Problem Set 6 (due Oct 30, 2018)

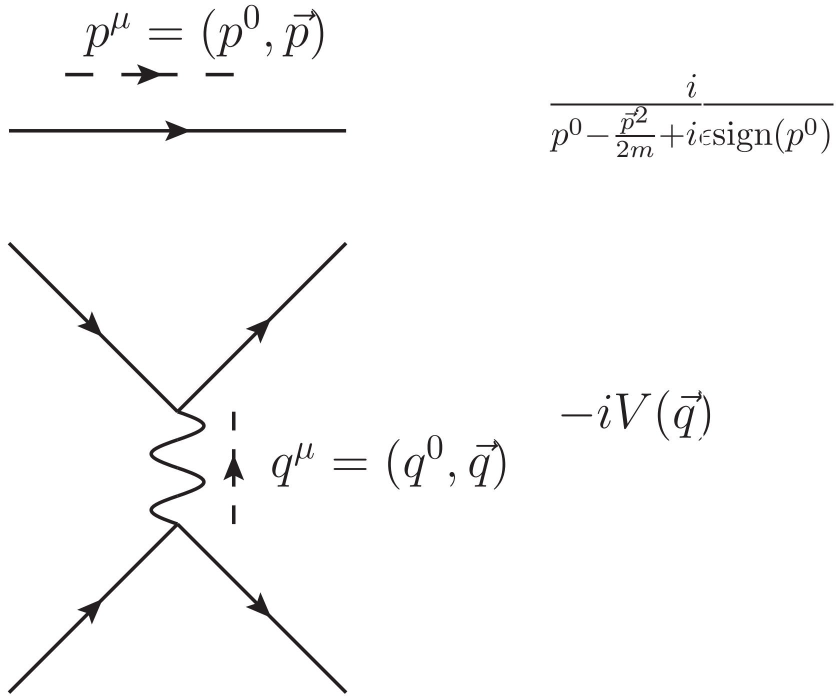

Figure 1: Feynman rules for Eq. 1 in momentum space. The arrows on the dashed lines represent momentum flow while the arrows on the solid lines represent particle number flow. If the arrow direction is opposite to the particle flow direction in the propagator (i.e. opposite to the convention shown in the figure above), p^μ in the propagator should be replaced by $-p^\mu$. The other Feynman rules are (1) Assume momentum conservation at each vertex (2) Assume particle conservation on each vertex (3) All unconstrained momenta ($k^\mu = (k^0, \vec{k})$) are integrated over (4) For scattering amplitudes, the external legs corresponding to the propagation of the "in" states and the "out" states are not included ("amputated diagram") while for correlation functions they are (5) For fermions, for each fermion line intersections multiply the expression by a factor -1 (6) For fermions, for each fermion loop multiply the expression by a factor -1.

1. **Scattering in non-relativistic field theory** The lagrangian density for a spin 0 particle in Fock space can be written as with a two particle potential $V(r)$ is given

by

$$\mathcal{L} = \psi^\dagger(r) i\partial_t \psi(r) - \psi^\dagger(r)[-\frac{\nabla^2}{2m}]\psi(r) - \frac{1}{2}\int d^3r' \psi^\dagger(r)\psi^\dagger(r')[V(r-r')]\psi(r')\psi(r) \quad . \tag{1}$$

The Feynman rules are given in Fig. 1

(a) Draw the two diagrams corresponding to the lowest Born level scattering amplitude for the scattering process $p_a, p_b \to p_c, p_d$. Write down the total scattering amplitude $iT(p_a, p_b \to p_c, p_d)$? The "in" and "out" momenta are on-shell; meaning that $p^0 = \vec{p}^2/(2m)$. The momenta of the intermediate propagators need not be on-shell and are only constrained by momentum-energy conservation.

(b) Consider the special case where $V(r) = \lambda\delta(r)$ where $\lambda < 0$ for attractive and $\lambda > 0$ for repulsive interaction. For the rest of the problem assume this "contact" form of the potential. What is the scattering amplitude in the previous part for the contact potential?

(c) Draw the diagram for the second Born level amplitude. Label the momenta and write the expression for the diagram. Choose the loop momentum to be k^μ. It is useful to introduce variables $p_a^0 + p_b^0 = E_{in}$ and $\vec{p}_a^0 + \vec{p}_b^0 = \vec{p}_{in}$

(d) Evaluate the k^0 integral using contour integration. This requires a very careful consideration of the two poles in the propagator: the two poles in the complex plane should on the opposite side of the real axis.

(e) The integral over $\vec{k}$ is divergent (at large $|\vec{k}|$). Cutoff the integration at $|\vec{k}| = \Lambda$ to find the regularized expression for the second Born level amplitude

(f) Show that going to higher order Born amplitudes gives a geometric series. Sum the series.

(g) By matching the form of the T-matrix at small p^a, p^b, find the scattering length and the range. What is the condition for the scattering length to be infinite?

2. **Finite chemical potential at $T = 0$.** Consider now the case of non-interacting fermions: $V = 0$. In the presence of a chemical potential, the propagator for the fermions is

$$\frac{i}{p^0 - (\frac{p^2}{2m}) + \mu + i\epsilon\mathrm{sign}(p^0)} \tag{2}$$

Calculate the following integrals where $\mu > 0$ is a constant and $\epsilon > 0$ and $0^+ > 0$ are infinitesimal constants.

(a)

$$\int \frac{dp^0}{(2\pi)} \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i0^+ p^0} \frac{i}{p^0 - \frac{\mathbf{p}^2}{2m} + \mu + i\epsilon\mathrm{sign}(p^0)} \tag{3}$$

Use intuition from the statistical mechanics of Fermi systems to interpret the result.

3. **Self energy corrections to the fermionic propagator** The lagrangian density for a spin 1/2 particle in Fock space in Euclidean space at a finite chemical potential μ can be written

$$\mathcal{L} = \psi_s^\dagger(r)\partial_\tau\psi_s(r) + \psi_s^\dagger(r)[-\frac{\nabla^2}{2m} - \mu]\psi_s(r) + \frac{1}{2}\int d^3r'\psi_s^\dagger(r)\psi_t^\dagger(r')[V(r-r')]\psi_t(r')\psi_s(r) \quad . \tag{4}$$

$s, t = -1/2, +1/2$. The partition function is

$$Z = \int \mathcal{D}\psi^\dagger\mathcal{D}\psi e^{-\int d^4x\mathcal{L}} \ . \tag{5}$$

At finite T and μ the Feynman diagrams are the same as given in Fig. 1 but the Feynman rules are now in imaginary time. In particular

- The propagator is $\frac{1}{p^0+\frac{p^2}{2m}-\mu}|_{p^0=i(2n+1)\pi T}$
- The interaction vertex is $-V(\vec{q})$
- Integration over internal frequencies, k^0, is replaced by

$$T\sum_{k^0=i(2n+1)\pi T} \tag{6}$$

(a) Show the diagrams which contribute to the correction in the fermionic propagator to the lowest order in V (There are two diagrams called Hartree and Fock contributions.)

(b) Write down the expressions for the diagrams in momentum space for the two contributions Σ. You do not need to compute Σ

(c) Show that going to higher order terms in V gives a geometric series. Sum the series in terms of the self energy correction Σ.

(d) For $V(r) = \lambda\delta(r)$ calculate Σ and show that the chemical potential is modified. Does it increase or decrease? What is the physical significance of the change?