High energy factorization in Nucleus-Nucleus collisions

François Gelis

CERN and CEA/Saclay

Outline

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Single gluon spectrum
Initial field perturbation
JIMWLK Hamiltonian
Factorization
Final remarks

- Introduction
- Single gluon spectrum at LO and NLO
- Expression as variations of the initial fields
- Leading log divergences and JIMWLK Hamiltonian
- Leading Log factorization
- Final remarks

(FG, T. Lappi and R. Venugopalan, in preparation)

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Saturation domain

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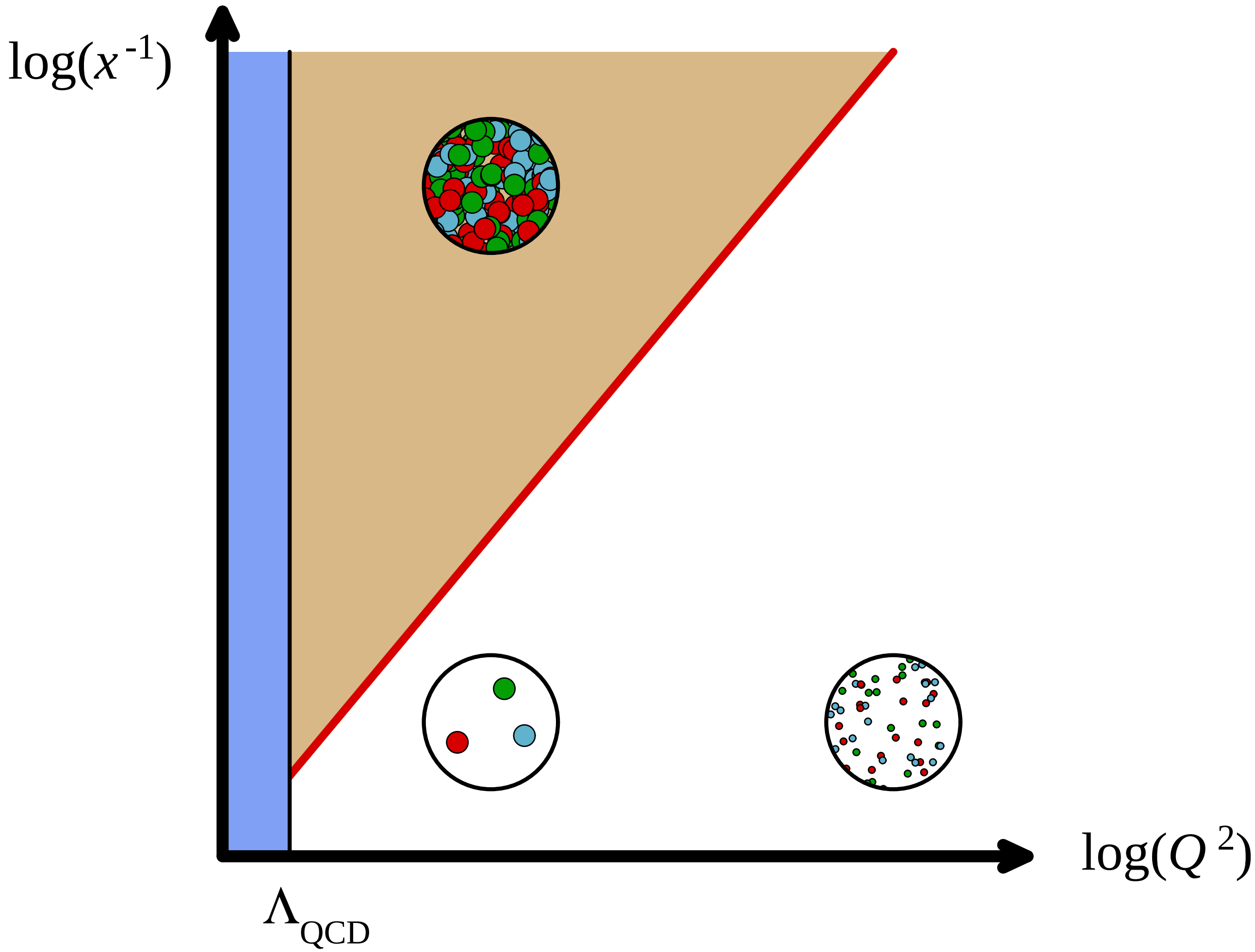

CGC degrees of freedom

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- The fast partons (large x) are frozen by time dilation
  $\triangleright$ described as static color sources on the light-cone :

$$J_a^\mu = \delta^{\mu+}\delta(x^-)\rho_a(\vec{\boldsymbol{x}}_\perp) \qquad\qquad (x^- \equiv (t-z)/\sqrt{2})$$

- Slow partons (small x) cannot be considered static over the time-scales of the collision process $\triangleright$ they must be treated as the usual gauge fields

  Since they are radiated by the fast partons, they must be coupled to the current J_a^μ by a term : $A_\mu J^\mu$

- The color sources ρ_a are random, and described by a distribution functional $W_Y[\rho]$, with Y the rapidity that separates “soft” and “hard”

CGC evolution

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- Evolution equation (JIMWLK) :

$$\frac{\partial W_Y[\rho]}{\partial Y} = \mathcal{H}[\rho] \;\; W_Y[\rho]$$

$$\mathcal{H}[\rho] = \int_{\vec{\boldsymbol{x}}_\perp} \sigma(\vec{\boldsymbol{x}}_\perp) \frac{\delta}{\delta\rho(\vec{\boldsymbol{x}}_\perp)} + \frac{1}{2} \int_{\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp} \chi(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \frac{\delta^2}{\delta\rho(\vec{\boldsymbol{x}}_\perp)\delta\rho(\vec{\boldsymbol{y}}_\perp)}$$

- σ and χ are non-linear functionals of ρ
- This evolution equation resums the powers of $\alpha_s \ln(1/x)$ and of $Q_s/p_\perp$ that arise in loop corrections
- This equation simplifies into the BFKL equation when the color density ρ is small (one can expand σ and χ in ρ)

Nucleus-nucleus collisions

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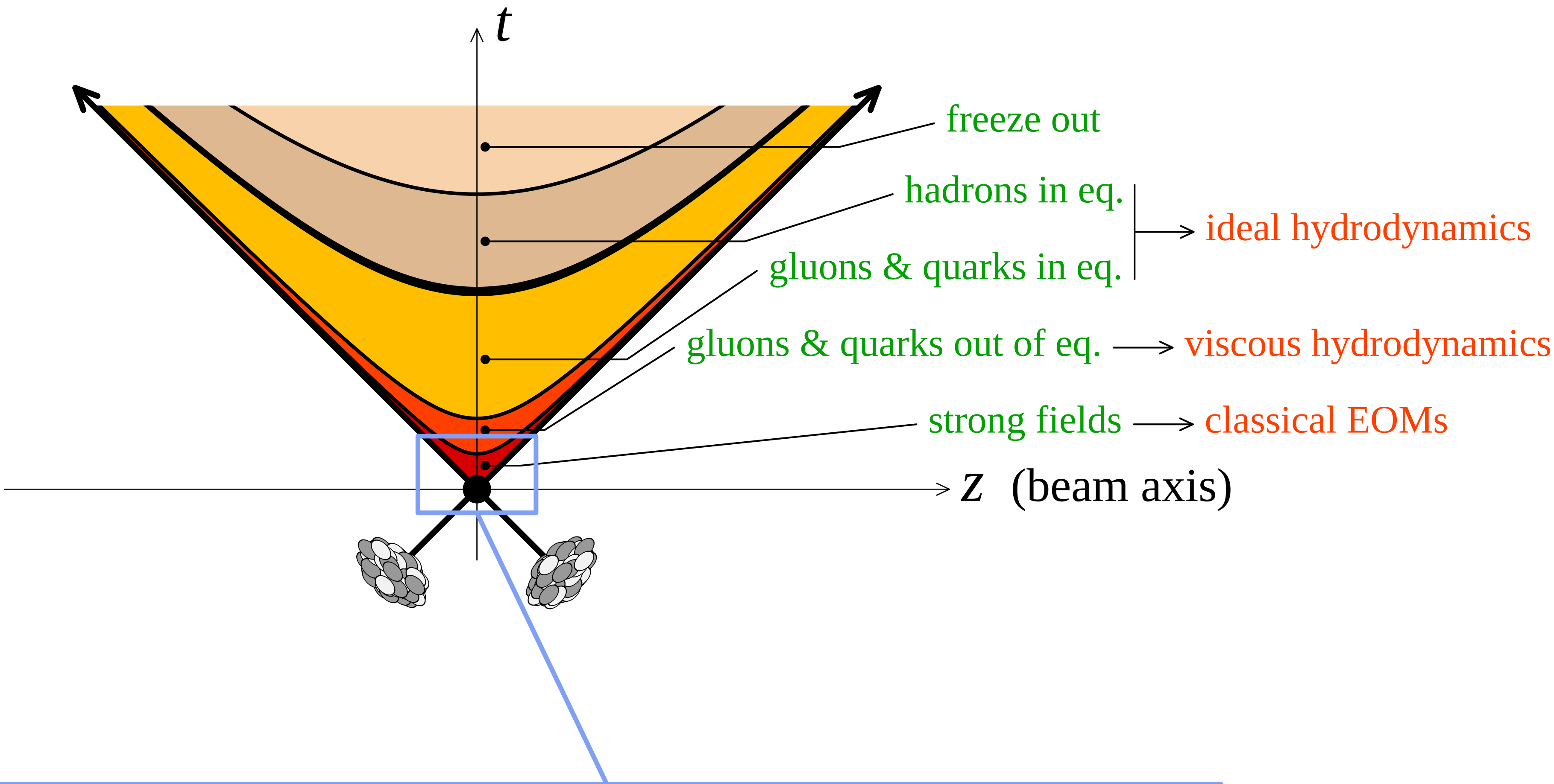

- calculate the initial production of semi-hard particles
- provide initial conditions for hydrodynamics

CGC and Nucleus-Nucleus collisions

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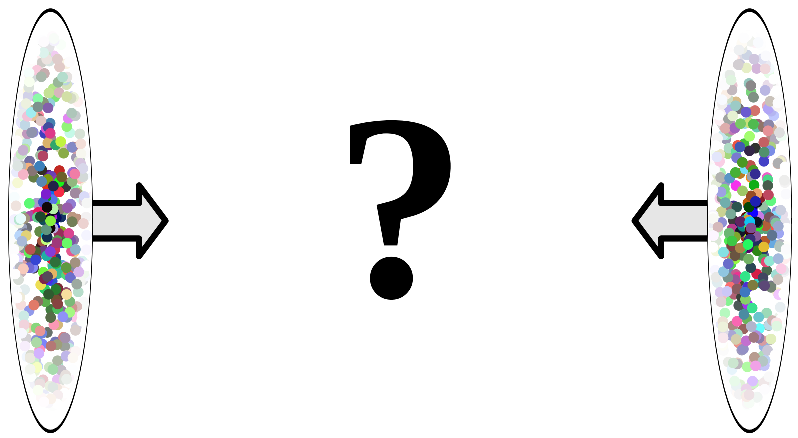

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu} + \underbrace{(J_1^\mu + J_2^\mu)}_{J^\mu} A_\mu$$

- Given the sources $\rho_{1,2}$ in each projectile, how do we calculate observables? Is there some kind of perturbative expansion?
- Loop corrections and factorization?

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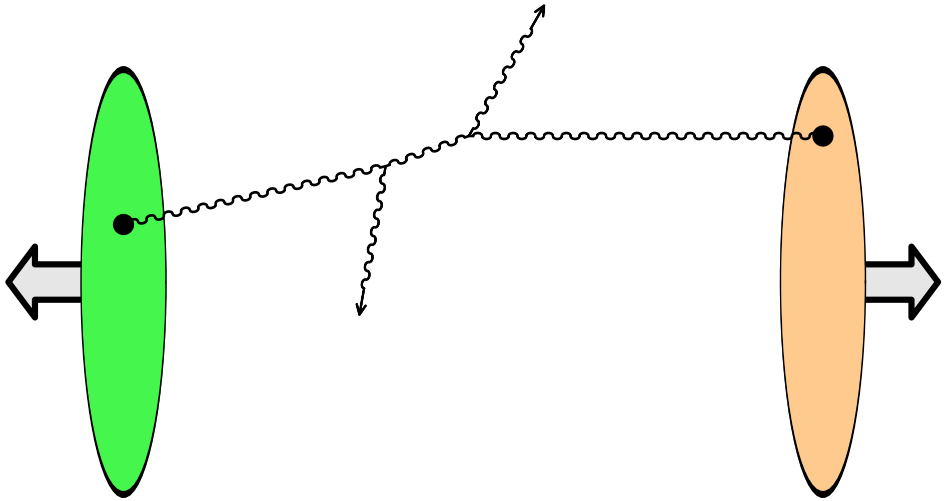

- Dilute regime : one parton in each projectile interact

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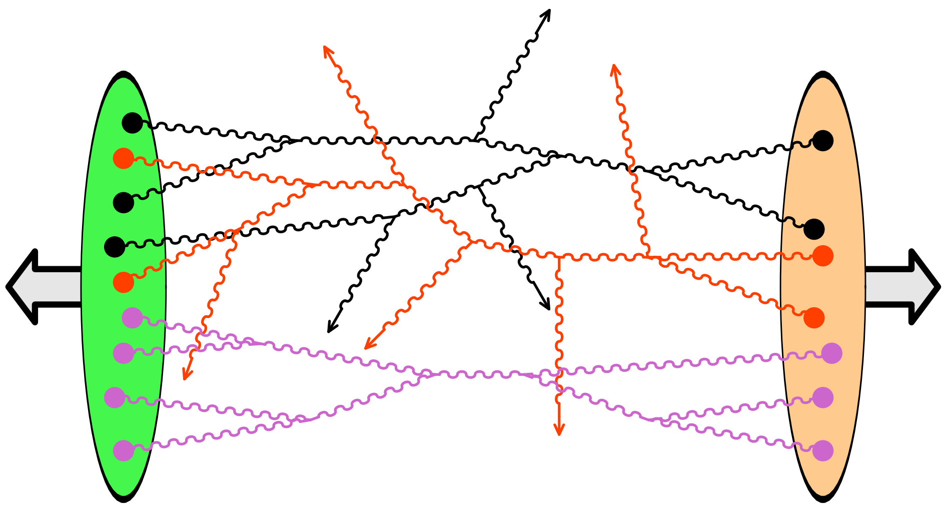

- Dilute regime : one parton in each projectile interact
- Dense regime : multiparton processes become crucial

  (+ pileup of many partonic scatterings in each AA collision)

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What is factorization ?

- A factorization formula divides an observable into a perturbatively calculable part (involving quarks and gluons) and a non-perturbative part describing the partonic content of hadrons or nuclei :

$$\mathcal{O} = F \otimes \mathcal{O}_{\rm partonic}$$

- Factorization has no predictive power unless the distributions F are intrinsic properties of the incoming projectiles :
  - F cannot depend on the observable
  - F of one projectile cannot depend on the second projectile
- Factorization can accommodate certain resummations :
  - Loop corrections in QCD generate corrections of the form $[\alpha_s \log(\cdot)]^n$, that are large in some parts of the phase-space
  - When these corrections do not depend on the observable and projectiles, they can be absorbed in the definition of F via an universal evolution equation

Factorization in the linear regime

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- Factorization in the linear small-x regime is known as k_{T}-factorization
- It was introduced in the discussion of heavy quark production near threshold, when $s \gg 4m_{\mathrm{q}}^2$, to resum large logs of $1/x_{1,2}$
  Collins, Ellis (1991), Catani, Ciafaloni, Hautmann (1991)
  Levin, Ryskin, Shabelski, Shuvaev (1991)
- In this framework, cross-sections read :

$$\frac{d\sigma}{dY\,d^2\vec{\boldsymbol{P}}_\perp} \propto \int\limits_{\vec{\boldsymbol{k}}_{1\perp},\vec{\boldsymbol{k}}_{2\perp}} \delta(\vec{\boldsymbol{k}}_{1\perp}+\vec{\boldsymbol{k}}_{2\perp}-\vec{\boldsymbol{P}}_\perp)\;\varphi_1(x_1,k_{1\perp})\;\varphi_2(x_2,k_{2\perp})\;\frac{|\mathcal{M}|^2}{k_{1\perp}^2\,k_{2\perp}^2}$$

$$x_{1,2} = \frac{M_\perp}{\sqrt{s}}\;e^{\pm Y}$$

- The small-x leading logs are resummed into the non-integrated gluon distributions $\varphi_{1,2}$ by letting them evolve according to the BFKL equation

Factorization in the nonlinear regime

- In the nonlinear regime, observables are sensitive to parton correlations beyond 2-point correlations. The distributions $\varphi_{1,2}$ do not provide this information, but it is present in the source distributions $W[\rho_{1,2}]$ of the CGC
- Factorization in the nonlinear regime at small-x has been established for DIS. The leading logs can be absorbed into $W[\rho]$ by letting it evolve according to the JIMWLK equation
- In the collision of two dense projectiles :
  - The large logs have a coefficient that depends in a complicated way on the sources of both nuclei. One must show that they can still be absorbed in one of the two $W[\rho]$'s
  - The dependence of the observable on the sources $\rho_{1,2}$ is not known analytically, already at LO
  - Even less is known about loop corrections...

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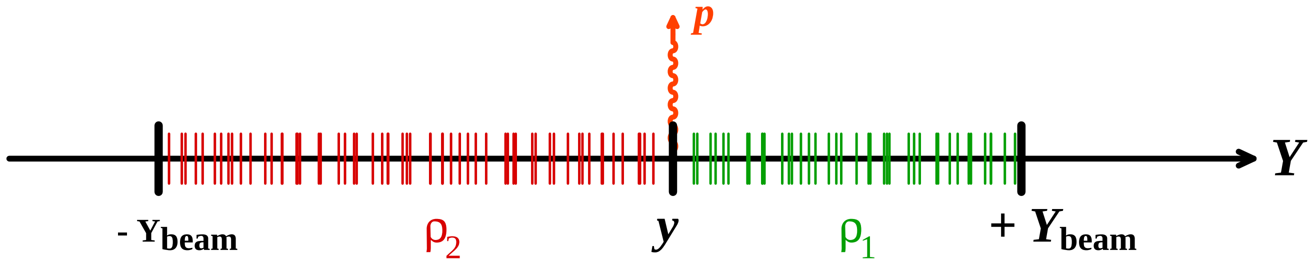
- For the single gluon spectrum in AA collisions, one would like to establish a formula such as :

$$\left\langle \frac{dN}{d^3\vec{\boldsymbol{p}}} \right\rangle \underset{\text{LLog}}{=} \int \big[D\rho_1\, D\rho_2\big]\, W_{Y_{\text{beam}}-y}\big[\rho_1\big]\, W_{y+Y_{\text{beam}}}\big[\rho_2\big]\;\; \frac{dN}{d^3\vec{\boldsymbol{p}}}\bigg|_{\text{LO}}$$

with $\displaystyle \frac{\partial}{\partial Y} W_Y = \mathcal{H}\, W$



- All the leading logs of $1/x_{1,2}$ should be absorbed in the $W's$
- The $W's$ should obey the JIMWLK evolution equation

Factorization in four easy steps

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- I : Express the single gluon spectrum at LO and NLO in terms of classical fields and small field fluctuations. Check that their boundary conditions are retarded

- II : Write the NLO terms as a perturbation of the initial value of the classical fields on the light-cone :

$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm NLO} = \left[\frac{1}{2}\int\limits_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\in{\rm LC}} \boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\,\mathbb{T}_{\boldsymbol{u}}\mathbb{T}_{\boldsymbol{v}} + \int\limits_{\vec{\boldsymbol{u}}\in{\rm LC}} \boldsymbol{\beta}(\vec{\boldsymbol{u}})\,\mathbb{T}_{\boldsymbol{u}}\right] \left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm LO}$$

- III : For $\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}$ on the same branch of the light-cone, one has :

$$\frac{1}{2}\int\limits_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\in{\rm LC}} \boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\,\mathbb{T}_{\boldsymbol{u}}\mathbb{T}_{\boldsymbol{v}} + \int\limits_{\vec{\boldsymbol{u}}\in{\rm LC}} \boldsymbol{\beta}(\vec{\boldsymbol{u}})\,\mathbb{T}_{\boldsymbol{u}} = \log\left(\frac{\Lambda^+}{p^+}\right)\times\mathcal{H} + \text{finite terms}$$

- IV : These are the only logs. Factorization follows trivially

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Single gluon spectrum

at LO and NLO

Single gluon spectrum at LO

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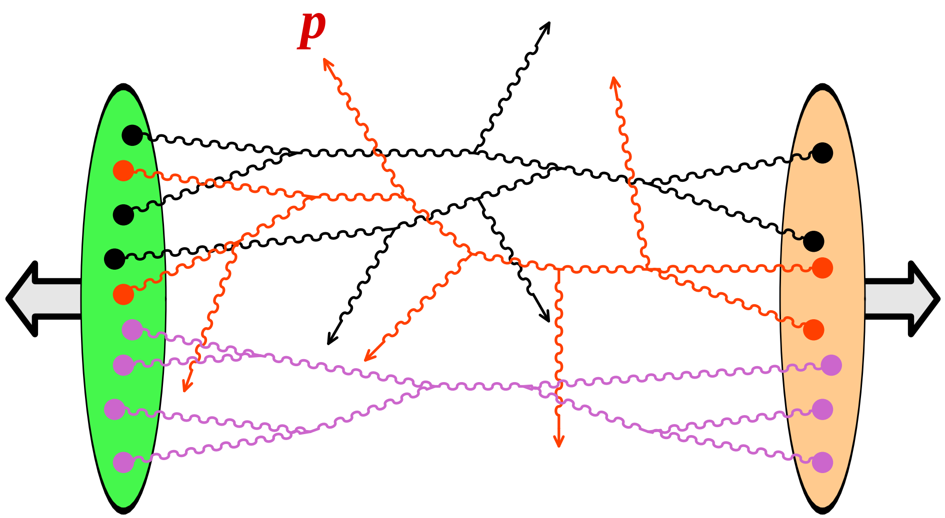


- Leading Order = tree diagrams only
- Tag one gluon of momentum $\vec{\boldsymbol{p}}$
- Integrate out the phase-space of all the other gluons

$$\frac{dN}{d^3\vec{\boldsymbol{p}}} \sim \sum_{n=0}^{\infty} \frac{1}{n!} \int \left[d^3\vec{\boldsymbol{p}}_1 \cdots d^3\vec{\boldsymbol{p}}_n\right] \left|\left\langle \vec{\boldsymbol{p}}\ \vec{\boldsymbol{p}}_1 \cdots \vec{\boldsymbol{p}}_n \middle| 0 \right\rangle\right|^2$$

Single gluon spectrum at LO

- LO results for the single gluon spectrum :
  - Disconnected graphs cancel in the inclusive spectrum
  - At LO, the single gluon spectrum can be expressed in terms of classical solutions of the field equation of motion
  - These classical fields obey retarded boundary conditions

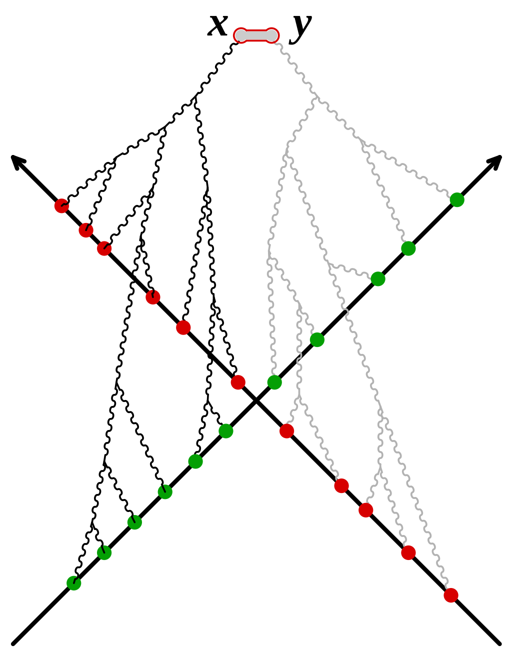
$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm LO} = \lim_{t\to+\infty} \int d^3\vec{\boldsymbol{x}} d^3\vec{\boldsymbol{y}}\; e^{i\vec{\boldsymbol{p}}\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})} \;\cdots \mathcal{A}^\mu(t,\vec{\boldsymbol{x}})\; \mathcal{A}^\nu(t,\vec{\boldsymbol{y}})$$

$$\left[\mathcal{D}_\mu, \mathcal{F}^{\mu\nu}\right] = J^\nu$$

$$\lim_{t\to-\infty} \mathcal{A}^\mu(t,\vec{\boldsymbol{x}}) = 0$$

Single gluon spectrum at LO

- Retarded classical fields are sums of tree diagrams :



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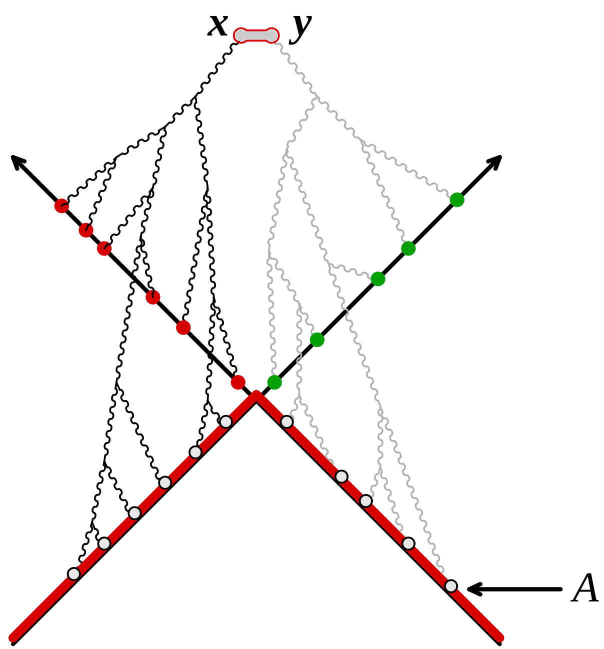

- Note : the gluon spectrum is a functional of the value of the classical field just above the backward light-cone :

$$\frac{dN}{d^3\vec{\boldsymbol{p}}} = \mathcal{F}[\mathcal{A}]$$

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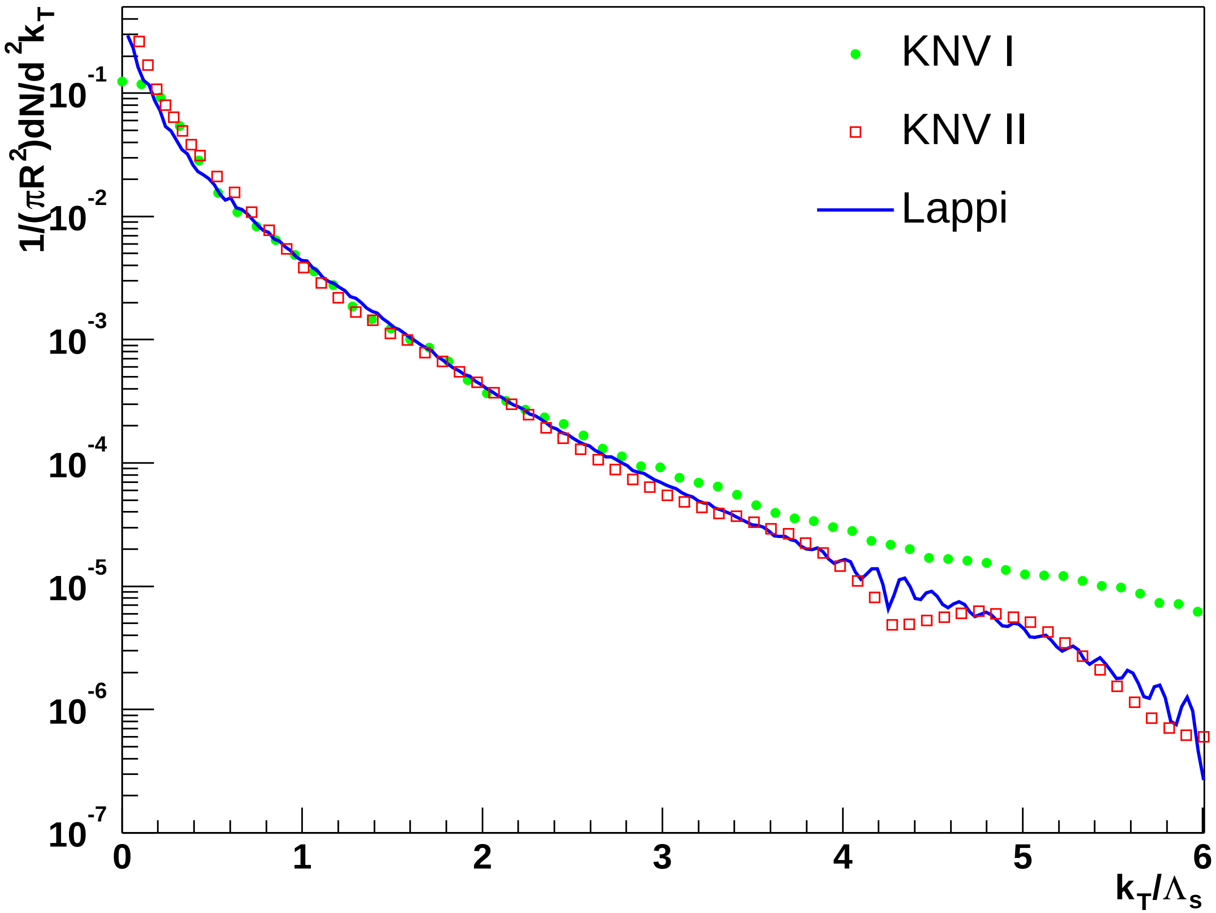
Krasnitz, Nara, Venugopalan (1999 – 2001), Lappi (2003)



- Important softening at small $k_\perp$ compared to pQCD (saturation)

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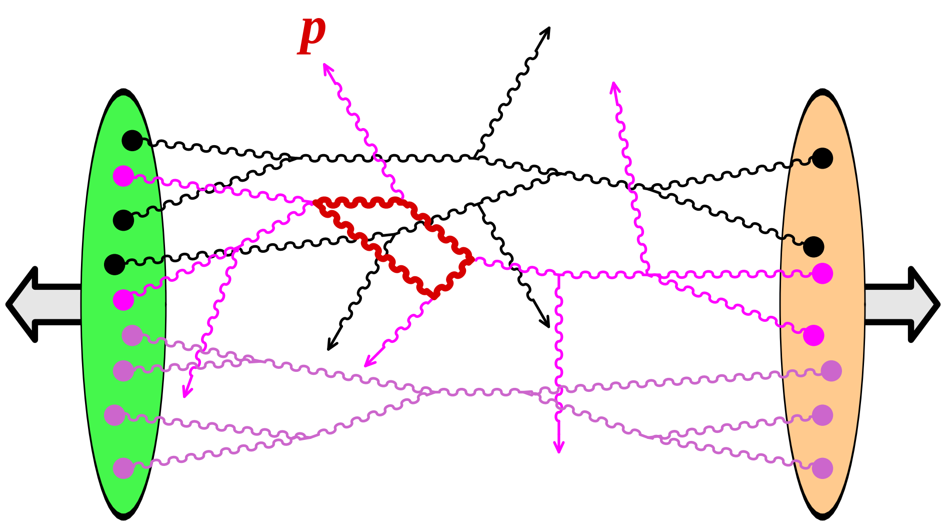

- Next to Leading Order = 1-loop diagrams
- Connected diagrams only
- Expressible in terms of classical fields, and small fluctuations about the classical field, both with retarded boundary conditions

Single gluon spectrum at NLO

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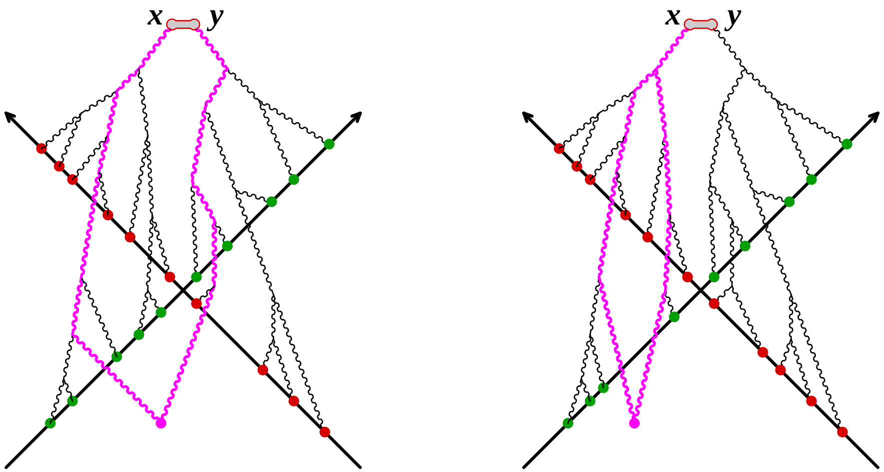
- 1-loop graphs contributing to the gluon spectrum at NLO :



$$\frac{dN}{d^3\vec{\boldsymbol{p}}}\bigg|_{\rm NLO} = \lim_{t\to+\infty} \int d^3\vec{\boldsymbol{x}}d^3\vec{\boldsymbol{y}}\; e^{i\vec{\boldsymbol{p}}\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})}\;\cdots\Big[\mathcal{G}^{\mu\nu}_{-+}(x,y) + \beta^\mu(t,\vec{\boldsymbol{x}})\;\mathcal{A}^\nu(t,\vec{\boldsymbol{y}}) + \mathcal{A}^\mu(t,\vec{\boldsymbol{x}})\;\beta^\nu(t,\vec{\boldsymbol{y}})\Big]$$

- $\mathcal{G}^{\mu\nu}_{-+}$ is a 2-point function
- β^μ is a small field fluctuation driven by a 1-loop source

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Single gluon spectrum at NLO

- The equation of motion for β^μ reads

$$\big[\mathcal{D}_\mu,\big[\mathcal{D}^\mu,\beta^\nu\big]\big]-\big[\mathcal{D}_\mu,\big[\mathcal{D}^\nu,\beta^\mu\big]\big]+ig\big[\mathcal{F}_\mu{}^\nu,\beta^\mu\big]=$$

$$=\frac{1}{2}\underbrace{\frac{\partial^3\mathcal{L}_{YM}(\mathcal{A})}{\partial\mathcal{A}^\nu(x)\partial\mathcal{A}^\rho(x)\partial\mathcal{A}^\sigma(x)}}_{\text{3-gluon vertex in the background } \mathcal{A}}\;\mathcal{G}^{\rho\sigma}_{++}(x,x)$$

- The 2-point functions $\mathcal{G}^{\mu\nu}_{-+}$ and $\mathcal{G}^{\mu\nu}_{++}$ can be written as

$$\mathcal{G}^{\mu\nu}_{-+}(x,y)=\int\frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3 2E_{\boldsymbol{k}}}\;\eta^\mu_{-\boldsymbol{k}}(x)\;\eta^\nu_{+\boldsymbol{k}}(y)$$

$$\mathcal{G}^{\mu\nu}_{++}(x,x)=\frac{1}{2}\int\frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3 2E_{\boldsymbol{k}}}\;\Big[\eta^\mu_{-\boldsymbol{k}}(x)\;\eta^\nu_{+\boldsymbol{k}}(x)+\eta^\mu_{+\boldsymbol{k}}(x)\;\eta^\nu_{-\boldsymbol{k}}(x)\Big]$$

with
$$\left\{\begin{aligned}&\big[\mathcal{D}_\mu,\big[\mathcal{D}^\mu,\eta^\nu_{\pm\boldsymbol{k}}\big]\big]-\big[\mathcal{D}_\mu,\big[\mathcal{D}^\nu,\eta^\mu_{\pm\boldsymbol{k}}\big]\big]+ig\big[\mathcal{F}_\mu{}^\nu,\eta^\mu_{\pm\boldsymbol{k}}\big]=0\\&\lim_{t\to-\infty}\eta^\mu_{\pm\boldsymbol{k}}(t,\vec{\boldsymbol{x}})=\epsilon^\mu(\boldsymbol{k})\;e^{\pm ik\cdot x}\end{aligned}\right.$$

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Expression as a perturbation of the initial classical field

Single gluon spectrum at NLO

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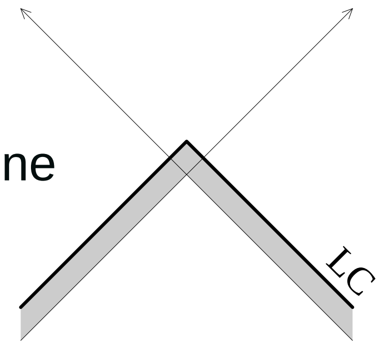
- For a small field fluctuation a^μ (not driven by a source) propagating on top of the classical field $\mathcal{A}^\mu$, one can prove :

$$a^\mu(x) = \Big[\int_{\vec{\boldsymbol{u}} \in \mathrm{LC}} a(u) \cdot \mathbb{T}_{\boldsymbol{u}} \Big] \ \mathcal{A}^\mu(x)$$



  - 'LC' denotes a surface just above the backward light-cone
  - $\mathbb{T}_{\boldsymbol{u}}$ is the generator of shifts of the initial value of the fields on this surface :

$$\mathcal{F}[\mathcal{A} + a] \equiv \exp \Big[\int_{\vec{\boldsymbol{u}} \in \mathrm{LC}} a(u) \cdot \mathbb{T}_{\boldsymbol{u}} \Big] \ \mathcal{F}[\mathcal{A}]$$

Note : this construction is possible only because the objects involved in the problem obey retarded boundary conditions

![CERN logo]

Single gluon spectrum at NLO

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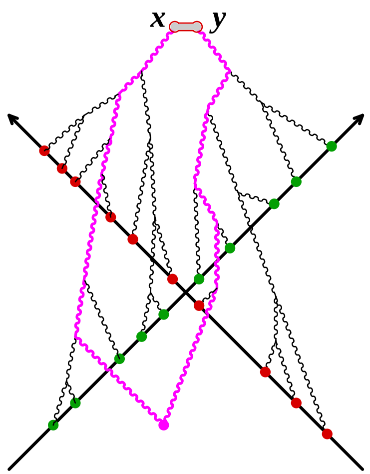
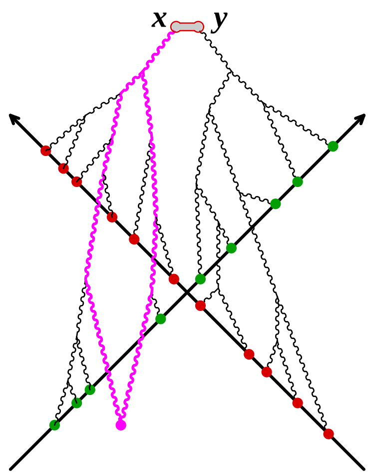
- 1-loop graphs contributing to the gluon spectrum at NLO :





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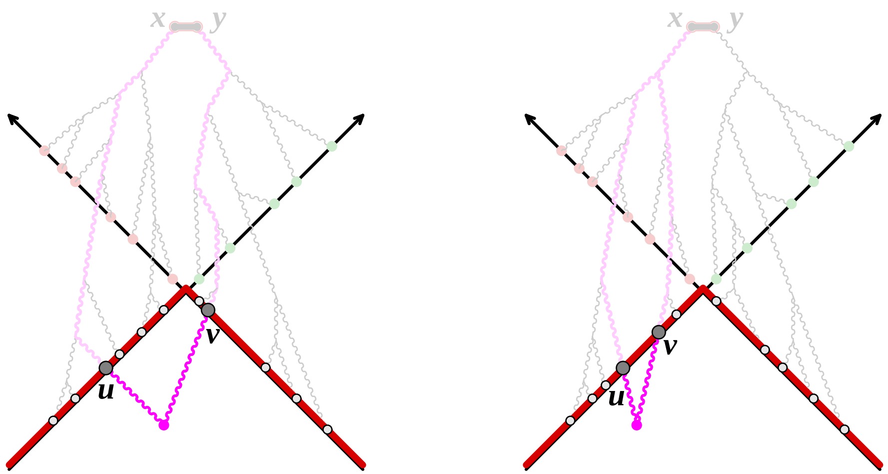
- 1-loop graphs contributing to the gluon spectrum at NLO :



- They can be written as a perturbation of the LC initial fields :

$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm NLO} = \left[\frac{1}{2}\int\limits_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\in{\rm LC}} \boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\,\mathbb{T}_{\boldsymbol{u}}\mathbb{T}_{\boldsymbol{v}}\right]\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm LO}$$

$$\boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) \equiv \int\frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3 2E_{\boldsymbol{k}}}\;\eta_{-\boldsymbol{k}}(u)\,\eta_{+\boldsymbol{k}}(v)$$

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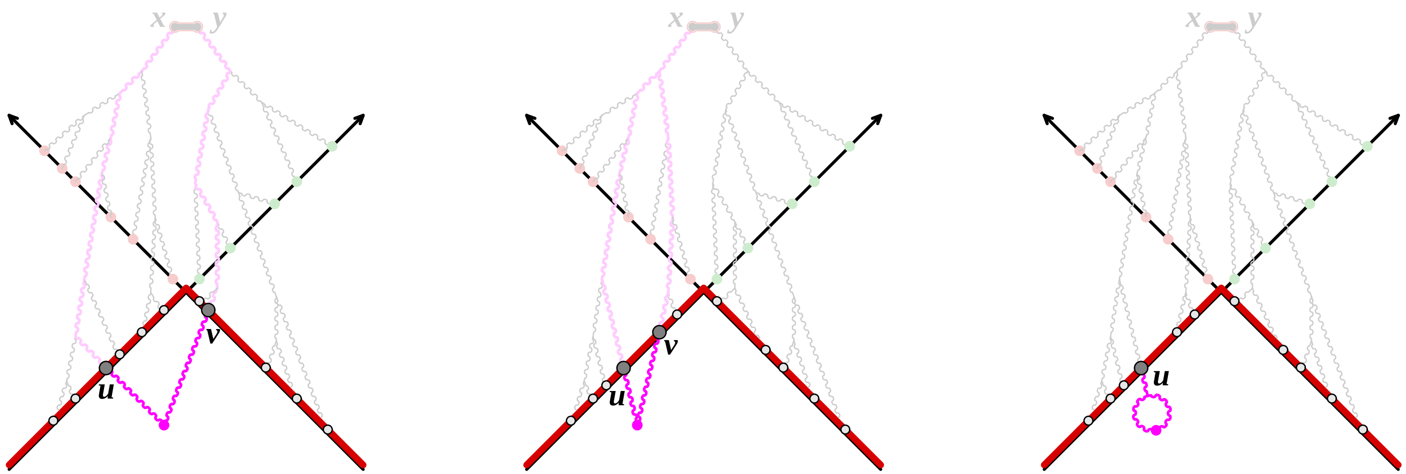

- The loop correction can also be below the light-cone :

$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\mathrm{NLO}} = \left[\frac{1}{2}\int\limits_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\in\mathrm{LC}} \boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\,\mathbb{T}_{\boldsymbol{u}}\mathbb{T}_{\boldsymbol{v}} + \int\limits_{\vec{\boldsymbol{u}}\in\mathrm{LC}} \boldsymbol{\beta}(\vec{\boldsymbol{u}})\,\mathbb{T}_{\boldsymbol{u}}\right] \left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\mathrm{LO}}$$

▷ the functions $\boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})$ and $\beta(\vec{\boldsymbol{u}})$ can be evaluated analytically

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JIMWLK Hamiltonian

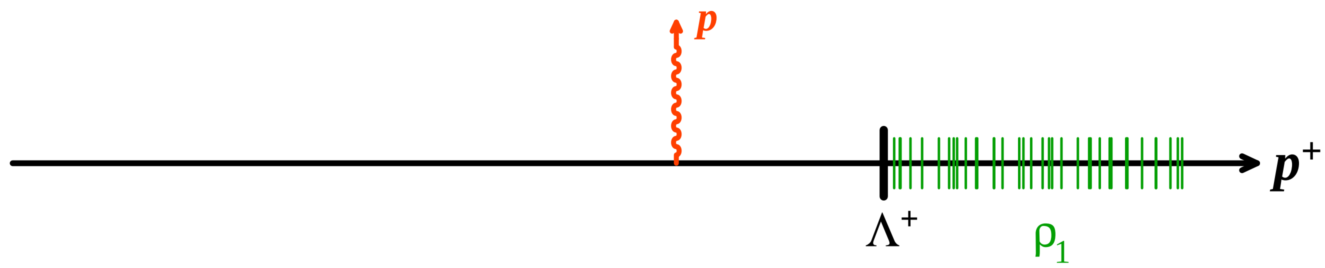
Divergences

- If $\vec{u}, \vec{v}$ belong to the same branch of the LC (e.g. $u^- = v^- = \epsilon$), the function $\boldsymbol{\Sigma}(\vec{u}, \vec{v})$ contains

$$\boldsymbol{\Sigma}(\vec{u}, \vec{v}) \sim \int_0^{+\infty} \frac{dk^+}{k^+} \cdots e^{ik^-(u^+-v^+)} \qquad \text{with} \quad k^- \equiv \frac{\boldsymbol{k}_\perp^2}{2k^+}$$

▷ the integral converges at $k^+ = 0$ but not when $k^+ \to +\infty$

Note : the log is a $\log(\Lambda^+/p^+)$, where Λ^+ is the boundary between the hard color sources and the fields, and p^+ the longitudinal momentum of the produced gluon



- Similar considerations apply when $\vec{u}, \vec{v}$ both belong to the other branch of the LC, leading to a $\log(\Lambda^-/p^-)$

Leading Log approximation

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- In the LC gauge $\mathcal{A}^+ = 0$, the operator $\eta(u) \cdot \mathbb{T}_{\boldsymbol{u}}$ is

$$\eta(u)\cdot\mathbb{T}_{\boldsymbol{u}} \equiv (\partial^- \eta_a^i(u))\frac{\delta}{\delta(\partial^- \mathcal{A}_a^i(u))} + \eta_a^-(u)\frac{\delta}{\delta\mathcal{A}_a^-(u)} + (\partial_\mu \eta_a^\mu(u))\frac{\delta}{\delta(\partial_\mu \mathcal{A}_a^\mu(u))}$$

- An explicit calculation of $\partial^- \eta^i_{\pm\boldsymbol{k}}$ and $\eta^-_{\pm\boldsymbol{k}}$ shows that these components have an extra $1/k^+$ when $k^+ \to +\infty$

- At leading log, it seems sufficient to consider :

$$\eta(u)\cdot\mathbb{T}_{\boldsymbol{u}} \underset{\mathrm{LLog}}{=} (\partial_\mu \eta_a^\mu(u))\frac{\delta}{\delta(\partial_\mu \mathcal{A}_a^\mu(u))}$$

This is almost correct, but not quite...

Leading Log approximation

- The space-time above the LC contains a classical background field,

$$\mathcal{A}^{\pm} = 0 \quad , \quad \mathcal{A}^{i} = \frac{i}{g}\Omega^{\dagger}\partial^{i}\Omega$$

▷ the interaction of the fluctuation with a background field can turn terms that are not divergent on the LC into divergent terms ! (factors of k^+ can arise in the 3-gluon derivative coupling)

- Because the background is a pure gauge, this problem is circumvented by using $\Omega_{ab}\eta_b$ instead of η_a as the initial condition :

$$\eta(u)\cdot\mathbb{T}_{\boldsymbol{u}} \equiv (\partial^{-}\Omega_{ab}\eta_b^{i}(u))\frac{\delta}{\delta(\partial^{-}\Omega_{ab}\mathcal{A}_b^{i}(u))} + \Omega_{ab}\eta_b^{-}(u)\frac{\delta}{\delta\Omega_{ab}\mathcal{A}_b^{-}(u)}$$
$$\underline{+(\partial_{\mu}\Omega_{ab}\eta_b^{\mu}(u))\frac{\delta}{\delta(\partial_{\mu}\Omega_{ab}\mathcal{A}_b^{\mu}(u))}}$$

▷ at leading log, only the last term matters

JIMWLK Hamiltonian

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- The coefficient of the leading log does not depend on u^+, v^+
- Derivatives with respect to $\partial_\mu \Omega_{ab} \mathcal{A}_b^\mu(u)$ can be mapped to derivatives with respect to the slowest color sources :

$$\int du^+ \frac{\delta}{\delta(\partial_\mu \Omega_{ab} \mathcal{A}_b^\mu(u))} = \int d^2\vec{\boldsymbol{x}}_\perp \left\langle \vec{\boldsymbol{u}}_\perp \left| \frac{1}{\boldsymbol{\partial}_\perp^2} \right| \vec{\boldsymbol{x}}_\perp \right\rangle \frac{\delta}{\delta \widetilde{\mathcal{A}}_a^+(\epsilon, \vec{\boldsymbol{x}}_\perp)}$$

with $\quad -\boldsymbol{\partial}_\perp^2 \, \widetilde{\mathcal{A}}^+(\epsilon, \vec{\boldsymbol{x}}_\perp) = \rho(\epsilon, \vec{\boldsymbol{x}}_\perp)$

- When $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}$ are on the same branch of the LC, we have

$$\frac{1}{2} \int_{\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathrm{LC}} \boldsymbol{\Sigma}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) \, \mathbb{T}_{\boldsymbol{u}} \mathbb{T}_{\boldsymbol{v}} \underset{\mathrm{LLog}}{=} \frac{1}{2} \log\left(\frac{\Lambda^+}{p^+}\right) \int_{\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp} \eta_{ab}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \frac{\delta^2}{\delta \widetilde{\mathcal{A}}_a^+(\epsilon, \vec{\boldsymbol{x}}_\perp) \delta \widetilde{\mathcal{A}}_b^+(\epsilon, \vec{\boldsymbol{y}}_\perp)}$$

with $\eta_{ab}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \equiv \frac{1}{\pi} \int \frac{d^2\vec{\boldsymbol{z}}_\perp}{(2\pi)^2} \frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp) \cdot (\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}})^2 (\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$

$$\times \Big[1 + \Omega(x)\Omega^\dagger(y) - \Omega(x)\Omega^\dagger(z) - \Omega(z)\Omega^\dagger(y) \Big]_{ab}$$

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JIMWLK Hamiltonian

- In principle, one could evaluate the term involving $\beta(\vec{u})$ by solving explicitly its EOM
- Shortcut: by using the Green's formula for this fluctuation, one can show directly that

$$\int_{\vec{u}\in \mathrm{LC}} \boldsymbol{\beta}(\vec{u})\,\mathbb{T}_{\boldsymbol{u}} \underset{\mathrm{LLog}}{=} \frac{1}{2}\log\left(\frac{\Lambda^+}{p^+}\right)\int_{\vec{x}_\perp}\left(\int_{\vec{y}_\perp}\frac{\delta\eta_{ab}(\vec{x}_\perp,\vec{y}_\perp)}{\delta\widetilde{\mathcal{A}}_b^+(\epsilon,\vec{y}_\perp)}\right)\frac{\delta}{\delta\widetilde{\mathcal{A}}_a^+(\epsilon,\vec{x}_\perp)}$$

- Combining the real and virtual terms :

$$\left[\frac{1}{2}\int_{\vec{u},\vec{v}\in \mathrm{LC}} \boldsymbol{\Sigma}(\vec{u},\vec{v})\,\mathbb{T}_{\boldsymbol{u}}\mathbb{T}_{\boldsymbol{v}} + \int_{\vec{u}\in \mathrm{LC}} \boldsymbol{\beta}(\vec{u})\,\mathbb{T}_{\boldsymbol{u}}\right]$$

$$\underset{\mathrm{LLog}}{=} \log\left(\frac{\Lambda^+}{p^+}\right)\underbrace{\frac{1}{2}\int_{\vec{y}_\perp}\frac{\delta}{\delta\widetilde{\mathcal{A}}_b^+(\epsilon,\vec{y}_\perp)}\eta_{ab}(\vec{x}_\perp,\vec{y}_\perp)\frac{\delta}{\delta\widetilde{\mathcal{A}}_a^+(\epsilon,\vec{x}_\perp)}}_{\text{JIMWLK }\mathcal{H}}$$

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- The configuration where $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}$ are on the first branch of the LC can be rewritten as

$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\text{NLO}} \underset{\text{LLog}}{=} \log\left(\frac{\Lambda^+}{p^+}\right) \mathcal{H}_1 \; \left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\text{LO}}$$

with $\mathcal{H}_1$ the JIMWLK Hamiltonian for the first nucleus

- Including also the configuration where both $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}$ are on the second branch of the LC, we get

$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\text{NLO}} \underset{\text{LLog}}{=} \left[\log\left(\frac{\Lambda^+}{p^+}\right) \mathcal{H}_1 + \log\left(\frac{\Lambda^-}{p^-}\right) \mathcal{H}_2\right] \; \left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\text{LO}}$$

Leading Log divergences

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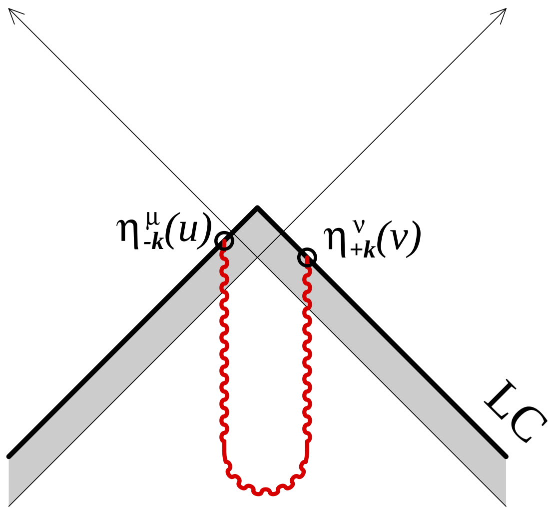
- The only remaining possibility is to have $\vec{u}$ and $\vec{v}$ on different branches of the LC



However, there is no log divergence in this case, since the k^+ integral is of the form :

$$\int \frac{dk^+}{k^+} \cdots e^{ik^+(u^- - v^-)}\; e^{ik^-(u^+ - v^+)}$$

▷ no mixing of the divergences of the two nuclei

Leading Log factorization

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- All the above discussion is for one given configuration of the sources $\rho_{1,2}$ (or of the fields $\widetilde{\mathcal{A}}^{\pm}_{1,2}$). Averaging over all the configurations of the sources in the two projectiles, and using the hermiticity of the JIMWLK Hamiltonian, we get

$$\left\langle \frac{dN}{d^3\vec{\boldsymbol{p}}} \right\rangle_{\mathrm{LO+NLO}} \underset{\mathrm{LLog}}{=} \int \left[D\widetilde{\mathcal{A}}_1^+ \, D\widetilde{\mathcal{A}}_2^-\right] \times \left(\left[1 + \log\left(\frac{\Lambda^+}{p^+}\right) \mathcal{H}_1 + \log\left(\frac{\Lambda^-}{p^-}\right) \mathcal{H}_2 \right] W[\widetilde{\mathcal{A}}_1^+] \, W[\widetilde{\mathcal{A}}_2^-] \right) \left. \frac{dN}{d^3\vec{\boldsymbol{p}}} \right|_{\mathrm{LO}}$$

- This is a 1-loop result. Using RG arguments, this leads to the following factorized formula for the resummation of the leading log terms to all orders :

$$\left\langle \frac{dN}{d^3\vec{\boldsymbol{p}}} \right\rangle \underset{\mathrm{LLog}}{=} \int \left[D\widetilde{\mathcal{A}}_1^+ \, D\widetilde{\mathcal{A}}_2^-\right] W_{Y_1}[\widetilde{\mathcal{A}}_1^+] \, W_{Y_2}[\widetilde{\mathcal{A}}_2^-] \left. \frac{dN}{d^3\vec{\boldsymbol{p}}} \right|_{\mathrm{LO}}$$

with $\dfrac{\partial}{\partial Y} W_Y = \mathcal{H}\, W \quad , \quad Y_1 = \log(P_1^+/p^+) \quad , \quad Y_2 = \log(P_2^-/p^-)$

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Requirements for factorization

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- The fact that the observable is bilinear in the fields is not essential. The formula

$$\mathcal{O}_{\rm NLO} = \Big[\frac{1}{2}\int\limits_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\in{\rm LC}} \boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\,\mathbb{T}_{\boldsymbol{u}}\mathbb{T}_{\boldsymbol{v}} + \int\limits_{\vec{\boldsymbol{u}}\in{\rm LC}} \boldsymbol{\beta}(\vec{\boldsymbol{u}})\,\mathbb{T}_{\boldsymbol{u}}\Big]\,\mathcal{O}_{\rm LO}$$

  can be established for more general observables, provided their expectation value depends on retarded fields only

- Crucial ingredients for factorization :
  - Only connected diagrams should contribute
  - One should have an initial value problem
    $\triangleright$ retarded boundary conditions seem essential
  - The observable should involve only one rapidity scale. Otherwise, there are extra large corrections in $\alpha_s(y_1 - y_2)$ that are not captured in the evolution of the $W[\widetilde{\mathcal{A}}^\pm]$'s

Quantities that do factorize

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- Energy-momentum tensor $T^{\mu\nu}(\tau,\eta,\vec{\boldsymbol{x}}_\perp)$:

$$\langle T^{\mu\nu}(\tau,\eta,\vec{\boldsymbol{x}}_\perp)\rangle \underset{\rm LLog}{=} \int \big[D\widetilde{\mathcal{A}}_1^+ \, D\widetilde{\mathcal{A}}_2^-\big]\; W_{Y_1}\big[\widetilde{\mathcal{A}}_1^+\big]\, W_{Y_2}\big[\widetilde{\mathcal{A}}_2^-\big]\; \Big[T^{\mu\nu}(\tau,\eta,\vec{\boldsymbol{x}}_\perp)\Big]_{\rm LO}$$

with $\quad Y_1 = Y_{\rm beam} - \eta \quad , \quad Y_2 = Y_{\rm beam} + \eta$

  - ▷ CGC initial conditions for hydrodynamics
  - ▷ Note : this cannot be used for studying fluctuations

- Higher moments (the connected part only) of the multiplicity distribution in a small slice in rapidity :

  - Moments of the multiplicity distribution are expressible in terms of retarded quantities (FG, Venugopalan)
  - If the p particles in the moment of order p are in the same small slice of rapidity, the locality requirement is satisfied

Quantities that do not factorize

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- For some quantities, an extension of the above form of factorization may be able to resum all the leading logs. Example : 2-gluon correlations with a large rapidity separation between the gluons (work in progress with T. Lappi and R. Venugopalan)

- More exclusive quantities seem out of reach of this form of factorization :
  - Example : survival probability of rapidity gaps

$\triangleright$ for such quantities, the main obstruction is the impossibility to write them as formulas involving only retarded objects

$\triangleright$ $W[\widetilde{\mathcal{A}}^{\pm}]$ may not contain enough information about the projectiles in order to compute these more detailed observables ($W[\widetilde{\mathcal{A}}^{\pm}]$ is only the diagonal part of the initial density matrix of the incoming nucleus)