Notes on Feynman rules in quantum mechanics

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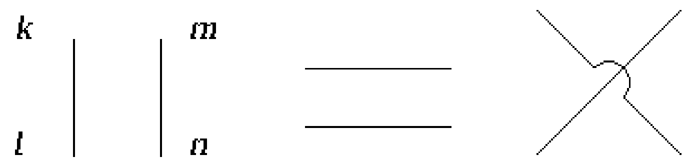

Figure 1: Feynman diagrams.

1 Correlations in gaussian variables

We start from the result

$$Z[y] = \int \Pi \; dx_i \; e^{-\frac{1}{2}x^T A x + y^T x} = \sqrt{\frac{\pi^N}{\det A}} e^{\frac{1}{2}y^T A^{-1} y} \tag{1}$$

We define correlations as

$$\begin{aligned} \langle x_k x_l x_m x_n ....\rangle &= \frac{1}{Z[0]} \int \Pi \; dx_i \; x_k x_l x_m x_n .... e^{-\frac{1}{2}x^T A x} \\ &= \frac{1}{Z[0]} \frac{\delta Z[y]}{\delta y_k \delta y_l \delta y_m \delta y_n ....}|_{y=0} \end{aligned} \tag{2}$$

One can deduce by differentiating with respoect to $-y^m$ multilpe times that

$$\langle x_k x_l x_m x_n ....\rangle = \sum_{\text{All pairwise combinations}} A^{-1}_{kl} A^{-1}_{mn} .... \tag{3}$$

Diagrammatically, one can represent the correlations by pairwise lines representing propagators. Each propagator corresponds to A^{-1}_{mn}. See Fig. for an example.

Note that pairwise combinations are not possible for an odd number of x's. These terms are 0 by parity symmetry.

2 Correlation functions for the harmonic oscillator

Now consider the case where the lagrangian is that of a harmonic oscillator,

$$Z[y] = \int \mathcal{D}x \; e^{i\frac{1}{2}\int dt x(t) m[-\frac{d^2}{dt^2} - \omega^2]x(t) + \int dt y(t)x(t)} = \sqrt{\frac{1}{\det \hat{A}}} e^{\frac{1}{2}\int dt y(t)\hat{A}^{-1}y(t)} \tag{4}$$

where

$$\hat{A} = im[\frac{d^2}{dt^2} + \omega^2] \; . \tag{5}$$

Following the previous section, we define correlations as

$$\begin{aligned} \langle T\{x(t_1)x(t_2)x(t_3)x(t_4)....\}\rangle &= \frac{1}{Z[0]}\int \mathcal{D}x \; x(t_1)x(t_2)x(t_3)x(t_4)....e^{-\frac{1}{2}\int \; dt\; x(t)\hat{A}x(t)} \\ &= \frac{1}{Z[0]} \frac{\delta Z[y]}{\delta y(t_1)\delta y(t_2)\delta y(t_3)\delta y(t_4)....}|_{y=0} \end{aligned} \tag{6}$$

The continuum version of the previous section gives,

$$\langle T\{x(t_1)x(t_2)x(t_3)x(t_4)....\}\rangle = \sum_{\text{All pairwise combinations}} A^{-1}(t_1,t_2)A^{-1}(t_3,t_4).... \tag{7}$$

The inverse of the operator A can be easily found in Fourier space,

$$A^{-1}(t_1,t_2) = \int \frac{dp^0}{2\pi} \frac{i}{m[(p^0)^2 - \omega^2 + i\epsilon]} e^{-ip^0(t_1-t_2)} \; , \tag{8}$$

where $i\epsilon$ avoids the singularity at $\pm\omega$.

Using contour integration one can show that,

$$A^{-1}(t_1,t_2) = \frac{1}{2m\omega}[\theta(t_1-t_2)e^{-i\omega(t_1-t_2)} + \theta(t_2-t_1)e^{i\omega(t_1-t_2)}] \tag{9}$$

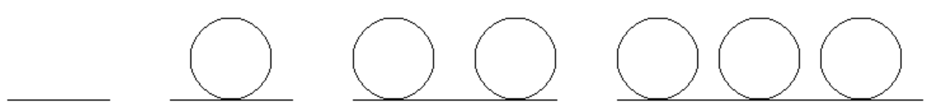

Figure 2: Feynman diagrams for the correction to the propagator.

3 Anharmonic oscillator

Now consider the case where the lagrangian where we add corrections to a harmonic oscillator,

$$Z[y] = \int \mathcal{D}x \; e^{i\frac{1}{2}\int dt x(t) m[-\frac{d^2}{dt^2} - \omega^2]x(t) - i\int dt \frac{g_4}{4!} x(t)^3 + \int dt y(t)x(t)} \tag{10}$$

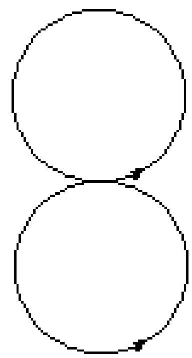

Figure 3: Feynman diagrams for the energy of the zero occupation state.

This can not be evaluated as before since the lagrangian is not quadratic. But quantities can be evaluated perturbatively in g_4. For example, let us consider the two point correlation function,

$$\begin{aligned} \langle T\{x(t_1)x(t_2)\}\rangle &= \frac{1}{Z[0]}\int \mathcal{D}x\; x(t_1)x(t_2)e^{i\frac{1}{2}\int dt x(t)m[-\frac{d^2}{dt^2}-\omega^2]x(t)-i\int dt\frac{g_4}{4!}x(t)^3} \\ &= \frac{1}{Z[0]}\frac{\delta Z[y]}{\delta y(t_1)\delta y(t_2)}|_{y=0} \end{aligned} \quad (11)$$

In Fourier space, to the lowest order in g_4 ($(g_4)^0$), the correlation function is given by the "non-interacting" propagator

$$\frac{i}{m[(p^0)^2-\omega^2+i\epsilon]} \quad (12)$$

The corrections to the propagator form a geometric series, Fig. 2

$$\begin{aligned} &\frac{i}{m[(p^0)^2-\omega^2+i\epsilon]}+\frac{i}{m[(p^0)^2-\omega^2+i\epsilon]}(-i\Sigma)\frac{i}{m[(p^0)^2-\omega^2+i\epsilon]} \\ &+\frac{i}{m[(p^0)^2-\omega^2+i\epsilon]}(-i\Sigma)\frac{i}{m[(p^0)^2-\omega^2+i\epsilon]}(-i\Sigma)\frac{i}{m[(p^0)^2-\omega^2+i\epsilon]}+\cdot\cdot \\ &=\frac{i}{m[(p^0)^2-\omega^2-\Sigma/m+i\epsilon]} \end{aligned} \quad (13)$$

Σ gives the correction to the excitation energy of the state with occupation number 1. In particular,

$$\Delta E_1 = \frac{\Sigma}{2\omega m}\ . \quad (14)$$

The calculation of Σ gives,

$$-i\Sigma = \frac{-ig_4}{2}\int\frac{dk^0}{2\pi}\frac{i}{m[(k^0)^2-\omega^2+i\epsilon]}\ . \quad (15)$$

This integration can be performed analytically and gives the result,

$$\Sigma = \frac{g_4}{4m\omega} \, . \tag{16}$$

Then, the change in the energy can be obtained from Eq. 14. This can be compared to the result from time independent perturbation theory.

The correction to the ground state energy is given by the diagram Fig. 3.