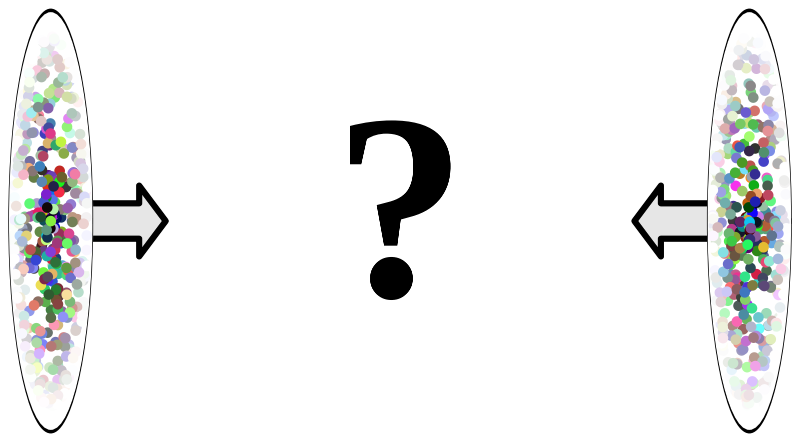
$$J^\mu \equiv \delta^{\mu+}\delta(x^-)\,\rho_1(\vec{\boldsymbol{x}}_\perp) + \delta^{\mu-}\delta(x^+)\,\rho_2(\vec{\boldsymbol{x}}_\perp)$$

- Average over the sources ρ_1, ρ_2

$$\left\langle\mathcal{O}\right\rangle_Y = \int [D\rho_1]\,[D\rho_2]\,W_{Y_{\text{beam}}-Y}[\rho_1]\,W_{Y+Y_{\text{beam}}}[\rho_2]\,\mathcal{O}[\rho_1,\rho_2]$$

- How to compute $\mathcal{O}[\rho_1,\rho_2]$ in the saturation regime ?
- Can this factorization formula be justified ?
- For which observables does it work ?

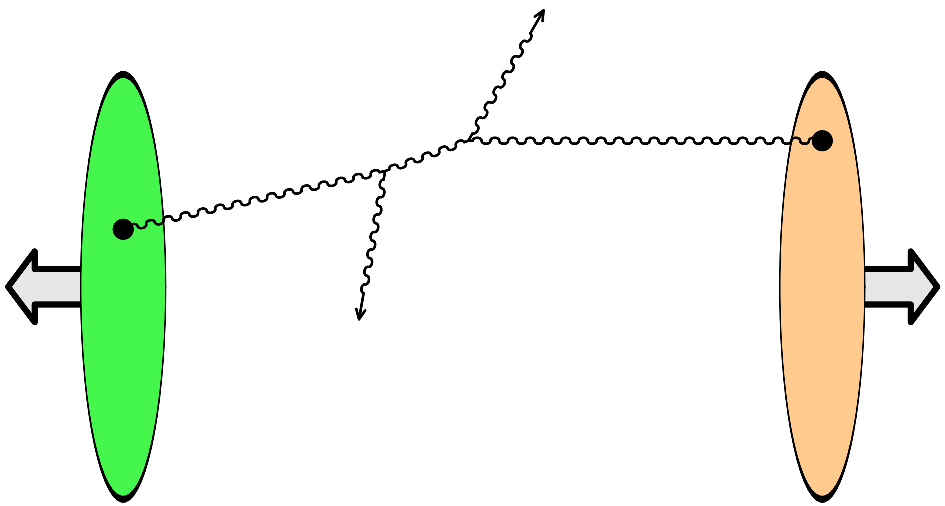
CGC and Nucleus-Nucleus collisions



$$\mathcal{L} = -\frac{1}{2}\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu} + \underbrace{(J_1^\mu + J_2^\mu)}_{J^\mu} A_\mu$$

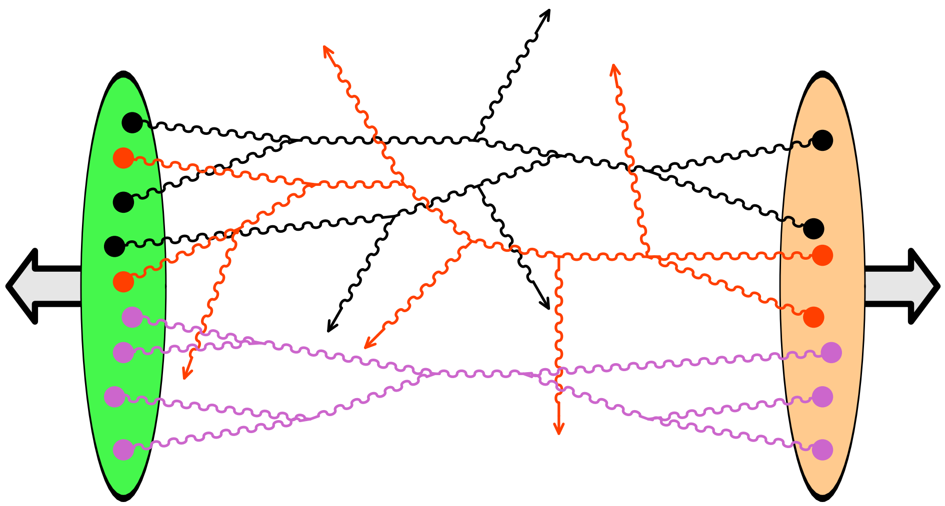
- Given the sources $\rho_{1,2}$ in each projectile, how do we calculate observables? Is there some kind of perturbative expansion?
- Loop corrections and factorization?

Initial particle production



- Dilute regime : one parton in each projectile interact

Initial particle production



- Dilute regime : one parton in each projectile interact
- Dense regime : multiparton processes become crucial

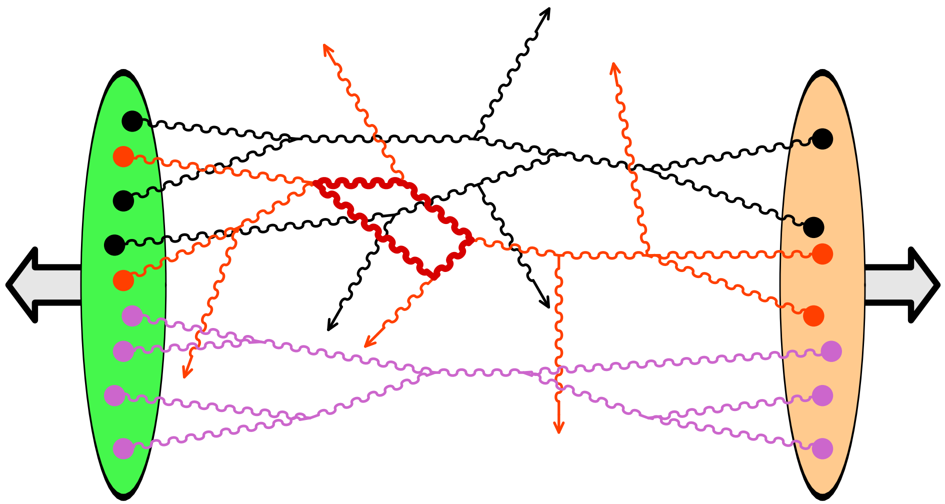
  (+ pileup of many partonic scatterings in each AA collision)

Goals

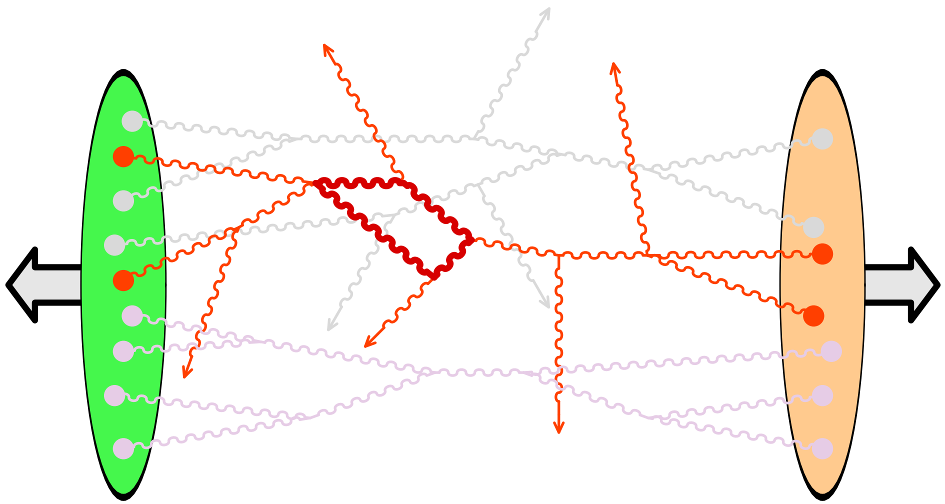
- Why can the inclusive gluon spectrum be obtained from classical solutions of Yang-Mills equations ?
- Why are the boundary conditions retarded ? What would it mean to choose different boundary conditions ?
- Is this a controlled approximation, i.e. the first term in a more systematic expansion ?
- Is it possible to go beyond this computation, and study the 1-loop corrections ? Logs(1/x) and factorization ?
- What are the final state interactions? Do they lead to local thermalization ?

Power counting and Bookkeeping

Power counting



Power counting

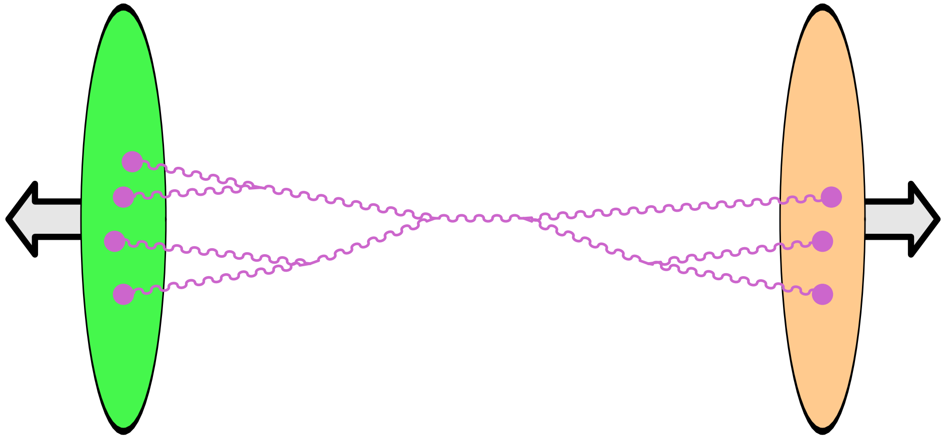


- In the saturated regime, the sources are of order $1/g$ (because $\langle\rho\rho\rangle \sim$ occupation number $\sim 1/\alpha_s$)
- The order of each connected diagram is given by :

$$\frac{1}{g^2}\quad g^{\#\text{ produced gluons}}\quad g^{2(\#\text{ loops})}$$

- The total order of a graph is the product of the orders of its disconnected subdiagrams

Vacuum diagrams



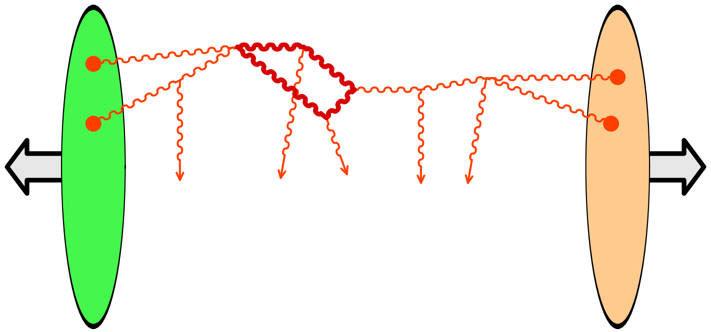
- Vacuum diagrams do not produce any gluon. They are contributions to the vacuum to vacuum amplitude $\langle 0_{\rm out} | 0_{\rm in} \rangle$
- The order of a connected vacuum diagram is given by :

$$g^{-2} \quad g^{2(\#\ \text{loops})}$$

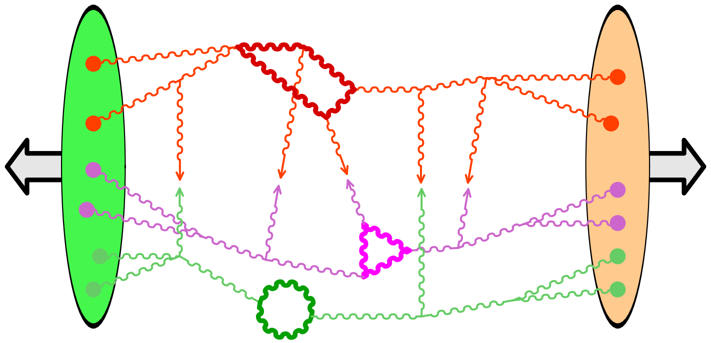
- Relation between connected and non connected vacuum diagrams :

$$\sum \begin{pmatrix} \text{all the vacuum} \\ \text{diagrams} \end{pmatrix} = \exp\left\{ \sum \begin{pmatrix} \text{simply connected} \\ \text{vacuum diagrams} \end{pmatrix} \right\} = e^{iV[j]}$$

Bookkeeping

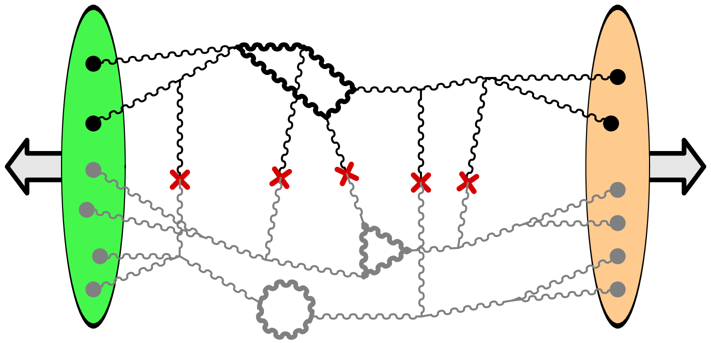


Bookkeeping



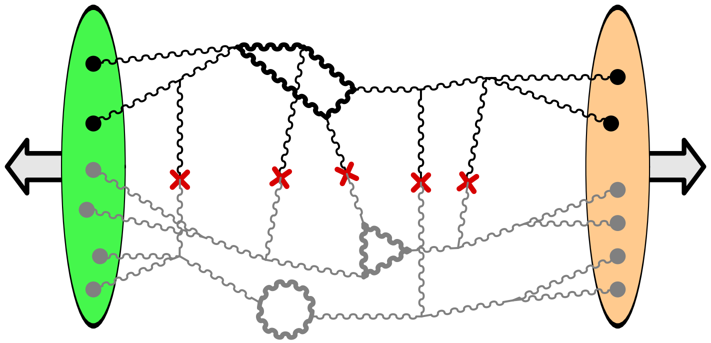
- Consider squared amplitudes (including interference terms) rather than the amplitudes themselves

Bookkeeping



- Consider squared amplitudes (including interference terms) rather than the amplitudes themselves
- See them as cuts through vacuum diagrams
  cut propagator : $2\pi\theta(-p^0)\delta(p^2)$

Bookkeeping



- Consider squared amplitudes (including interference terms) rather than the amplitudes themselves
- See them as cuts through vacuum diagrams
  cut propagator : $2\pi\theta(-p^0)\delta(p^2)$
- The sum of the vacuum diagrams, $\exp(iV[j])$, is the generating functional for time-ordered products of fields :

$$\left\langle 0_{\rm out}\middle| T A(x_1)\cdots A(x_n)\middle|0_{\rm in}\right\rangle = \frac{\delta}{\delta j(x_1)}\cdots\frac{\delta}{\delta j(x_n)}\; e^{iV[j]}$$

Bookkeeping

- The probability of producing exactly n particles is :

$$P_n = \frac{1}{n!} \int \frac{d^3\vec{\boldsymbol{p}}_1}{(2\pi)^3 2E_1} \cdots \frac{d^3\vec{\boldsymbol{p}}_n}{(2\pi)^3 2E_n} \left| \langle \vec{\boldsymbol{p}}_1 \cdots \vec{\boldsymbol{p}}_{n\,\mathrm{out}} | 0_{\mathrm{in}} \rangle \right|^2$$

- There is an operator $\mathcal{C}$ such that
  - P_n is given by $P_n = \frac{1}{n!}\, \mathcal{C}^n \; e^{iV[j_+]}\; e^{-iV^*[j_-]} \Big|_{j_+=j_-=j}$

with

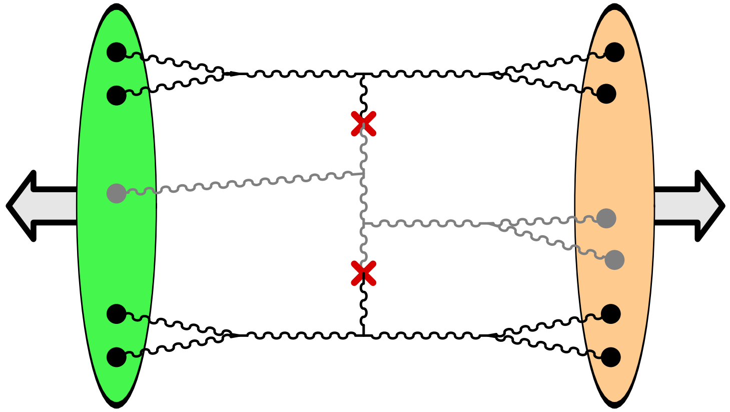
$$\begin{cases} \mathcal{C} \equiv \int_{x,y} G^0_{+-}(x,y)\, \Box_x \Box_y \, \frac{\delta}{\delta j_+(x)} \frac{\delta}{\delta j_-(y)} \\ G^0_{+-}(x,y) \equiv \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, 2\pi\theta(-p^0)\delta(p^2) \end{cases}$$

Bookkeeping

- Reminder :

$$\mathcal{C} \equiv \int_{x,y} G^0_{+-}(x,y)\, \square_x \square_y \,\frac{\delta}{\delta j_+(x)}\frac{\delta}{\delta j_-(y)}$$

- Consider a generic cut vacuum diagram :



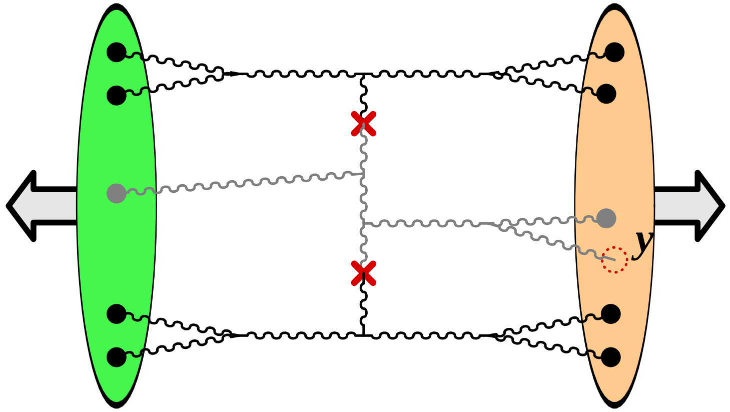
Bookkeeping

- Reminder :

$$\mathcal{C} \equiv \int_{x,y} G^0_{+-}(x,y)\,\square_x \square_y \,\frac{\delta}{\delta j_+(x)}\frac{\delta}{\delta j_-(y)}$$

- Consider a generic cut vacuum diagram :

$$\frac{\delta}{\delta j_-(y)} \quad \longrightarrow$$



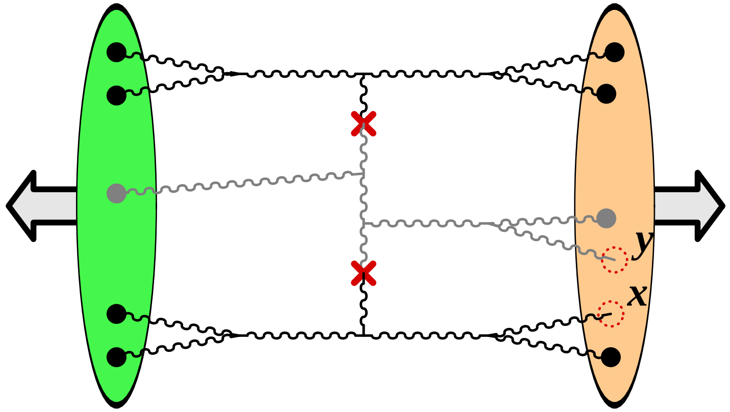
Bookkeeping

- Reminder :

$$\mathcal{C} \equiv \int_{x,y} G^0_{+-}(x,y)\, \square_x \square_y \, \frac{\delta}{\delta j_+(x)} \frac{\delta}{\delta j_-(y)}$$

- Consider a generic cut vacuum diagram :

$$\frac{\delta}{\delta j_+(x)} \quad \longrightarrow$$



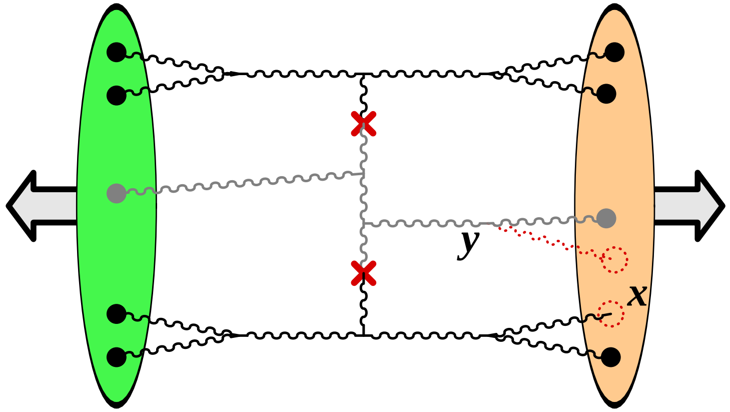
Bookkeeping

- Reminder :

$$\mathcal{C} \equiv \int_{x,y} G^0_{+-}(x,y)\, \square_x \square_y \frac{\delta}{\delta j_+(x)} \frac{\delta}{\delta j_-(y)}$$

- Consider a generic cut vacuum diagram :

$$\square_y \quad \longrightarrow$$



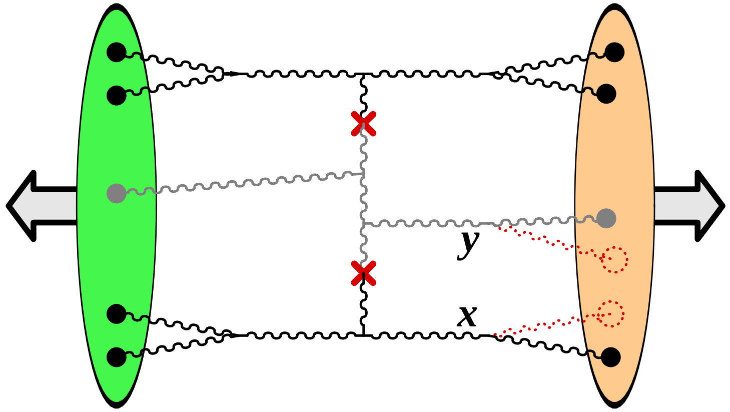
Bookkeeping

- Reminder :

$$\mathcal{C} \equiv \int_{x,y} G^0_{+-}(x,y)\, \square_x \square_y \, \frac{\delta}{\delta j_+(x)} \frac{\delta}{\delta j_-(y)}$$

- Consider a generic cut vacuum diagram :

$$\square_x \quad \longrightarrow$$



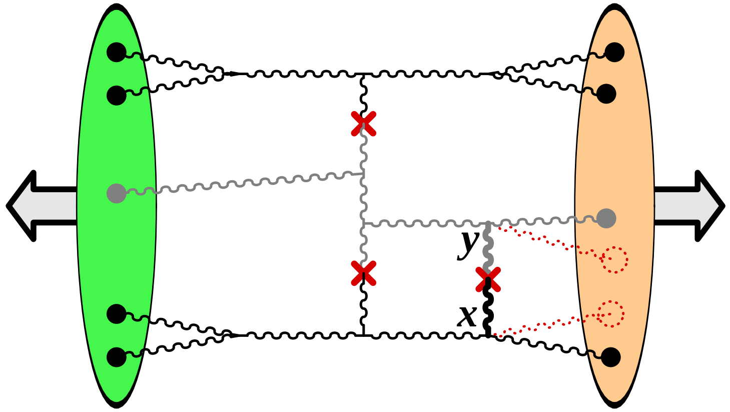
Bookkeeping

- Reminder :

$$\mathcal{C} \equiv \int_{x,y} G^0_{+-}(x,y)\,\square_x\square_y\,\frac{\delta}{\delta j_+(x)}\frac{\delta}{\delta j_-(y)}$$

- Consider a generic cut vacuum diagram :

$$G^0_{+-}(x,y) \quad \longrightarrow$$



▷ the operator $\mathcal{C}$ removes two sources (one in the amplitude and one in the complex conjugated amplitude), and creates a new cut propagator

Bookkeeping

- The sum of all the cut vacuum diagrams, with sources j_+ on one side of the cut and j_- on the other side, can be written as :

$$\sum \left(\begin{array}{c} \text{all the cut} \\ \text{vacuum diagrams} \end{array} \right) = e^{\mathcal{C}} \; e^{iV[j_+]} \; e^{-iV^*[j_-]}$$

- If we set $j_+ = j_- = j$, then this is $\sum_n P_n = 1$

  (this property is a direct consequence of the “largest time equation” in Cutkosky’s cutting rules)

Bookkeeping

- The operator $\mathcal{C}$ can be used to derive many useful formulas :

$$F(z)=\sum_{n=0}^{+\infty} z^n\, P_n = e^{z\mathcal{C}}\; e^{iV[j_+]}\; e^{-iV^*[j_-]}\Big|_{j_+=j_-=j}$$

▷ sum of all cut vacuum graphs, where each cut is weighted by z

$$\overline{N}=F'(1)=\mathcal{C}\; e^{\mathcal{C}}\; e^{iV[j_+]}\; e^{-iV^*[j_-]}\Big|_{j_+=j_-=j}$$

$$\overline{N(N-1)}=F''(1)=\mathcal{C}^2\; e^{\mathcal{C}}\; e^{iV[j_+]}\; e^{-iV^*[j_-]}\Big|_{j_+=j_-=j}$$

- Benefits :
  - The tracking of infinite sets of Feynman diagrams has been replaced by simple algebraic manipulations
  - The use of the identity $e^{\mathcal{C}}\; e^{iV[j_+]}\; e^{-iV^*[j_-]}\Big|_{j_+=j_-}=1$ renders automatic an important cancellation that would be hard to see at the level of diagrams (somewhat related to AGK)

Inclusive gluon spectrum

Gluon multiplicity at LO

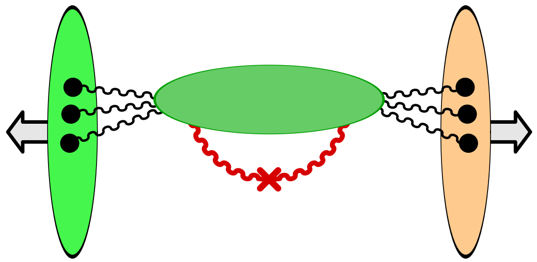
- It is easy to express the average multiplicity as :

$$\overline{N} = \sum_n n\, P_n = \mathcal{C}\Big\{ \underbrace{e^{\mathcal{C}}\, e^{iV[j_+]}\, e^{-iV^*[j_-]}} \Big\}_{j_+=j_-=j}$$

sum of all the cut vacuum diagrams : $e^{iW[j_+,j_-]}$

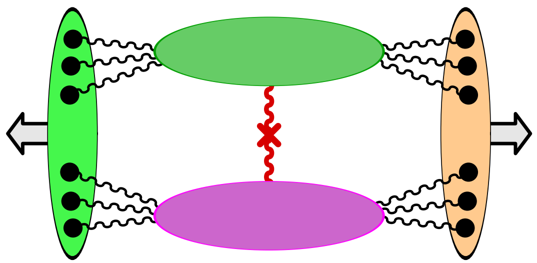
- There are two types of terms :
  - $\mathcal{C}$ picks two sources in the same connected cut diagram

$$\frac{\delta^2 iW}{\delta j_+(x)\delta j_-(y)} \longrightarrow$$



  - $\mathcal{C}$ picks two sources in two distinct connected cut diagrams

$$\frac{\delta iW}{\delta j_+(x)}\;\frac{\delta iW}{\delta j_-(y)} \longrightarrow$$

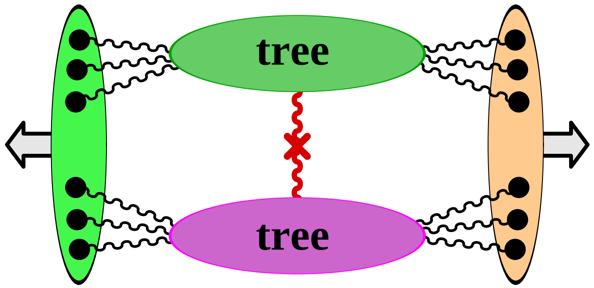


Gluon multiplicity at LO

- At LO, only tree diagrams contribute
  - ▷ the first type of topologies can be neglected (they have at least one loop)

- In each blob, we must sum over all the tree diagrams, and over all the possible cuts :

$$\overline{N}_{LO} = \sum_{\text{trees}} \sum_{\text{cuts}}$$



- Reminder : at the end, the sources on both sides of the cut must be set equal :

$$j_+ = j_-$$

Sum over the cuts

- In the previous diagrams, one must sum over all the possible ways of cutting lines inside the blobs
- This can be achieved via Cutkosky's cutting rules :
  - A vertex is $-ig$ on one side of the cut, and $+ig$ on the other side
  - There are four propagators, depending on the location w.r.t. the cut of the vertices they connect :

$$G^0_{++}(p) = i/(p^2 - m^2 + i\epsilon) \qquad \text{(standard Feynman propagator)}$$

$$G^0_{--}(p) = -i/(p^2 - m^2 - i\epsilon) \qquad \text{(complex conjugate of } G^0_{++}(p)\text{)}$$

$$G^0_{+-}(p) = 2\pi\theta(-p^0)\delta(p^2 - m^2)$$

$$G^0_{-+}(p) = 2\pi\theta(p^0)\delta(p^2 - m^2)$$

  - At each vertex of a given diagram, sum over the types $+$ and $-$ (2^n terms for a diagram with n vertices)

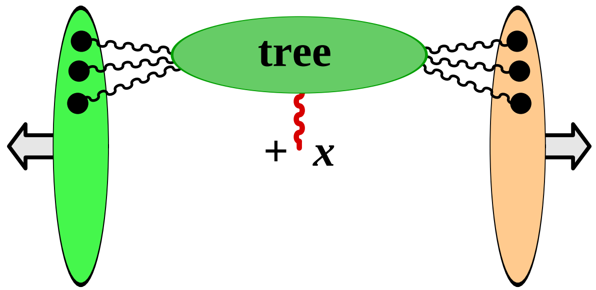
Expression in terms of classical fields

- The gluon spectrum at LO is given by :

$$\frac{d\overline{N}_{LO}}{dY\,d^2\vec{\boldsymbol{p}}_\perp} = \frac{1}{16\pi^3}\int_{x,y} e^{ip\cdot(x-y)}\,\square_x\square_y\sum_\lambda \epsilon_\mu^{(\lambda)}\epsilon_\nu^{(\lambda)}\;\mathcal{A}_+^\mu(x)\mathcal{A}_-^\nu(y)$$

- $\mathcal{A}_\pm^\mu(x)$ are sums of cut tree diagrams ending at the point x :

$$\mathcal{A}_+^\mu(x) = \sum_{\text{trees}}\sum_{\text{cuts}}$$



- This sum of graphs is given by an integral equation :
  (written here in a scalar theory for simplicity)

$$\mathcal{A}_+(x) = \int d^4y\,\Big[G^0_{++}(x,y)\Big[j(y)-U'(\mathcal{A}_+(y))\Big] - G^0_{+-}(x,y)\Big[j(y)-U'(\mathcal{A}_-(y))\Big]\Big]$$

(there is a similar equation for $\mathcal{A}_-$)

Boundary conditions

- $\mathcal{A}_\pm$ is a solution of the classical EOM ($\square\mathcal{A} + U'(\mathcal{A}) = j$)
- However, this fact alone does not determine $\mathcal{A}_+^\mu(x)$ uniquely
  ▷ we need the boundary conditions
- Any solution of the EOM can be written as :

$$\mathcal{A}_+(x) = \int d^4y \Big[G^0_{++}(x,y)\Big[j(y) - U'(\mathcal{A}_+(y))\Big] - G^0_{+-}(x,y)\Big[j(y) - U'(\mathcal{A}_-(y))\Big]\Big]$$
$$+ \int d^3\vec{\boldsymbol{y}} \Big[G^0_{++}(x,y)\,\overleftrightarrow{\partial^0_y}\,\mathcal{A}_+(y) - G^0_{+-}(x,y)\,\overleftrightarrow{\partial^0_y}\,\mathcal{A}_-(y)\Big]^{+\infty}_{-\infty}$$

where $\overleftrightarrow{\partial^0_y} \equiv \overrightarrow{\partial^0_y} - \overleftarrow{\partial^0_y}$ (Green's formula)

- The first line is identical to the integral equation for $\mathcal{A}_+$
  ▷ the boundary term on the second line must be zero :

$$\int d^3\vec{\boldsymbol{y}} \Big[G^0_{++}(x,y)\,\overleftrightarrow{\partial^0_y}\,\mathcal{A}_+(y) - G^0_{+-}(x,y)\,\overleftrightarrow{\partial^0_y}\,\mathcal{A}_-(y)\Big]^{+\infty}_{-\infty} = 0$$

Boundary conditions

- Note : these boundary conditions are non local in space
- Because the propagators $G^0_{\pm\pm}$ are linear combinations of plane waves, things become simpler when written for the Fourier modes of the fields :

$$\mathcal{A}_\epsilon(x) \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}} \left[a_\epsilon^{(+)}(x_0,\vec{\boldsymbol{p}})\, e^{-ip\cdot x} + a_\epsilon^{(-)}(x_0,\vec{\boldsymbol{p}})\, e^{+ip\cdot x}\right]$$

- The boundary conditions can be rewritten as :

$$a_+^{(+)}(-\infty,\vec{\boldsymbol{p}}) = a_-^{(-)}(-\infty,\vec{\boldsymbol{p}}) = 0$$
$$a_-^{(+)}(+\infty,\vec{\boldsymbol{p}}) = a_+^{(+)}(+\infty,\vec{\boldsymbol{p}})$$
$$a_+^{(-)}(+\infty,\vec{\boldsymbol{p}}) = a_-^{(-)}(+\infty,\vec{\boldsymbol{p}})$$

- These conditions imply $\mathcal{A}_+ = \mathcal{A}_- = \mathcal{A}$, and $\mathcal{A}(x^0 = -\infty) = 0$

Classical fields and tree diagrams

- The link between classical solutions and tree diagrams can also be obtained by a more pedestrian method

- Proof for a scalar theory: The classical EOM reads

$$(\square + m^2)\,\varphi(x) + \frac{g}{2}\varphi^2(x) = j(x)$$

- Write the Green's formula for the retarded solution that obeys $\varphi(x) = 0$ at $x^0 = -\infty$:

$$\varphi(x) = \int d^4y\; G_R^0(x-y)\left[j(y) - \frac{g}{2}\varphi^2(y)\right]$$

where $G_R^0(x-y)$ is the free retarded propagator

Retarded classical solution

- One can construct the solution iteratively, by using in the r.h.s. the solution found in the previous orders

- Order g^0 :

$$\varphi_{(0)}(x) = \int d^4y \; G_R^0(x-y)\, j(y)$$

- Order g^1 :

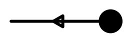
$$\varphi_{(0)}(x) + \varphi_{(1)}(x) = \int d^4y \; G_R^0(x-y) \left[j(y) - \frac{g}{2}\varphi_{(0)}^2(y) \right]$$

i.e.

$$\varphi_{(1)}(x) = -\frac{g}{2} \int d^4y \; G_R^0(x-y) \left[\int d^4z \; G_R^0(y-z)\, j(z) \right]^2$$

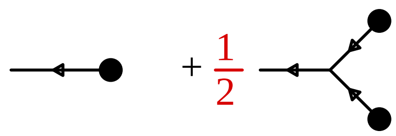
Retarded classical solution

- The diagrammatic expansion of this classical solution is :



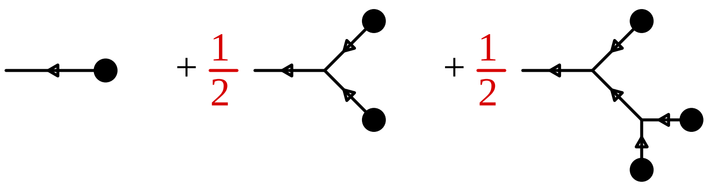
Retarded classical solution

- The diagrammatic expansion of this classical solution is :



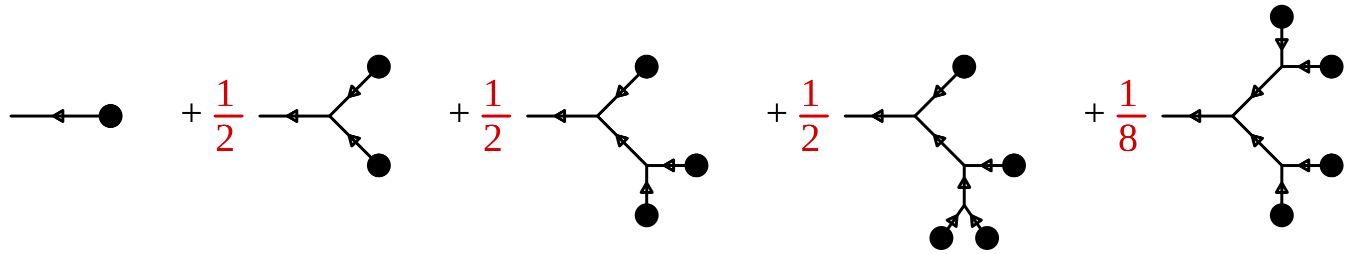
Retarded classical solution

- The diagrammatic expansion of this classical solution is :



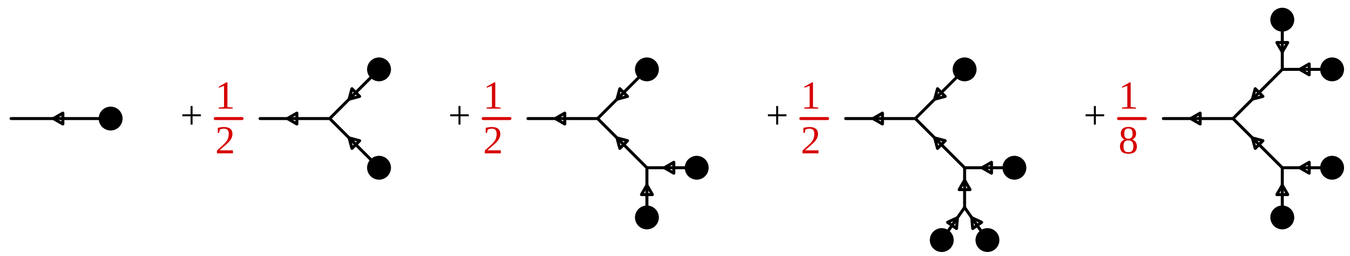
Retarded classical solution

- The diagrammatic expansion of this classical solution is :



Retarded classical solution

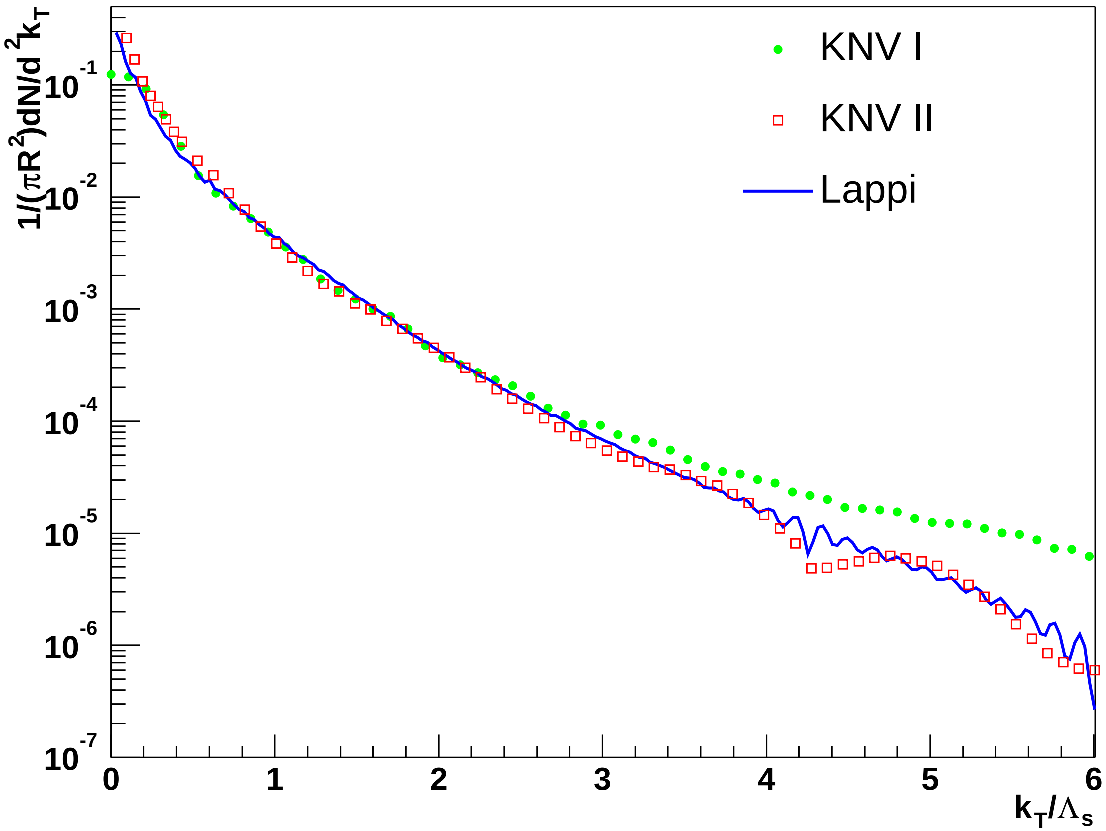
- The diagrammatic expansion of this classical solution is :



- The retarded classical solution is given by the sum of all the tree diagrams with retarded propagators

Gluon spectrum at LO

Krasnitz, Nara, Venugopalan (1999 – 2001), Lappi (2003)

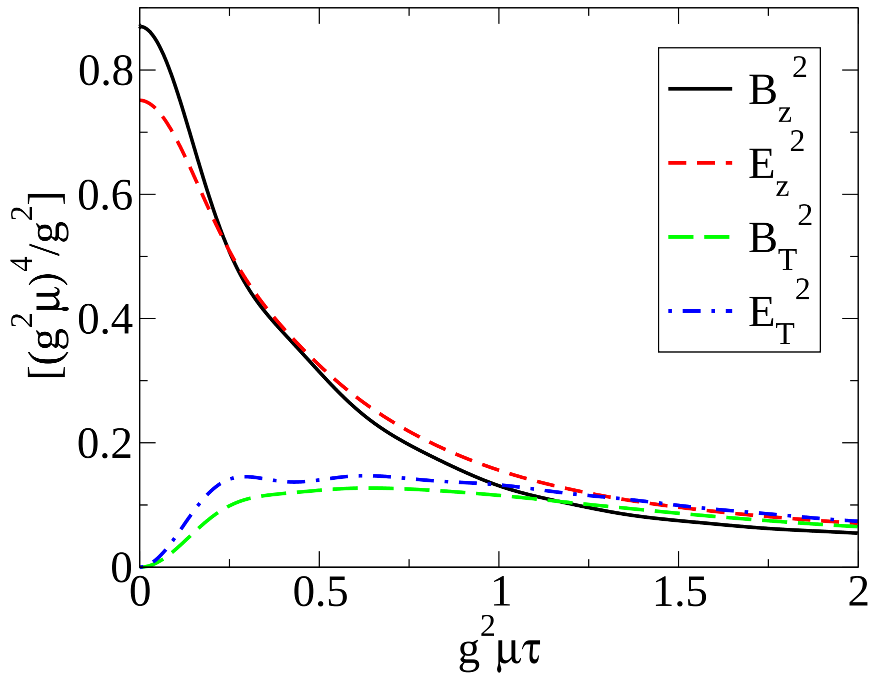


- Important softening at small $k_\perp$ compared to pQCD (saturation)
- See the following lecture by T. Lappi for many more details

Initial fields

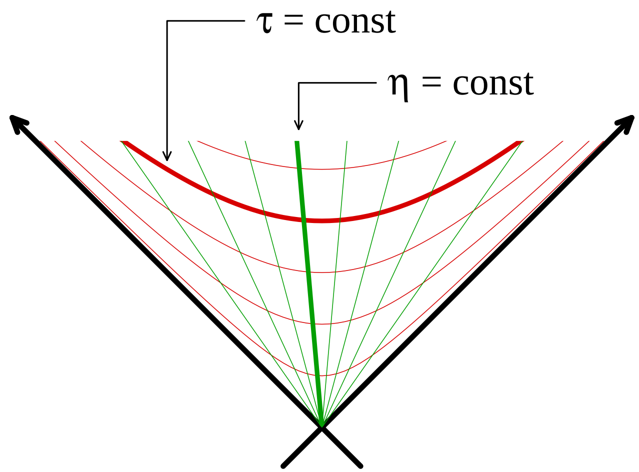
Lappi, McLerran (2006)

- Before the collision, the chromo-$\vec{E}$ and $\vec{B}$ fields are localized in two sheets transverse to the beam axis
- Immediately after the collision, the chromo-$\vec{E}$ and $\vec{B}$ fields have become longitudinal :



Initial conditions and boost invariance

- Gauge condition : $x^+\mathcal{A}^- + x^-\mathcal{A}^+ = 0$



$$\Rightarrow \quad \mathcal{A}^\pm(x) \quad = \quad \pm\, x^\pm\, \beta(\tau, \eta, \vec{\boldsymbol{x}}_\perp)$$

- Initial values at $\tau = 0^+$: $\mathcal{A}^i(0^+, \eta, \vec{\boldsymbol{x}}_\perp)$ and $\beta(0^+, \eta, \vec{\boldsymbol{x}}_\perp)$ do not depend on the rapidity η

  ▷ $\mathcal{A}^i$ and β remain independent of η at all times (invariance under boosts in the z direction)

  ▷ numerical resolution performed in $1+2$ dimensions

Generating functional

Definition

- Consider a function $z(\vec{\boldsymbol{p}})$, and define the functional

$$F[z] \equiv \frac{1}{n!}\sum_{n=0}^{+\infty}\int d\Phi_1\cdots d\Phi_n\; z(\vec{\boldsymbol{p}}_1)\cdots z(\vec{\boldsymbol{p}}_n)\;\left|\left\langle\vec{\boldsymbol{p}}_1\cdots\vec{\boldsymbol{p}}_{n\,\mathrm{out}}\middle|0_{\mathrm{in}}\right\rangle\right|^2$$

- Any physical quantity can be obtained from $F[z]$
  - Single inclusive spectrum :

$$\frac{d\overline{N}}{d^3\vec{\boldsymbol{p}}} = \left.\frac{\delta F[z]}{\delta z(\vec{\boldsymbol{p}})}\right|_{z=1}$$

  - Double inclusive spectrum (correlated part) :

$$\mathcal{C}(\vec{\boldsymbol{p}}_1,\vec{\boldsymbol{p}}_2) = \left.\frac{\delta^2 F[z]}{\delta z(\vec{\boldsymbol{p}}_1)\delta z(\vec{\boldsymbol{p}}_2)} - \frac{\delta F[z]}{\delta z(\vec{\boldsymbol{p}}_1)}\,\frac{\delta F[z]}{\delta z(\vec{\boldsymbol{p}}_2)}\right|_{z=1}$$

Diagrammatic expansion

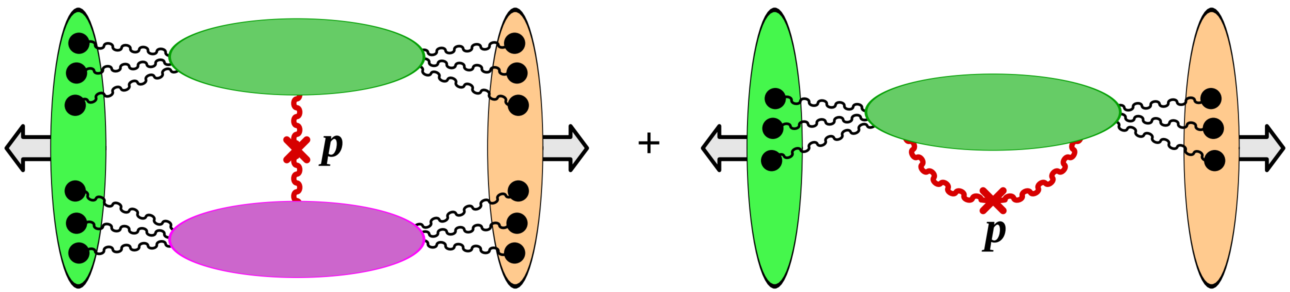
- Write : $\mathcal{C} = \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}} \underbrace{\int_{x,y} e^{ip\cdot(x-y)}\, \Box_x \Box_y \frac{\delta}{\delta j_+(x)} \frac{\delta}{\delta j_-(y)}}_{\mathcal{C}_{\boldsymbol{p}}}$

- The functional $F[z]$ can be written as :

$$F[z] = \exp\left\{ \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}}\, z(\vec{\boldsymbol{p}})\, \mathcal{C}_{\boldsymbol{p}} \right\} e^{iV[j_+]}\, e^{-iV^*[j_-]}\Big|_{j_+=j_-}$$

- By analogy with $\overline{N}$, we have :

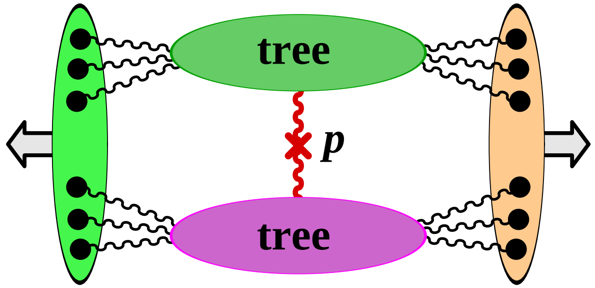
$$\frac{1}{F[z]} \frac{\delta F[z]}{\delta z(\vec{\boldsymbol{p}})} =$$



Note : cut propagators are modified : $G^0_{+-}(p) \to z(\vec{\boldsymbol{p}}) G^0_{+-}(p)$

- At leading order, this is given by the same topologies of diagrams as those involved in $\overline{N}$:

$$\frac{1}{F[z]}\frac{\delta F[z]}{\delta z(\vec{\boldsymbol{p}})} = \sum_{\text{trees}} \sum_{\text{cuts}}$$



but the internal cut propagators are multiplied by $z(\vec{\boldsymbol{p}})$

- One can also write it in terms of two “fields” $\mathcal{A}_\pm(x)$ as :

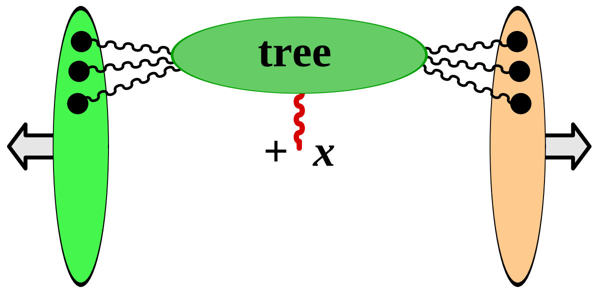
$$\frac{1}{F[z]}\frac{\delta F[z]}{\delta z(\vec{\boldsymbol{p}})} = \int_{x,y} e^{ip\cdot(x-y)} \cdots \mathcal{A}_+^\mu(x)\mathcal{A}_-^\nu(y)$$

Note : this formula is formally identical to the formula for the inclusive spectrum, but the fields $\mathcal{A}_\pm$ it contains are different

F[z] at Leading Order

- $\mathcal{A}_+^\mu(x)$ is the sum of cut tree diagrams ending at the point x :

$$\mathcal{A}_+^\mu(x) = \sum_{\text{trees}} \sum_{\text{cuts}}$$



- $\mathcal{A}_\pm$ is a solution of the classical EOMs
- It obeys the boundary condition :

$$\int d^3\vec{\boldsymbol{y}} \left[G^0_{++}(x,y) \overleftrightarrow{\partial^0_y} \mathcal{A}_+(y) - G^0_{+-}(x,y) \overleftrightarrow{\partial^0_y} \mathcal{A}_-(y) \right]_{-\infty}^{+\infty} = 0$$

Note : although everything is formally identical to the case of the single inclusive spectrum, the boundary conditions are quite different because all the cut propagators are now multiplied by $z(\boldsymbol{p})$

F[z] at Leading Order

- The boundary conditions can be rewritten in terms of the Fourier coefficients as :

$$a_+^{(+)}(-\infty,\vec{\boldsymbol{p}}) = a_-^{(-)}(-\infty,\vec{\boldsymbol{p}}) = 0$$
$$a_-^{(+)}(+\infty,\vec{\boldsymbol{p}}) = z(\vec{\boldsymbol{p}})\; a_+^{(+)}(+\infty,\vec{\boldsymbol{p}})$$
$$a_+^{(-)}(+\infty,\vec{\boldsymbol{p}}) = z(\vec{\boldsymbol{p}})\; a_-^{(-)}(+\infty,\vec{\boldsymbol{p}})$$

- The function $z(\vec{\boldsymbol{p}})$ enters only via the boundary conditions

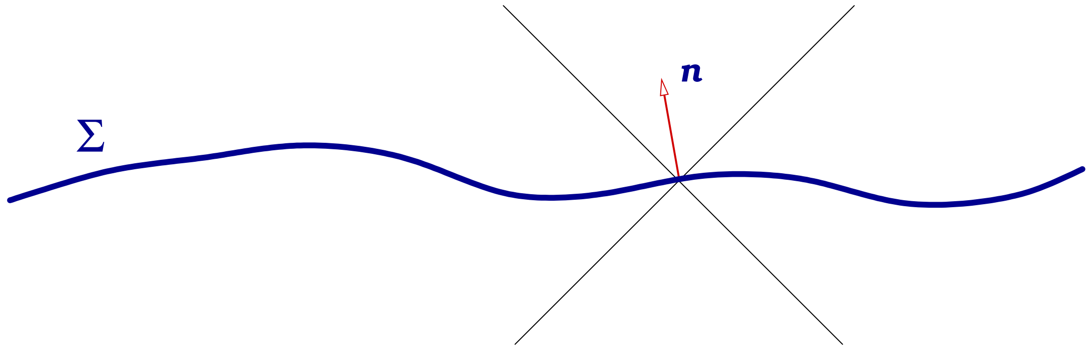
- The only difference at leading order between inclusive quantities and more exclusive ones comes from the boundary conditions

- These boundary conditions are not retarded
  - ▷ very difficult to solve numerically

Surgery of retarded graphs

Inclusive observables

- Inclusive observables share two interesting properties :
  - at LO, they are expressible in terms of retarded fields
  - they depend only on the $t \to +\infty$ limit of these fields
- Retarded fields are fully determined by their value on some arbitrary (locally space-like) initial surface :



Green's formula for classical fields

- The formal relationship between the value of a retarded classical field and its value on the initial surface is provided by a Green's formula :

$$\mathcal{A}(x) = \int_{\Sigma^+} d^4y \, G^0_R(x,y) \Big[j(y) - U'(\mathcal{A}(y)) \Big] + \int_{\Sigma} d^3_\Sigma \vec{\boldsymbol{u}} \, G^0_R(x,u) \, [n \cdot \overleftrightarrow{\partial}_u \,] \mathcal{A}(u)$$

Σ^+	domain above the surface Σ
$d^3_\Sigma \vec{\boldsymbol{y}}$	measure on the surface Σ
n^μ	vector normal to Σ at the point y
$U(\mathcal{A}(y))$	interaction potential of the fields
$G^0_R(x,y)$	free retarded propagator

Small fluctuations over the classical field

- Consider a small fluctuation $a(x)$ propagating over $\mathcal{A}(x)$. Its EOM is :

$$\Big[\square_x + U''(\mathcal{A}(x))\Big]\, a(x) = 0$$

- If it also obeys retarded boundary conditions, one has the following Green's formula :

$$\begin{aligned}\mathcal{A}(x) \quad &= \quad \int_{\Sigma+} d^4y\, G^0_{_R}(x,y)\,\Big[-U''(\mathcal{A}(y))a(y)\Big] \\ &+ \int_{\Sigma} d^3_{_\Sigma}\vec{\boldsymbol{u}}\; G^0_{_R}(x,u)\,\big[n\cdot \overleftrightarrow{\partial}_u\,\big]a(u)\end{aligned}$$

Small fluctuations over the classical field

- Apply a linear differential operator $\boldsymbol{T}$ on $\mathcal{A}$'s Green's formula :

$$\boldsymbol{T}\mathcal{A}(x) = \int_{\Sigma^+} d^4y\, G^0_R(x,y)\Big[-U''(\mathcal{A}(y))(\boldsymbol{T}\mathcal{A}(y))\Big] + \boldsymbol{T}\int_\Sigma d^3_\Sigma \vec{\boldsymbol{u}}\, G^0_R(x,u)\,[n\cdot \overset{\leftrightarrow}{\partial}_u\,]\mathcal{A}(u)$$

▷ we can identify $a(x) = \boldsymbol{T}\mathcal{A}(x)$ provided that

$$\boldsymbol{T}\int_\Sigma d^3_\Sigma \vec{\boldsymbol{u}}\, G^0_R(x,u)\,[n\cdot \overset{\leftrightarrow}{\partial}_u\,]\mathcal{A}(u) = \int_\Sigma d^3_\Sigma \vec{\boldsymbol{u}}\, G^0_R(x,u)\,[n\cdot \overset{\leftrightarrow}{\partial}_u\,]a(u)$$

- This is fulfilled if

$$\boldsymbol{T} \equiv \int_\Sigma d^3_\Sigma \vec{\boldsymbol{u}}\,\underbrace{\Big[a(u)\frac{\delta}{\delta\mathcal{A}(u)} + (n\cdot\partial a(u))\frac{\delta}{\delta(n\cdot\partial\mathcal{A}(u))}\Big]}_{a\cdot\mathbb{T}_{\boldsymbol{u}}}$$

▷ $\mathbb{T}_{\boldsymbol{u}}$ is the generator of shifts of the field at the point $\vec{\boldsymbol{u}}$ on Σ

Useful trick

- Translation operator for the initial field on Σ :

$$\exp \int_{\Sigma} d^3\vec{\boldsymbol{u}}\; a(\vec{\boldsymbol{u}}) \cdot \mathbb{T}_{\boldsymbol{u}}$$

- Action of a functional $F[\mathbb{T}_{\boldsymbol{u}}]$ on a functional $G[\mathcal{A}]$:
  - $G[\mathcal{A}]$ can be seen as a functional of the initial field on Σ, thanks to the retarded boundary conditions :

$$G[\mathcal{A}] \quad \rightarrow \quad G[\mathcal{A}_\Sigma]$$

  - Introduce the "Laplace transform" of $F[\mathbb{T}_{\boldsymbol{u}}]$:

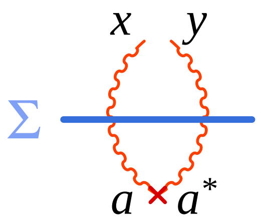
$$F[\mathbb{T}_{\boldsymbol{u}}] \equiv \int [Da(\vec{\boldsymbol{u}})]\; F[a(\vec{\boldsymbol{u}})]\; \exp \int_{\Sigma} d^3\vec{\boldsymbol{u}}\; a(\vec{\boldsymbol{u}}) \cdot \mathbb{T}_{\boldsymbol{u}}$$

  - Then, one gets :

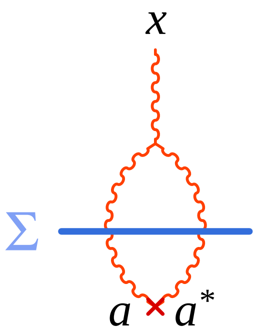
$$F[\mathbb{T}_{\boldsymbol{u}}]\; G[\mathcal{A}_\Sigma] = \int [Da(\vec{\boldsymbol{u}})]\; F[a(\vec{\boldsymbol{u}})]\; G[\mathcal{A}_\Sigma + a]$$

Loop corrections over the classical field

- The loop corrections involved in the NLO corrections of the inclusive gluon spectrum can be written in terms of the operator $\mathbb{T}_{\boldsymbol{u}}$:



$$= \int_{\Sigma} d^3_{\Sigma}\vec{\boldsymbol{u}}\, d^3_{\Sigma}\vec{\boldsymbol{v}}\, \Big[(a\cdot\mathbb{T}_{\boldsymbol{u}})\mathcal{A}(x)\Big]\Big[(a^*\cdot\mathbb{T}_{\boldsymbol{v}})\mathcal{A}(y)\Big]$$



$$= \int_{\Sigma} d^3_{\Sigma}\vec{\boldsymbol{u}}\, d^3_{\Sigma}\vec{\boldsymbol{v}}\, (a\cdot\mathbb{T}_{\boldsymbol{u}})(a^*\cdot\mathbb{T}_{\boldsymbol{v}})\mathcal{A}(x)$$

- The main reason why these formulas are extremely useful is that, even though we do not know $\mathcal{A}(x)$ analytically, we can calculate the fluctuation $a(x)$ on the surface Σ