Initial conditions in heavy ion collisions
Lecture III: Next to leading order, factorization

Tuomas Lappi
IPhT, CEA/Saclay
tuomas.lappi@cea.fr

Goa, September 2008

Outline

- Warmup: quark pair production
- Another warmup: the weak field limit and BFKL
- General NLO corrections
- JIMWLK factorization

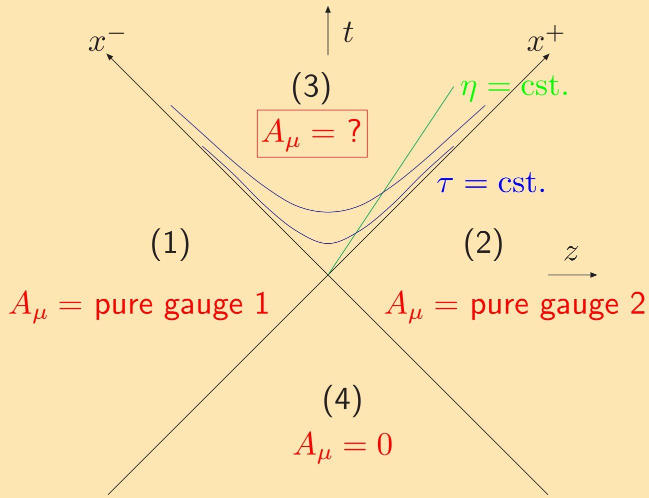
Recall: Weizsäcker-Williams color field, MV model

Separation of scales between small x and large x:

classical field **color charge**

$$[D_\mu, F^{\mu\nu}] \quad = \quad J^\nu$$

$$J^\mu = \delta^{\mu +}\rho_{(1)}(\boldsymbol{x})\delta(x^-) + \delta^{\mu -}\rho_{(2)}(\boldsymbol{x})\delta(x^+)$$



Charge density $\rho(\boldsymbol{x})$: stochastic variable, distribution

$$\boxed{W_y[\rho(\boldsymbol{x})]}$$

E.g. MV model [1]: $W[\rho(\boldsymbol{x})] \sim \exp\left[-\frac{1}{2}\int \mathrm{d}^2\boldsymbol{x}\rho^a(\boldsymbol{x})\rho^a(\boldsymbol{x})/g^2\mu^2\right]$

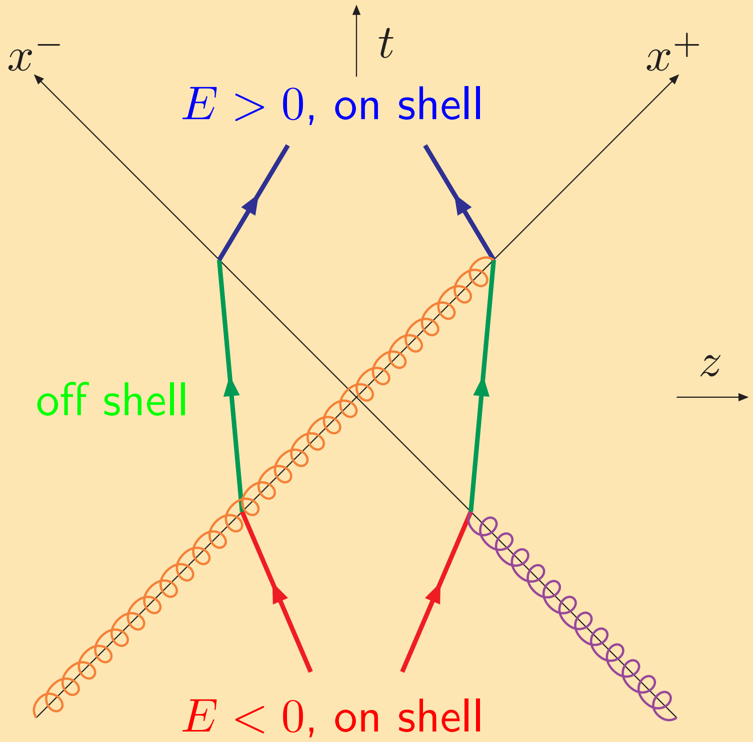
Evolution equation for $y = \ln 1/x$–dependence: **JIMWLK**.

[1] L. D. McLerran and R. Venugopalan, *Phys. Rev.* **D49** (1994) 2233 [hep-ph/9309289].

Dirac equation in external field

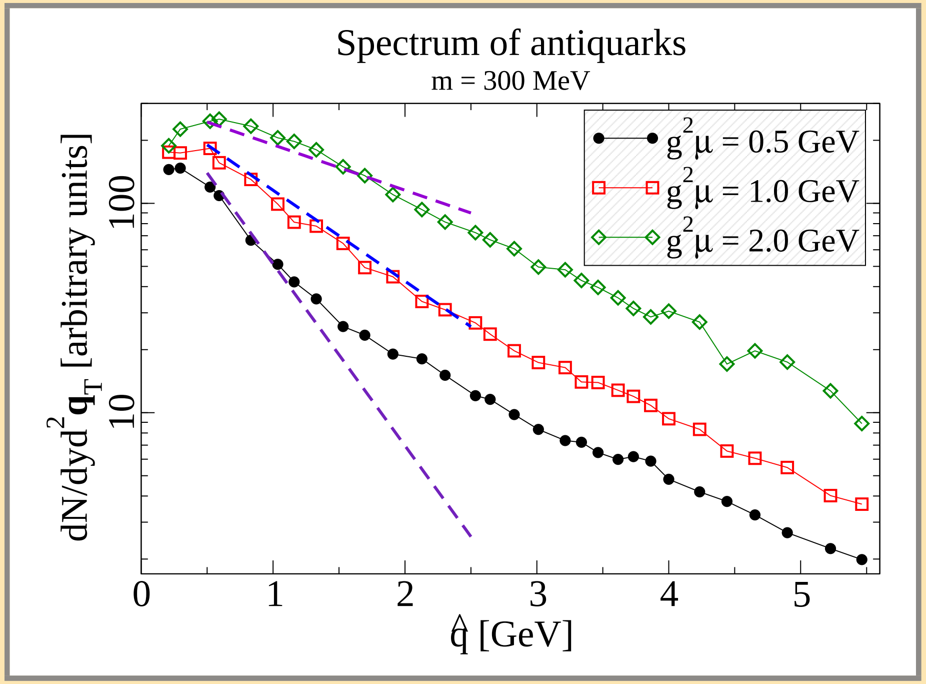
The QED calculation can be done analytically (see e.g. Baltz & McLerran [2] or Baltz, Gelis, McLerran & Peshier[3]).

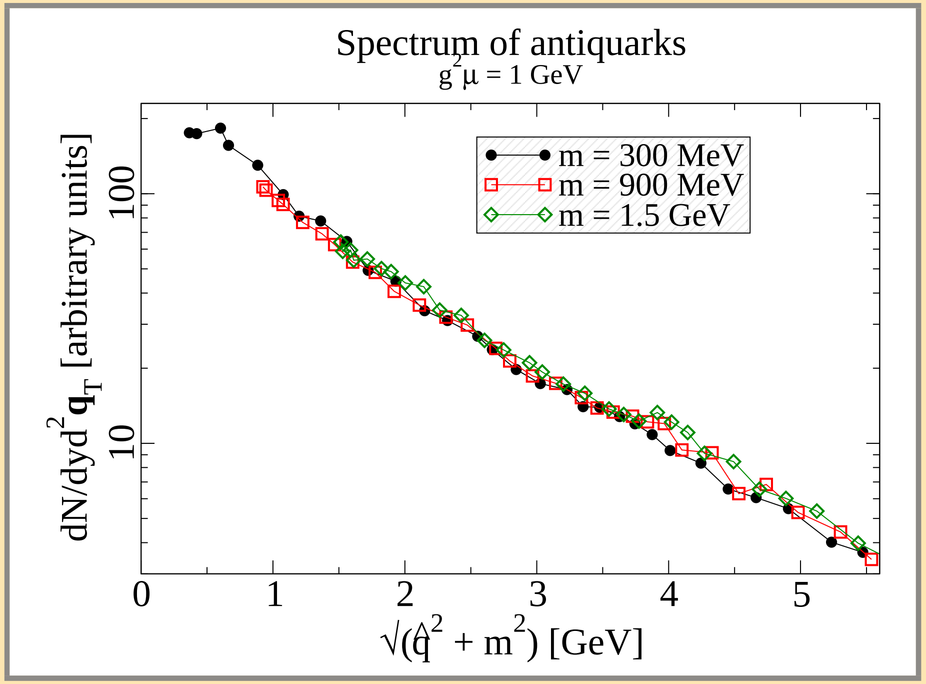
- Initial condition: negative energy plane wave.
- Solution numerical for $\tau > 0$, because of interaction with the background field that is known only numerically.
- Final state: project to positive energy plane wave.
- Two separate "branches" of the solution.



[2] A. J. Baltz and L. D. McLerran, *Phys. Rev.* **C58** (1998) 1679 [nucl-th/9804042].
[3] A. J. Baltz, F. Gelis, L. D. McLerran and A. Peshier, *Nucl. Phys.* **A695** (2001) 395 [nucl-th/0101024].
[4] F. Gelis, K. Kajantie and T. Lappi, *Phys. Rev.* **C71** (2005) 024904 [hep-ph/0409058].

(Anti)quark spectrum



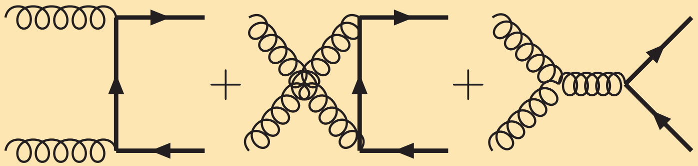


[5] F. Gelis, K. Kajantie and T. Lappi, *Phys. Rev. Lett.* **96** (2006) 032304 [hep-ph/0508229].

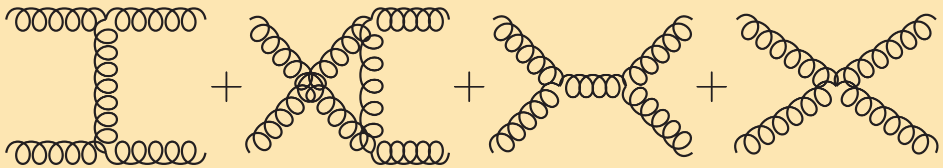
So, what about gluon pairs?

So we can produce pairs of quarks, how about pairs of gluons?

$2 \longrightarrow 2$ processes in weak field limit: diagrams:



vs.



$$s + u + t = 0$$

High energy limit: $t \sim p_T^2 \sim$ **const,** $s \sim -u \longrightarrow \infty$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} \sim |M|^2/s^2$$

Quarks: $|M|^2 \sim (s/t)$, $\frac{\mathrm{d}\sigma}{\mathrm{d}t} \sim 1/s$ Gluons: $|M|^2 \sim (s/t)^2$, $\frac{\mathrm{d}\sigma}{\mathrm{d}t} \sim 1$

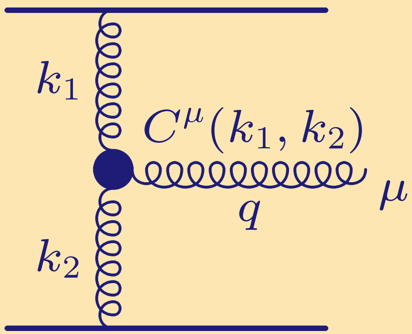
Spin of object in t-channel! Graviton exchange diverges like s.

This is why high energy scattering is so close to unitarity limit (and quantum is so difficult).

Reminder: weak field limit in covariant gauge

$$C(k_1, k_2) = \left(q^+ - \frac{\underset{\sim}{\boldsymbol{k}}_1^2}{q^-} \, , \, \frac{\underset{\sim}{\boldsymbol{k}}_2^2}{q^+} - q^- \, , \, \underset{\sim}{\boldsymbol{k}}_2 - \underset{\sim}{\boldsymbol{k}}_1 \right) \quad q^\mu C_\mu = 0 \quad C^\mu C_\mu = 4 \frac{\underset{\sim}{\boldsymbol{k}}_1^2 \underset{\sim}{\boldsymbol{k}}_2^2}{\underset{\sim}{\boldsymbol{q}}^2}$$

$$A^\mu(q) = J^+_{(1)}(k_1) J^-_{(2)}(k_2) \frac{g^{+-}}{\underset{\sim}{\boldsymbol{k}}_1^2} \frac{g^{+-}}{\underset{\sim}{\boldsymbol{k}}_1^2} C^\mu(k_1, k_2),$$



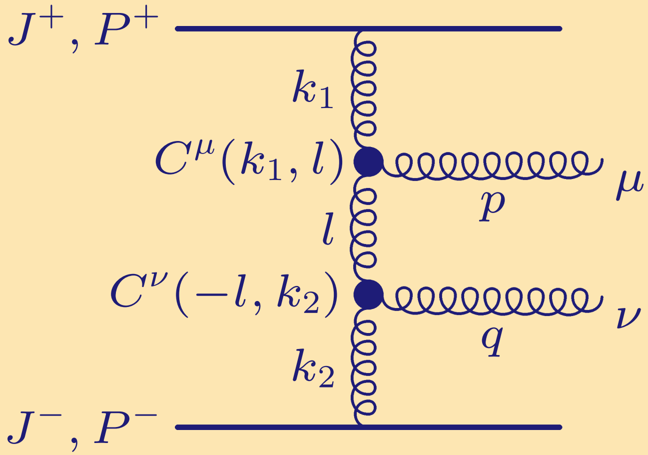
- Many diagrams, but leading high energy magically simplifies into effective **Lipatov vertex** C^μ
- Only $\pm$-component propagating down the t-channel
- $C^\mu C_\mu$ kills half of the t-channel propagators

Result:

$$\frac{\mathrm{d}N}{\mathrm{d}y \, \mathrm{d}^2\underset{\sim}{\boldsymbol{q}}} \sim \int_{\underset{\sim}{\boldsymbol{k}}_1, \underset{\sim}{\boldsymbol{k}}_2} \overbrace{\frac{\phi_y(\underset{\sim}{\boldsymbol{k}}_1)}{\underset{\sim}{\boldsymbol{k}}_1^2}}^{\mathcal{O}(1/\alpha_\mathrm{s})} \frac{\phi_y(\underset{\sim}{\boldsymbol{k}}_2)}{\underset{\sim}{\boldsymbol{k}}_2^2} \delta^2(\underset{\sim}{\boldsymbol{q}} - \underset{\sim}{\boldsymbol{k}}_1 - \underset{\sim}{\boldsymbol{k}}_2) \left[\alpha_\mathrm{s} \frac{\underset{\sim}{\boldsymbol{k}}_1^2 \underset{\sim}{\boldsymbol{k}}_2^2}{\underset{\sim}{\boldsymbol{q}}^2} \right]$$

Weak field limit: emit another gluon

- Emit another gluon from the ladder, with its own Lipatov vertex
- We're assuming rapidity ordering: $P^+ \gg p^+ \gg q^+ \gg P_T^2/P^-$
- Remember $\varphi(\boldsymbol{k}) \sim 1/\alpha_s$, so this is $\mathcal{O}(\alpha_s^0)$



$$\frac{\mathrm{d}N}{\mathrm{d}y_q\,\mathrm{d}^2\boldsymbol{q}\,\mathrm{d}y_p\,\mathrm{d}^2\boldsymbol{p}} \sim \int_{\boldsymbol{k}_1,\boldsymbol{k}_2} \frac{\phi_y(\boldsymbol{k}_1)}{\boldsymbol{k}_1^2}\frac{\phi_y(\boldsymbol{k}_2)}{\boldsymbol{k}_2^2} \overbrace{\frac{\delta^2(\boldsymbol{q}+\boldsymbol{p}-\boldsymbol{k}_1-\boldsymbol{k}_2)}{(\boldsymbol{k}_1-\boldsymbol{p})^2(\boldsymbol{k}_2-\boldsymbol{q})^2}}^{l\text{-propagator}^2} \left[\alpha_s^2 \overbrace{\frac{\boldsymbol{k}_1^2(\boldsymbol{k}_1-\boldsymbol{p})^2}{\boldsymbol{p}^2}}^{C^\mu C_\mu} \overbrace{\frac{\boldsymbol{k}_2^2(\boldsymbol{k}_2-\boldsymbol{q})^2}{\boldsymbol{q}^2}}^{C^\nu C_\nu}\right]$$

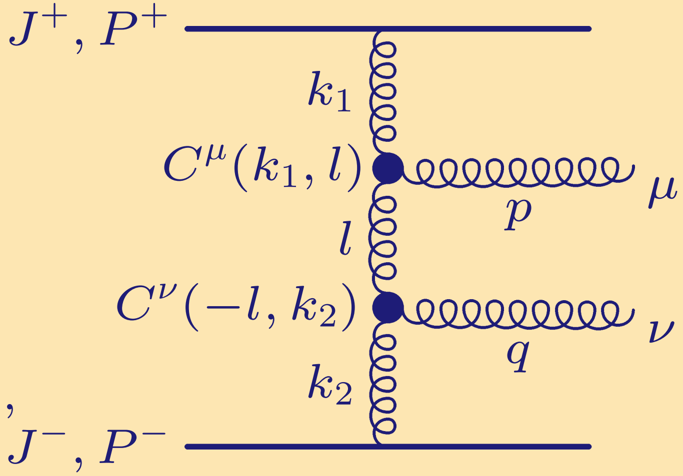
Suppose we want to integrate out p-gluon: **y_p-integral diverges**.

- Large p^+ corresponds to large k_1^+, physically this is of course cut off by finite $\sqrt{s}$, but at large $\sqrt{s}$ this is not the solution we're looking for.

▶ There is also a rapidity dependence in $\varphi_y(\boldsymbol{k}_1)$, this is where the divergence has to go.

Pair production and real part of BFKL

Regulate divergence with a cutoff Y and combine two leading terms



$$\frac{\mathrm{d}N}{\mathrm{d}y\,\mathrm{d}^2\mathbf{q}} \sim \alpha_\mathrm{s} \int_{\mathbf{k}_2} \frac{\phi_y(\mathbf{k}_2)}{\mathbf{q}^2}\Bigg[\int_{\mathbf{k}_1} \delta^2(\mathbf{q}-\mathbf{k}_1-\mathbf{k}_2)\phi_Y(\mathbf{k}_1) + \frac{N_\mathrm{c}\alpha_\mathrm{s}}{\pi^2}\int^Y \mathrm{d}y_p \int_{\mathbf{p},\mathbf{k}_1} \delta^2(\mathbf{p}+\mathbf{q}-\mathbf{k}_1-\mathbf{k}_2)\frac{\phi_Y(\mathbf{k}_1)}{\mathbf{p}^2}\Bigg],$$

There is no Y-dependence if

$$\partial_Y \phi_Y(\mathbf{q}-\mathbf{k}_2) = -\frac{N_\mathrm{c}\alpha_\mathrm{s}}{\pi^2}\int_{\mathbf{k}_1} \frac{\phi_Y(\mathbf{k}_1)}{(\mathbf{q}-\mathbf{k}_1-\mathbf{k}_2)^2}$$

This is the real part of the BFKL equation – sign because evolution is in the negative Y direction. cf. Cyrille

Then there are loop corrections, which in the same way factorize into the virtual term of BFKL.

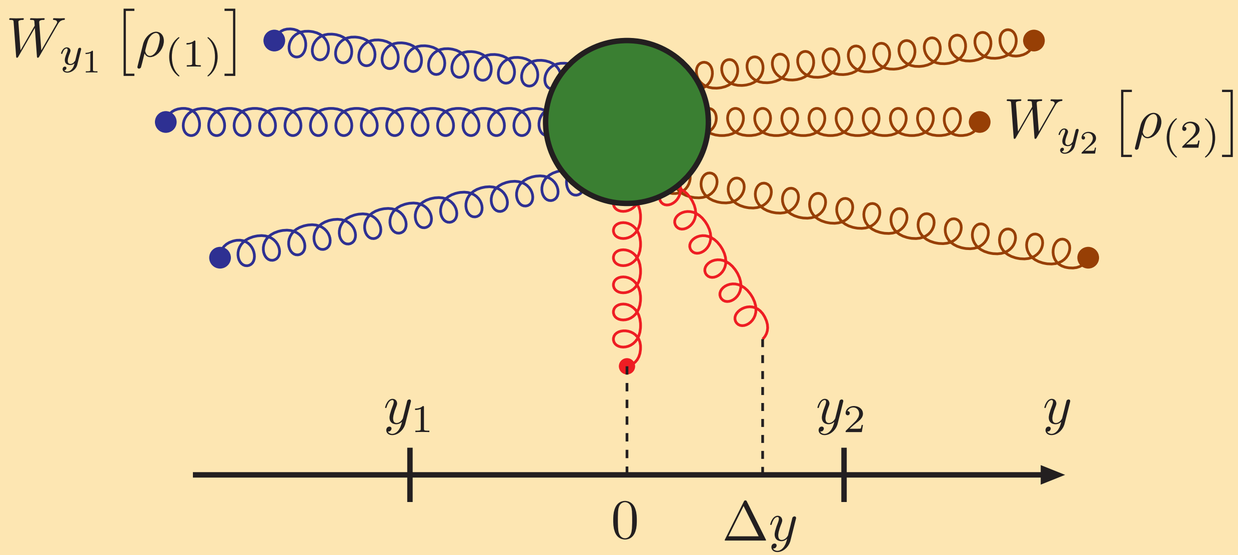
Physics is independent of the cutoff, to appropriate order in α_s

NLO corrections, factorization: JIMWLK

Restrict $y_1 < \Delta y < y_2$

▼

Physics indep. of $y_1,\ y_2$
(to appropriate order in α_{s}).



$$\boldsymbol{\nabla}_T{}^2 \mathcal{A}^+(\boldsymbol{y}) = -g\rho(\boldsymbol{y})$$

$$U(\boldsymbol{x}) = \mathrm{P}e^{i\int \mathrm{d}y^- \mathcal{A}^+(\boldsymbol{x},y^-)}$$

Sources W evolve with **JIMWLK Hamiltonian**:

$$\mathcal{H} \equiv \frac{1}{2}\int \mathrm{d}^2\boldsymbol{x}\,\mathrm{d}^2\boldsymbol{y}\frac{\delta}{\delta\widetilde{\mathcal{A}}_c^+(\boldsymbol{y})}\eta^{bc}(\boldsymbol{x},\boldsymbol{y})\frac{\delta}{\delta\widetilde{\mathcal{A}}_b^+(\boldsymbol{x})}$$

$$\eta^{bc}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\pi}\int\frac{d^2\boldsymbol{u}}{(2\pi)^2}\frac{(\boldsymbol{x}-\boldsymbol{u})\cdot(\boldsymbol{y}-\boldsymbol{u})}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{u})^2}$$
$$\times\left[U(\boldsymbol{x})U^\dagger(\boldsymbol{y}) - U(\boldsymbol{x})U^\dagger(\boldsymbol{u}) - U(\boldsymbol{u})U^\dagger(\boldsymbol{y}) + 1\right]_{bc}$$

JIMWLK factorization

I am not going through the derivation, in:

- F. Gelis, T. L. and R. Venugopalan, "High energy factorization in nucleus-nucleus collisions," arXiv:0804.2630 [hep-ph] (PRD tbp).
- F. Gelis, T. L. and R. Venugopalan, "High energy factorization in nucleus-nucleus collisions II — Multigluon correlations," arXiv:0807.1306 [hep-ph] (PRD tbp).

and very much based on the work in:

- F. Gelis and R. Venugopalan, "Particle production in field theories coupled to strong external sources," Nucl. Phys. A **776** (2006) 135 [arXiv:hep-ph/0601209].
- F. Gelis and R. Venugopalan, "Particle production in field theories coupled to strong external sources. II: Generating functions," Nucl. Phys. A **779** (2006) 177 [arXiv:hep-ph/0605246].

I will, in stead, try to describe a few ingredients that go into the calculation

- Schwinger-Keldysh, retarded propagation
- Propagators as functional derivatives w.r.t. the initial condition

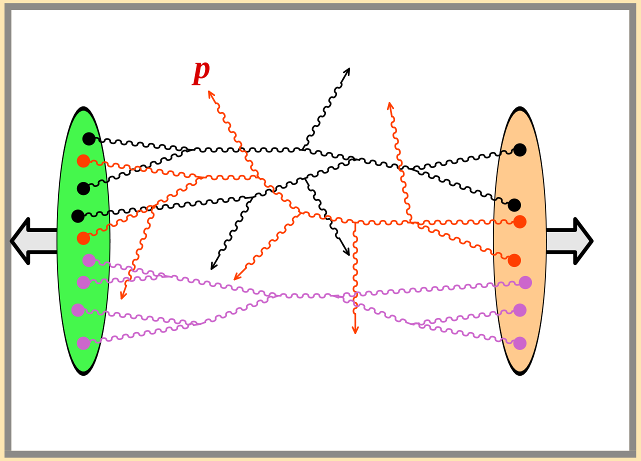
Gluon multiplicity as cut vacuum graphs

$$J^\mu \sim 1/g$$

Particle production with strong external sources Gelis, Venugopalan [6], compute multiplicity

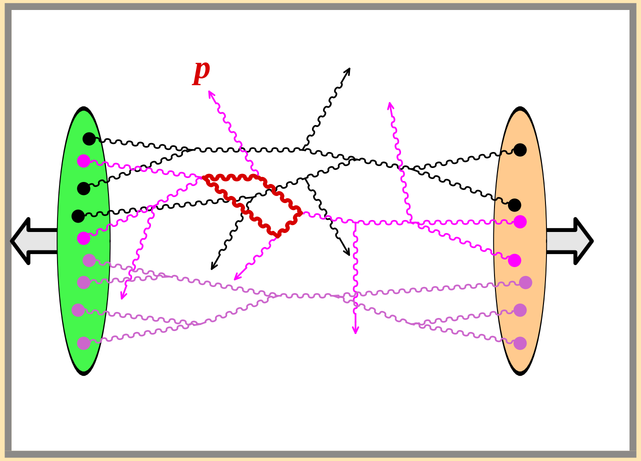
$$\frac{dN}{d^3\vec{\boldsymbol{p}}} \sim \sum_{n=0}^{\infty} \frac{1}{n!} \int \left[d^3\vec{\boldsymbol{p}}_1 \cdots d^3\vec{\boldsymbol{p}}_n \right] \; \left| \langle \vec{\boldsymbol{p}} \;\; \vec{\boldsymbol{p}}_1 \cdots \vec{\boldsymbol{p}}_n | 0 \rangle \right|^2$$

All insertions of source at same order



◀ LO: tree diagrams

NLO: 1 loop ▶



Integrate phase space of additional gluons.

[6] F. Gelis and R. Venugopalan, *Nucl. Phys.* **A776** (2006) 135 [hep-ph/0601209].

A small word on $i\varepsilon$

$$\frac{dN}{d^3\vec{\boldsymbol{p}}} \sim \int_{\vec{\boldsymbol{x}}\vec{\boldsymbol{y}}} e^{i\vec{\boldsymbol{p}}\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})} \underbrace{(\cdots)}_{\text{proj.}} \underbrace{\left[A^\mu(t,\vec{\boldsymbol{x}})A^\nu(t,\vec{\boldsymbol{y}})\right]}_{\text{no } T\text{-product!}}\Bigg|_{t\to\infty}$$

We are computing multiplicities from theory with sources, not scattering amplitudes

$$P_1 = \left|\langle 0_{\text{out}}|\, a^\dagger_{\text{out}}\, |0_{\text{in}}\rangle\right|^2$$

▼

Time-ordered, i.e. Feynman propagator, cross sections, analytical s-matrix, crossing symmetry

▼

Usual perturbative calculation

$$\langle n\rangle = \langle 0_{\text{in}}|\, a^\dagger_{\text{out}} a_{\text{out}}\, |0_{\text{in}}\rangle$$

▼

No time ordering. Use Schwinger-Keldysh formalism, leads to retarded propagators.

▼

Could be numerically doable in real time.

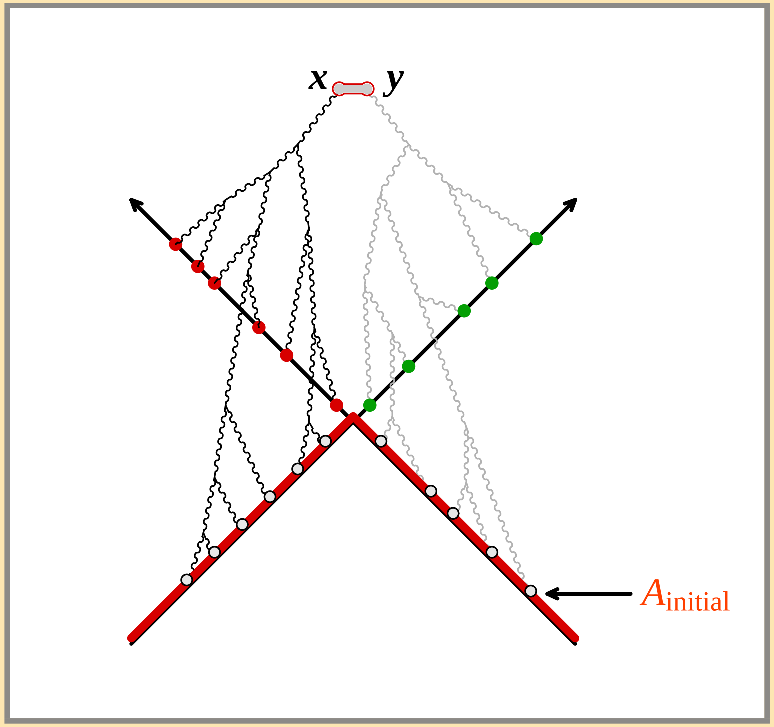
See e.g. Baltz, Gelis; McLerran [3] for discussion in fermion pair context

[3] A. J. Baltz, F. Gelis, L. D. McLerran and A. Peshier, *Nucl. Phys.* **A695** (2001) 395 [nucl-th/0101024].

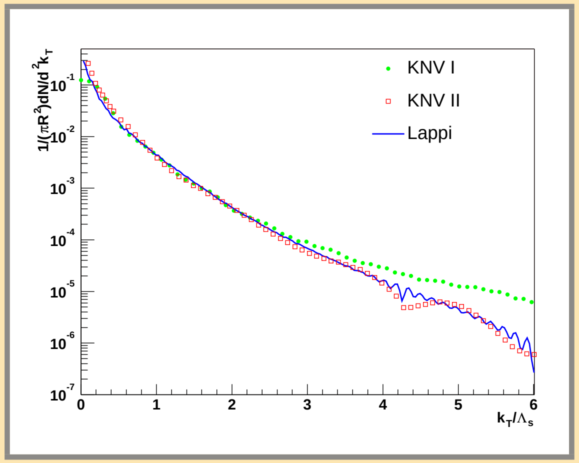
LO is classical field

Leading order multiplicity from **retarded** solution of classical field equations.

$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\mathrm{LO}} = \int_{\vec{\boldsymbol{x}}\vec{\boldsymbol{y}}} e^{i\vec{\boldsymbol{p}}\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})}\ (\cdots)\left[\mathcal{A}^{\mu}(t,\vec{\boldsymbol{x}})\mathcal{A}^{\nu}(t,\vec{\boldsymbol{y}})\right]\Big|_{t\to\infty}$$



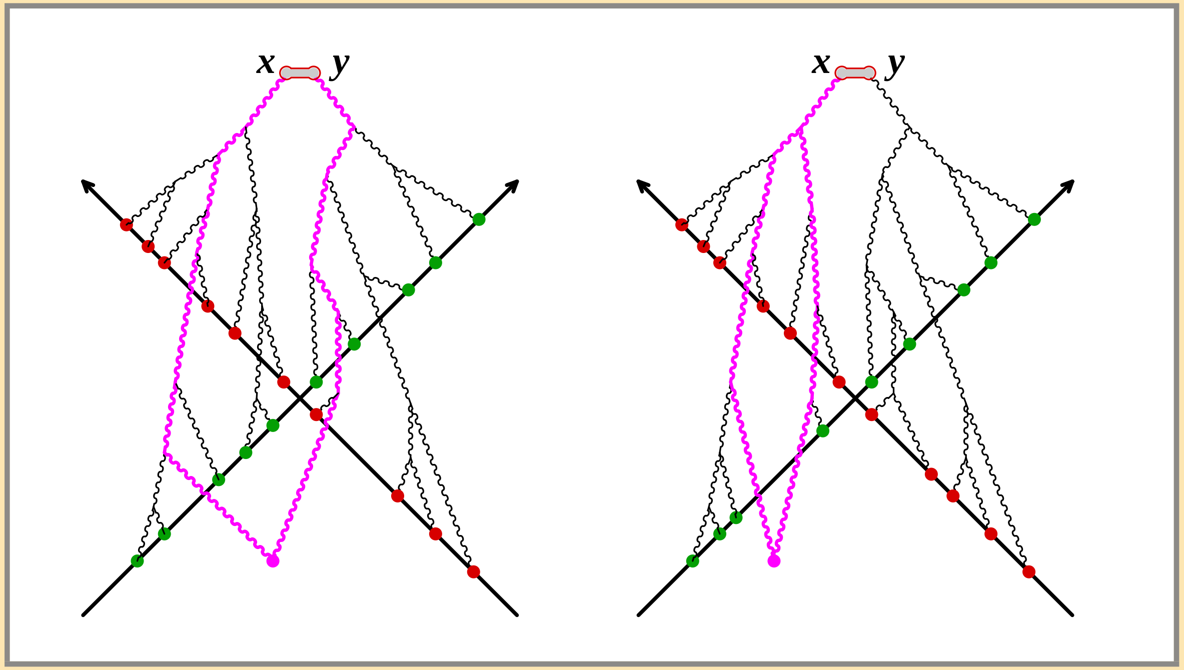
Gluon spectrum from numerical computation



View multiplicity as functional of classical field on initial surface.

NLO is 1 loop

“real”, pair production

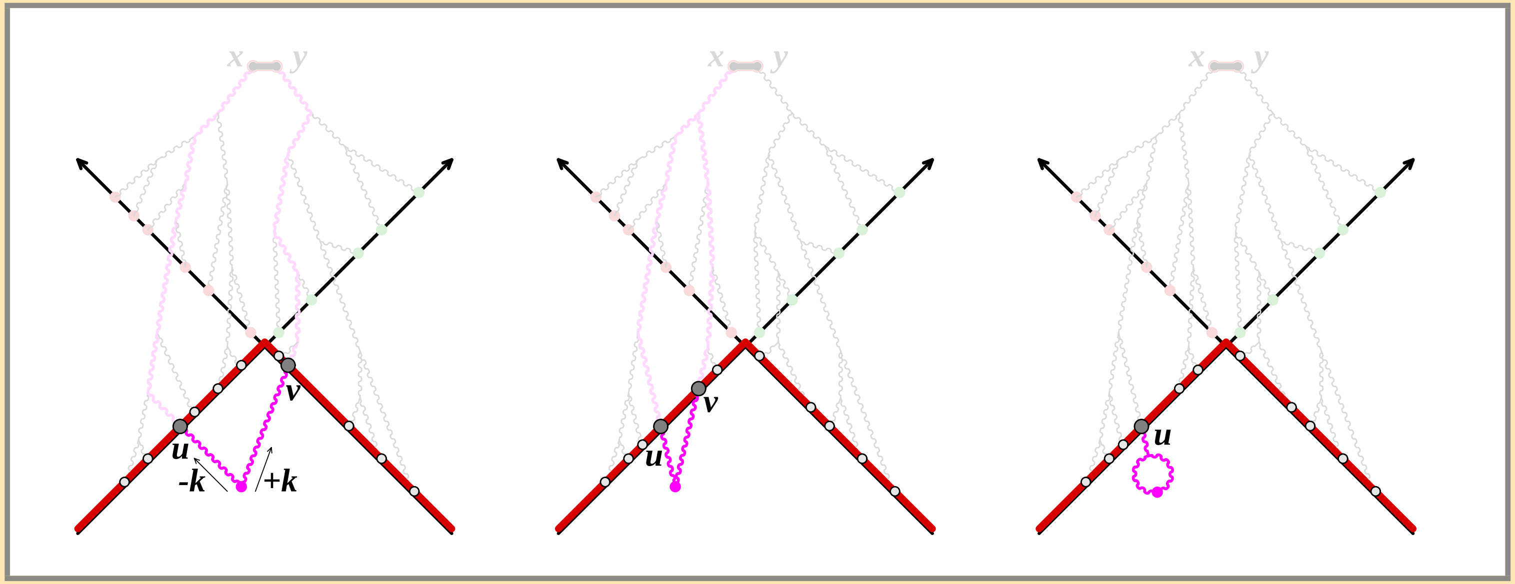


“virtual”, loop correction to field

$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\text{NLO}} = \int_{\vec{\boldsymbol{x}}\vec{\boldsymbol{y}}} e^{i\vec{\boldsymbol{p}}\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})}\,(\cdots)\Big[\mathcal{G}^{\mu\nu}(x,y) + \beta^{\mu}(t,\vec{\boldsymbol{x}})\,\mathcal{A}^{\nu}(t,\vec{\boldsymbol{y}}) + \mathcal{A}^{\mu}(t,\vec{\boldsymbol{x}})\,\beta^{\nu}(t,\vec{\boldsymbol{y}})\Big]\Big|_{t\to\infty}$$

- $\mathcal{G}^{\mu\nu}$ is a 2-point function on top of the classical field
- β^{μ} is a small field fluctuation driven by a 1-loop source

NLO: propagators as functional derivatives



$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\text{NLO}} = \underbrace{\left[\frac{1}{2}\int_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\in\text{LC}} \mathcal{G}(\underset{\sim}{\boldsymbol{u}},\underset{\sim}{\boldsymbol{v}})\mathbb{T}_{\underset{\sim}{\boldsymbol{u}}}\mathbb{T}_{\underset{\sim}{\boldsymbol{v}}} + \int_{\vec{\boldsymbol{u}}\in\text{LC}} \beta(\underset{\sim}{\boldsymbol{u}})\mathbb{T}_{\underset{\sim}{\boldsymbol{u}}}\right]}_{\text{below LC}} \underbrace{\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\text{LO}}}_{\text{above LC}}$$

$$a^\mu(x) = \int_{\vec{\boldsymbol{u}}\in\text{LC}} a(\vec{\boldsymbol{u}})\cdot\mathbb{T}_{\underset{\sim}{\boldsymbol{u}}}\mathcal{A}^\mu(x) \qquad \mathcal{G}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) \equiv \int \frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3 2E_{\boldsymbol{k}}}\,\eta_{-\boldsymbol{k}}(u)\,\eta_{+\boldsymbol{k}}(v)$$

Divergence from $\int \frac{\mathrm{d}k^+}{k^+}$ Functional derivative in $\mathbb{T}$ ▶ $\frac{\delta}{\delta\widetilde{\mathcal{A}}_c^+(\underset{\sim}{\boldsymbol{y}})}$ in JIMWLK

Dominant polarization

In general the functional derivative can be written as (on the right-movers' LC)

$$a \cdot \mathbb{T}_u = \partial^-(U(u)a^i(u))\frac{\delta}{\delta\big(\partial^-(U(u)\mathcal{A}^i(u))\big)}$$

$$+ U(u)a^-(u)\frac{\delta}{\delta\big(U(u)\mathcal{A}^-(u)\big)} + \overbrace{\partial^\mu(U(u)a_\mu(u))\frac{\delta}{\delta\big(\partial^\mu(U(u)\mathcal{A}_\mu(u))\big)}}^{\text{longit.}}$$

In the LC gauge, these 3 components fully specify the field on the LC.

Only the longitudianal component matters. Why: only one that is there in the WW-field

Some aspects of the factorization theorem

- High energy kinematics: fixed $Q^2 \sim Q_{\mathrm{s}}^2$, large $\sqrt{s} \sim e^y$ ▶ weak field limit is BFKL
- Not factorization of pdf's, but color charge distributions
- Power counting: sources $\sim 1/g$
  - Nonperturbative, all orders in classical field
  - NLO in weak coupling/loop expansion (not all orders)
- Work with multiplicities, not cross sections
  - Most natural thing to look at in multiparticle production
  - Retarded propagation
  - Diffractive observables **?**
- Express retarded propagators as functional derivatives wrt. initial condition ▶ relate to functional derivatives in JIMWLK Hamiltonian

JIMWLK evolution in Langevin form

Analogy: diffusion equation

$$\partial_t P(x,t) = D\partial_x P(\vec{x},t)$$

Equivalent to **Langevin equation**

$$\dot{x} = \sqrt{2D}\eta(t), \qquad \langle \eta(t)\eta(t')\rangle = \delta(t-t')$$

Blaizot, Iancu, Weigert [7]: can take $\sqrt{\cdot}$ of JIMWLK Hamiltonian:

$$\mathcal{H} \equiv \frac{1}{2}\int \mathrm{d}^2\underset{\sim}{\boldsymbol{x}}\,\mathrm{d}^2\underset{\sim}{\boldsymbol{y}}\,\mathrm{d}^2\underset{\sim}{\boldsymbol{z}} \frac{\delta}{\delta \widetilde{\mathcal{A}}_c^+(\underset{\sim}{\boldsymbol{y}})} \underset{\sim}{\boldsymbol{e}}^{ba}(\underset{\sim}{\boldsymbol{x}},\underset{\sim}{\boldsymbol{z}}) \cdot \underset{\sim}{\boldsymbol{e}}^{ca}(\underset{\sim}{\boldsymbol{y}},\underset{\sim}{\boldsymbol{z}}) \frac{\delta}{\delta \widetilde{\mathcal{A}}_b^+(\underset{\sim}{\boldsymbol{x}})},$$

$$\underset{\sim}{\boldsymbol{e}}^{ba}(\underset{\sim}{\boldsymbol{x}},\underset{\sim}{\boldsymbol{z}}) = \frac{1}{\sqrt{4\pi^3}} \frac{\underset{\sim}{\boldsymbol{x}}-\underset{\sim}{\boldsymbol{z}}}{(\underset{\sim}{\boldsymbol{x}}-\underset{\sim}{\boldsymbol{z}})^2} \left(1 - U^\dagger(\underset{\sim}{\boldsymbol{x}})U(\underset{\sim}{\boldsymbol{z}})\right)^{ba}$$

The JIMWLK equation can then be written in Langevin form.

$$U_{y+\mathrm{d}y}(\underset{\sim}{\boldsymbol{x}}) = U_y(\mathrm{d}y)e^{-i\,\mathrm{d}y\alpha(\underset{\sim}{\boldsymbol{x}},y)} \qquad \alpha^a(\underset{\sim}{\boldsymbol{x}},y) = \sigma^a(\underset{\sim}{\boldsymbol{x}},y) + \int_{\underset{\sim}{\boldsymbol{z}}} \underset{\sim}{\boldsymbol{e}}^{at}(\underset{\sim}{\boldsymbol{x}},\underset{\sim}{\boldsymbol{z}})\underset{\sim}{\boldsymbol{\eta}}^b(\underset{\sim}{\boldsymbol{z}},y)$$

(Time must be discrete for unique interpretation, Itô)

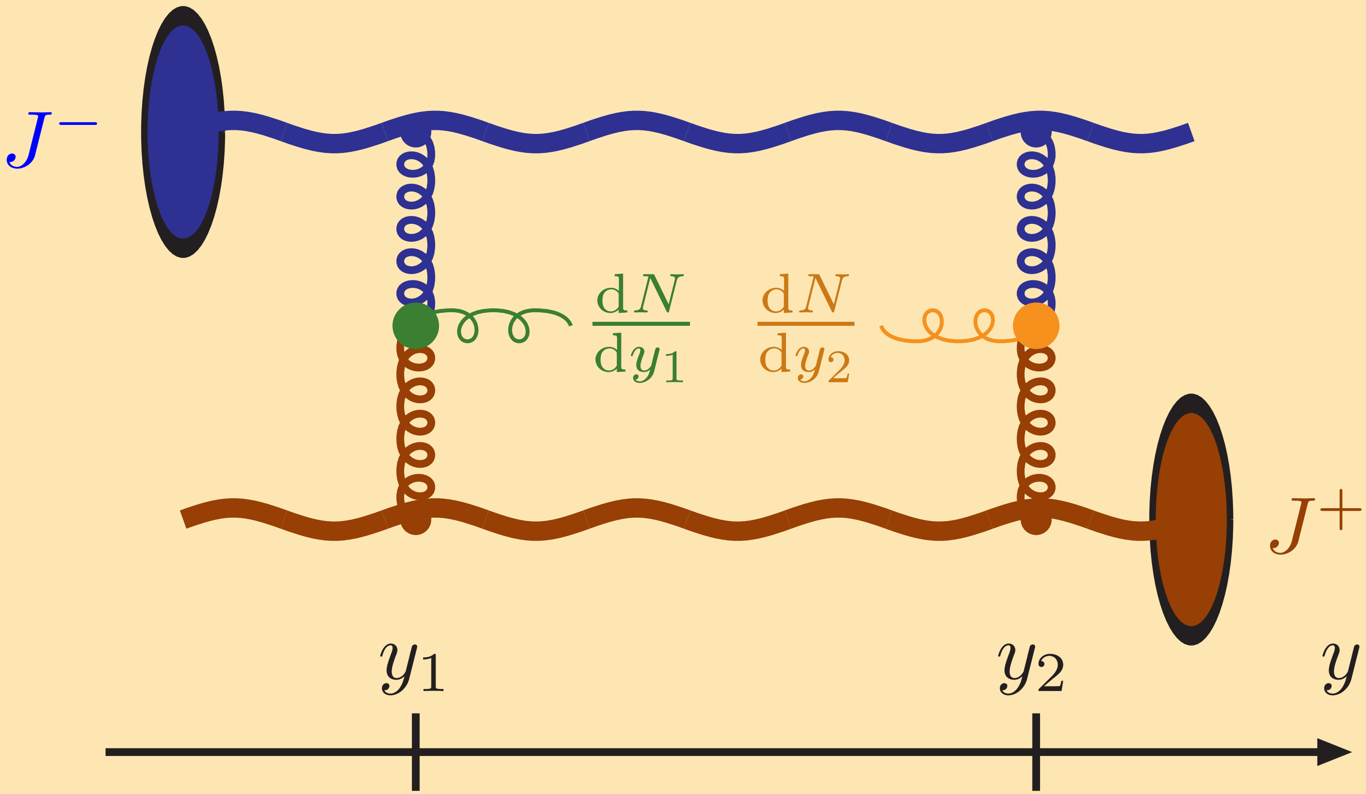
Basis for numerical sulution of JIMWLK Rummmukainen, Weigert[8]

[7] J.-P. Blaizot, E. Iancu and H. Weigert, *Nucl. Phys.* **A713** (2003) 441 [hep-ph/0206279].
[8] K. Rummukainen and H. Weigert, *Nucl. Phys.* **A739** (2004) 183 [hep-ph/0309306].

Rapidity correlations from Langevin evolution

Take the trajectories seriously: large rapidity correlations

One event = one Langevin trajectory



Conclusions

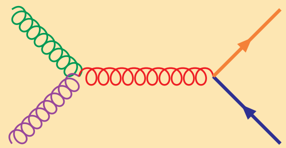
- Simpler part of NLO computation quark pair production
- Weak field limit and BFKL
- JIMWLK factorization

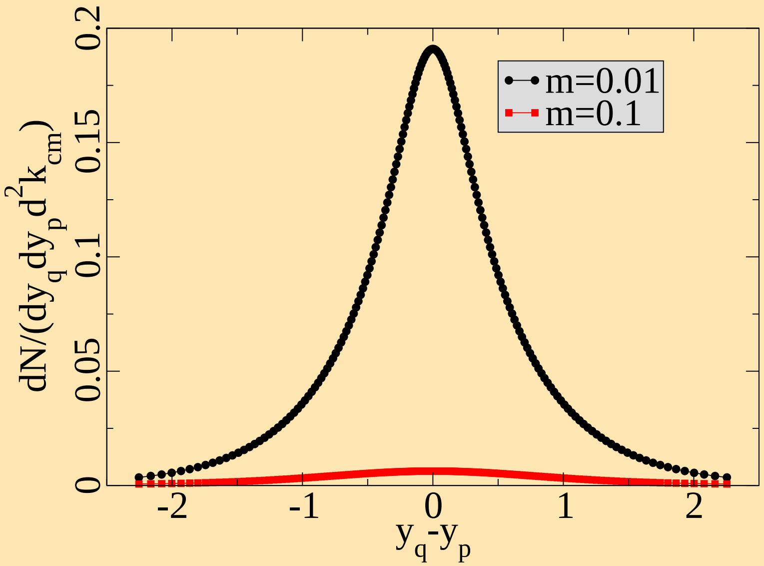
Full NLO gluon production . . . will be done some day.

Backup

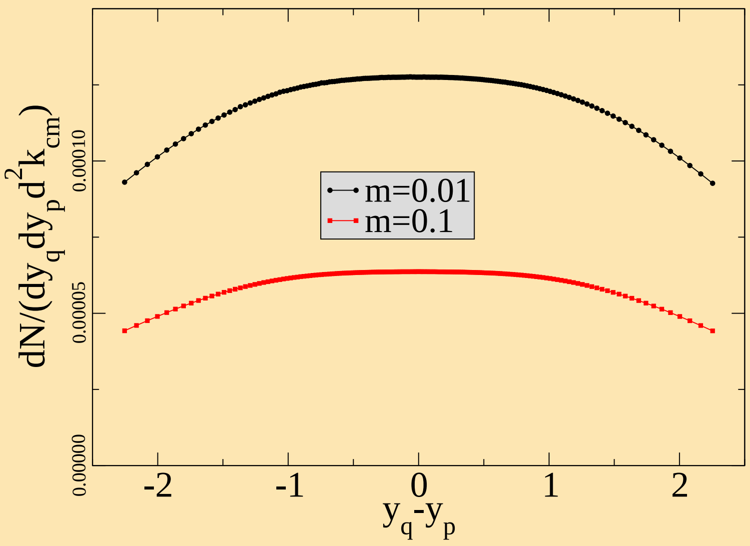
Difficulties in solving Dirac equation

- Coordinate system: we want to use **proper** time like in gluon case.
- Correlation $y_q - y_{\overline{q}} \leftrightarrow$ nontrivial η-dependence of the quark wave function even in η-independent background.
- Different divergences in the $m_q \longrightarrow 0$-limit in the Abelian and non-Abelian theories due to s-channel interaction:





Nonabelian



Abelian

Dirac equation: numerics

Temporal:

- Hard sources present only in the initial condition
  - initial condition defined at $\tau = 0$
  - Use **proper** time $\tau = \sqrt{t^2 - z^2}$ like in gluon case.

Longitudinal: Alternatives: $\eta, x^{\pm}, z$.

- η cannot parametrize $\tau = 0$–surface
- $x^{\pm}$ is not symmetric
  - Use z

Dirac in $\tau, z, \underset{\sim}{\boldsymbol{x}}$–coordinates:

$$\partial_\tau \psi = \underbrace{\frac{\sqrt{\tau^2 + z^2} + \gamma^0\gamma^3 z}{\tau}}_{\text{Coeff. depends on } \tau, z} \Big[-\gamma^0\gamma^3 \partial_z \psi + i\gamma^0 \overbrace{(i\boldsymbol{\gamma}_T \cdot \mathbf{D}_T - m)\,\psi}^{\text{Bg field in } \mathbf{D}_T = \boldsymbol{\nabla}_T + ig\mathbf{A}_T \text{ and } A_\eta} \Big] - i\gamma^0\gamma^3 \frac{A_\eta}{\tau}\psi$$

Longitudinal (z) direction discretized *implicitly* to handle the curved coordinate system.

1 and 2 point-functions

- The 2-point function $\mathcal{G}^{\mu\nu}$ can be written as

$$\mathcal{G}^{\mu\nu}(x,y) = \int \frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3 2E_{\boldsymbol{k}}}\, \eta^{\mu}_{-\boldsymbol{k}}(x)\ \eta^{\nu}_{+\boldsymbol{k}}(y)$$

$$\big[\mathcal{D}_\mu, \big[\mathcal{D}^\mu, \eta^{\nu}_{\pm\boldsymbol{k}}\big]\big] - \big[\mathcal{D}_\mu, \big[\mathcal{D}^\nu, \eta^{\mu}_{\pm\boldsymbol{k}}\big]\big] + ig\big[\mathcal{F}_\mu{}^\nu, \eta^{\mu}_{\pm\boldsymbol{k}}\big] = 0$$

$$\lim_{t\to-\infty} \eta^{\mu}_{\pm\boldsymbol{k}}(t,\vec{\boldsymbol{x}}) = \epsilon^\mu(\boldsymbol{k})\ e^{\pm ik\cdot x}$$

(obtained by writing the YM equation for $\mathcal{A} + \eta_{\pm\boldsymbol{k}}$, linearized in $\eta_{\pm\boldsymbol{k}}$)

- The equation of motion for β^μ reads

$$\big[\mathcal{D}_\mu, \big[\mathcal{D}^\mu, \beta^{\nu}\big]\big] - \big[\mathcal{D}_\mu, \big[\mathcal{D}^\nu, \beta^{\mu}\big]\big] + ig\big[\mathcal{F}_\mu{}^\nu, \beta^{\mu}\big] =$$

$$= \underbrace{\frac{\partial^3 \mathcal{L}_{YM}(\mathcal{A})}{\partial\mathcal{A}^\nu(x)\partial\mathcal{A}^\rho(x)\partial\mathcal{A}^\sigma(x)}}_{\text{3g vertex in the background } \mathcal{A}} \underbrace{\frac{1}{2}\int \frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3 2E_{\boldsymbol{k}}}\, \eta^{\mu}_{-\boldsymbol{k}}(x)\ \eta^{\nu}_{+\boldsymbol{k}}(x)}_{\text{value of the loop}}$$