Problem Set 1 (due Sept 21, 2015)

1. **Special relativity, tensors**

   (a) Show that if components of any four vector V^μ are transformed by the rule $V^\mu \to (V')^\mu = \Lambda^\mu{}_\nu V^\nu$ when one goes from a reference frame/coordinate system F to a new reference frame F', then the condition $\eta_{\mu\nu}\Lambda^\mu{}_\sigma \Lambda^\nu{}_\lambda = \eta_{\sigma\lambda}$ implies the transformation leaves the dot product of and two four vectors unchanged. For the special cases that the two vectors are the same, this condition ensures that the norms (for eg. the space-time interval $x^\mu x_\mu$ or the mass $p_\mu p^\mu$) are Lorentz invariant

   (b) Write $\Lambda^\mu{}_\nu$ that transforms coordinates in a reference frame F to a reference frame F' rotated by an angle θ about the x direction

   (c) Write $\Lambda^\mu{}_\nu$ that transforms coordinates in a reference frame F to a reference frame F' boosted by speed v about the $+x$ direction

   (d) $\Lambda_\nu{}^\mu$ is defined by $\Lambda_\nu{}^\mu = \eta^{\mu\sigma}\eta_{\nu\lambda}\Lambda^\lambda{}_\sigma$. Considering $\Lambda_\nu{}^\mu$ as a matrix $[L](\nu,\mu)$ and $[M](\lambda,\sigma) = \Lambda^\lambda{}_\sigma$ prove that $[L] = ([M]^{-1})^T$

   (e) Write the derivatives in the new coordinate system $\partial_\mu{}' = \frac{\partial}{\partial (X')^\mu}$ in terms of the derivatives in the old coordinate system $\partial_\mu = \frac{\partial}{\partial X^\mu}$ and the transformation Λ (Hint: The order of the upper and lower indices on Λ is important). For a scalar field ϕ show that $\partial_\mu \phi \partial_\nu \phi \eta^{\mu\nu}$ is a Lorentz scalar

2. **Antisymmetric tensors**

   (a) Show that a symmetric (antisymmetric) tensor S (A) satisfying $S^{\mu\nu} = S^{\nu\mu}$ ($A^{\mu\nu} = -A^{\nu\mu}$) remains symmetric (antisymmetric) on changing the reference frame

   (b) Show that $\partial_\alpha F_{\beta\gamma} + (\alpha\beta\gamma \to \gamma\alpha\beta) + (\alpha\beta\gamma \to \beta\gamma\alpha) = 0$

   (c) Define the fully antisymmetric tensor on 4 dimensions as

$$\varepsilon^{\mu\nu\sigma\lambda} = \left(\begin{array}{c} 1 \quad (\mu,\nu,\sigma,\lambda) \text{ are cyclic permutations of } (1,2,3,4) \\ -1 \quad (\mu,\nu,\sigma,\lambda) \text{ are anti} - \text{cyclic permutations of } (1,2,3,4) \\ 0 \quad \text{otherwise} \end{array} \right. \tag{1}$$

   Lower the indices and write $\varepsilon_{\mu\nu\sigma\lambda}$ in the form of Eq. 1

   (d) Define the dual stress-energy tensor $(F^*)^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\sigma\lambda}F_{\sigma\lambda}$ [this is called $\mathcal{F}$ in Classical Electrodynamics by Jackson (pg. 556)]. Write down the components of F^* in terms of E and B

3. **E and B** Jackson Problem 11.14 (a) and (b) [Pg. 571]
4. **Covariant form for the field of a point charge** Jackson Problem 11.17 [Pg. 572]
5. **Ultrarelativistic limit** Jackson Problem 11.18 [Pg. 573]