Problem Set 4 (due Sep 30, 2018)

1. **(Effective range expansion for arbitrary l)** Problem 3.19(a) in Advanced quantum theory, Paul Roman.

2. **(Partial waves for a repulsive spherical well potential)** Consider the potential in 3 space dimensions

$$U(r) = \begin{cases} U_0 & r < d \\ 0 & r \geq d \end{cases} , \tag{1}$$

where $U(r) = 2mV(r)$, and $r = |\mathbf{r}|$.
We consider scattering states with $E > 0$ and $k = \sqrt{2mE}$. We also define

$$f_l(k) = \frac{1}{k} e^{i\delta_l(k)} \sin \delta_l(k) , \tag{2}$$

so that

$$f(k, \theta) = \sum_l (2l + 1) f_l(k) P_l(\cos\theta) . \tag{3}$$

Therefore

$$\begin{aligned} \sigma(k) &= \int d\cos\theta d\phi |f(k, \theta)|^2 \\ &= \sum_l (2l + 1)(4\pi)|f_l(k)|^2 \\ &= \sum_l \sigma_l(k) . \end{aligned} \tag{4}$$

   (a) First consider general l.

      i. Write the interior ($r < d$) form of the Schrödinger equation and its solution. (Consider three cases (1) $U_0 < 0$ (2) $U_0 > k^2$ (3) $0 < U_0 < k^2$)

      ii. Write the exterior ($r > d$) form of the Schrödinger equation and its general solution. (Introduce the phase shift.)

      iii. Match the two solutions appropriately at $r = d$ to find a general equation for the phase shift as a function of the incident momentum k.

   (b) Consider the $l = 0$ phase shifts in the partial wave expansion.

      i. What is the functional behaviour of the phase shift in the limit of $|U_0| \gg k^2$? What is the scattering length. What is the effective range?

ii. Plot the range and the scattering length in units for d as a function of U_0 for $U_0 \in [-(10)^2/d^2, 1/d^2]$

iii. What is the functional behaviour of the phase shift in the limit of $|U| \ll k^2$? Consider two cases (1) $U_0 < 0$, $|U_0| \ll k^2$ (2) $0 < U_0 \ll k^2$

iv. Taking $U_0 = 2/d^2$ plot the phase shift, $f_0(k)$, and $\sigma_0(k)$ as a function of k up to reasonable values.

v. Taking $U_0 = -(4.8)^2/d^2$ plot the phase shift, $f_0(k)$, and $\sigma_0(k)$ as a function of k up to reasonable values.

vi. Levinson's theorem states that if the phase shift at $k \to \infty$ is taken to be 0, and the phase shifts are chosen to be monotonically increasing as k decreases, then

$$\begin{aligned} \delta_l(k=0) &= n\pi \ \text{ if l } > \text{ 0 or there is no l } = \text{ 0 bound state with E } = \text{ 0} \\ \delta_0(k=0) &= (n+1/2)\pi \ \text{ there is a l } = \text{ 0 bound state with E } = \text{ 0} \end{aligned} \tag{5}$$

n is the number of bound states ($E < 0$). Argue that the result for the phase shifts calculated in Parts 2(b)iv, 2(b)iv are consistent with Levinson's theorem.

(c) Repeat the exercises given in Part 2b for $l = 1$.