Problem Set 5 (due Oct 15, 2018)

1. **Field operators** The Fock space representation of a state for N spin 0 particles with a quantum mechanical wavefunction $\Phi(r_1, r_2, \cdot\cdot r_N)$ can be written as follows. We introduce field operators $\psi(r)$, $\psi^\dagger(r)$ satisfying the commutation relations

$$\begin{aligned} [\psi(r), \psi^\dagger(r')] &= \delta(r - r') \\ [\psi(r), \psi(r')] &= 0 \\ [\psi^\dagger(r), \psi^\dagger(r')] &= 0 \ . \end{aligned} \tag{1}$$

We define a basis of states,

$$|r_1, r_2, \cdot\cdot r_N\rangle = \psi^\dagger(r_1)\psi^\dagger(r_2) \cdot\cdot \psi^\dagger(r_N)|\Omega\rangle \tag{2}$$

$$|\Phi\rangle = \int \Pi_{i=1}^N dr_i \frac{1}{N!}\Phi(r_1, r_2, \cdot\cdot r_N)|r_1, r_2, \cdot\cdot r_N\rangle \tag{3}$$

where $\Phi(r_1, r_2, \cdot\cdot r_N)$ is symmetric under the exchange of its arguments. In other words, $\Phi(r_1, r_2, \cdot\cdot r_N)$ is the state in position representation,

$$\Phi(r_1, r_2, \cdot\cdot r_N) = \langle r_1, r_2, \cdot\cdot r_N|\Phi\rangle \tag{4}$$

(a) Generalize Eqs. 1, 2, 3, 4 for particles with spin 1/2. Note that there will be an additional index associated with the spin. What is the symmetry property of the wavefunctions under the permutation of its arguments?

(b) Write down the Hamiltonian $\hat{H}$ in the Fock space representation for a multi-particle system of spin 0 particles in terms of the field operators for a system of particles in the presence of a background potential $V(r)$. Explicitly show that the Hamiltonian acting on a $N-$particle state with a quantum mechanical wavefunction $\Phi(r_1, r_2, \cdot\cdot r_N)$ gives the desired result. I.e.

$$\langle r_1, r_2, \cdot\cdot r_N|\hat{H}|\Phi\rangle = [-\sum_i \frac{\nabla_i^2}{2m} + \sum_i V(r_i)]\Phi(r_1, r_2, \cdot\cdot r_N) \tag{5}$$

Generalize to spin 1/2 allowing for spin dependent $V(r)$.

(c) Add terms to the Hamiltonian (spin 0) to include a two-body potential interaction $V_2(r_1, r_2)$ and a three body potential $V_3(r_1, r_2, r_3)$

(d) What are the constraints on the potentials V_1, V_2, V_3 if the system is translationally invariant? (The Hamiltonian should not change if the location of each particle is shifted by the same amount.) Use the constraint to write $V_2(r_1, r_2)$ in momentum space. What is the additional constraint on V_2 if the system is rotationally invariant?

2. **(Galilean invariance of non-relativistic quantum mechanics)** Problem 1, Chapter 2, *Quantum Field Theory* by Lowell S. Brown