Problem Set 4 (due Nov 16, 2015)

1. **Relativistic Kinematics of Compton scattering**

   (a) Consider Comption scattering. A photon with momentum four vector $k^\mu = (k, 0, 0, k) = \hbar(\omega/c, 0, 0, \omega/c)$ collides with a massive particle initially at rest with four momentum $p^\mu = (mc, 0, 0, 0)$. After the collision the photon momentum vector is $k, \mu = (k', k' \sin\theta, 0, k' \cos\theta)$. Use momentum conservation to find k'/k in terms of m, ω, θ. This gives the corrected Compton differential cross-section [Eq. 14.127 Jackson] to the Thompson differential cross-section [Eq. 14.124 Jackson]. Show the steps.

2. **Relativistic Kinematics of Coulomb scattering**

   (a) Consider the scattering (to be concrete one can think of Coulomb scattering but the following kinematic results apply to any scattering that conserves four momentum) of two particles with mass m_1 and m_2 in the centre of mass frame. This implies that the initial spatial momenta of the particles are equal and opposite, $p_1^\mu = (\sqrt{\mathbf{p}^2 + m_1^2 c^2}, \mathbf{p})$, $p_2^\mu = (\sqrt{\mathbf{p}^2 + m_2^2 c^2}, -\mathbf{p})$. Rewrite $\mathbf{p}^2$ in terms of the total energy $E^2 = s^2 = (p_1^\mu + p_2^\mu)^2$ and the the masses. What is the magnitude of the spatial momentum after the collision $[(\mathbf{p}')^2]$?

   (b) Show that the kinetic energy transferred to an electron (T) during a collision with momentum transfer $K^\mu = p'^\mu - p^\mu$ is given by

   $$-(K)^2 = 2m_e T \ . \tag{1}$$

   (Conventionally, one defines $Q^2 = -K^2$ thus obtaining $Q^2 = 2m_e T$.) This is general and does not assume that the mass of the incident particle, M, is large compared to m_e. Now consider the incoming particle coming with a momentum $\mathbf{p} = M\gamma\beta c$ striking an electron initially at rest. Assuming $M \gg m_e$ and $M \gg |\mathbf{p}|c$ show that the maximum kinetic energy transferred to the electron, $T_{\max}$, is given by

   $$T_{\max} = 2\gamma^2\beta^2 c^2 m_e \ . \tag{2}$$

   Now relax the assumption that $M \gg m_e$. Then show that,

   $$T_{\max} = 2\gamma^2\beta^2 c^2 m_e \frac{1}{1 + 2m_e E/(M^2 c^2) + m_e^2/M^2} \ . \tag{3}$$

3. **Relativistic Kinematics in Bremsstrahlung**

(a) Show that a single incident particle with mass $M \neq 0$ and intial four momentum $p^\mu = (\sqrt{M^2 + \mathbf{p}^2}, \mathbf{p})$ can not split into a photon and another particle with the same mass. This is the quantum-mechanical interpretation of the classical statement that a charge moving without acceleration can not emit radiation. Furthermore, show that if $M = 0$, it can only split into a photon moving collinearly or anticollinearly with the initial momentum

(b) Now consider the scattering of the incident particle of mass M with a heavy nucleus $\mathcal{M} \gg M$. Let the four momentum transferred to the heavy nucleus be $K^\mu = (K^0, \mathbf{K})$. The quantum mechanical interpretation of Bremsstrahlung is that the momentum exchange with the nucleus can kinematically allow the emission of a photon — acceleration of M can lead to emission of radiation. In general, the kinematical constraint

$$p^\mu + (\mathcal{M}c, 0) \to (p')^\mu + (\mathcal{M}c + K^0, \mathbf{K}) + \hbar(k, \mathbf{k}) \ , \tag{4}$$

is quite complicated. (Google "three body phase space" and "Dalitz plot"). (Here, $|\mathbf{k}| = \omega/c$ is the photon wave-vector.)

But in the limit $\mathcal{M}c \gg |\mathbf{K}|$, show that $K^0 = 0$. Thus,

$$K^\mu = (0, \mathbf{p}' + \mathbf{k} - \mathbf{p}) \tag{5}$$

This implies $K^2 = -Q^2 = -(\mathbf{p}' + \mathbf{k} - \mathbf{p})^2$. This simplifies the calculation substantially, and we shall consider this limit in the following two parts

(c) First consider the non-relativistic case $Mc \gg |\mathbf{q}|$. The four momenta can then be written as

$$\begin{aligned} p^\mu &\approx (Mc + \frac{\mathbf{p}^2}{2Mc}, \mathbf{p}) \\ (p')^\mu &\approx (Mc + \frac{(\mathbf{p}')^2}{2Mc}, \mathbf{p}') \end{aligned} \tag{6}$$

The energy conservation equation gives

$$\frac{\mathbf{p}^2}{2M} = \frac{(\mathbf{p}')^2}{2M} + \hbar\omega \ . \tag{7}$$

Show that

$$Q^2 = (\mathbf{p}' + \mathbf{k} - \mathbf{p})^2 \approx (\mathbf{p}' - \mathbf{p})^2 \ . \tag{8}$$

The maximum (minimum) value of Q^2 arises when $\mathbf{p}'$ and $\mathbf{p}$ are anti-parallel (parallel).

$$\begin{aligned} Q^2_{\max} &= (|\mathbf{p}'| + |\mathbf{p}|)^2 = (\sqrt{2ME'} + \sqrt{2ME})^2 \\ Q^2_{\min} &= (|\mathbf{p}'| - |\mathbf{p}|)^2 = (\sqrt{2ME'} - \sqrt{2ME})^2 \ , \end{aligned} \tag{9}$$

where $E = \mathbf{p}^2/(2M)$ and $E' = (\mathbf{p}')^2/(2M)$. Show that,

$$\frac{Q_{\max}}{Q_{\min}} = \frac{(\sqrt{E} + \sqrt{E'})^2}{\hbar\omega} \tag{10}$$

(d) Now consider the highly relativistic case $Mc \ll |\mathbf{q}|$. The four momenta can then be written as

$$\begin{aligned} p^\mu &\approx (p + \frac{(Mc)^2}{2p}, \mathbf{p}) \\ (p')^\mu &\approx (p' + \frac{(Mc)^2}{2p'}, \mathbf{p}') \end{aligned} \tag{11}$$

The energy conservation equation gives

$$p + \frac{(Mc)^2}{2p} = p' + \frac{(Mc)^2}{2p'} + \hbar\omega \ . \tag{12}$$

The maximum (minimum) value of Q^2 arises when $\mathbf{p}'$ and $\mathbf{p}, \mathbf{k}$ are anti-parallel (parallel).

$$\begin{aligned} Q^2_{\text{max}} &= (|\mathbf{p}'| + |\mathbf{p}| + |\mathbf{k}|)^2 \\ Q^2_{\text{min}} &= (|\mathbf{p}'| - |\mathbf{p}| - |\mathbf{k}|)^2 \ . \end{aligned} \tag{13}$$

Write down Q_{max} and Q_{min} in terms of p and ω.

(Note that the discussion also applies to the emission of other massless or very light particles, for eg. neutrinos instead of photons. Ergo, neutrino bremsstrahlung)

4. **Coulomb scattering** Problem 13.1 Jackson

5. **Bremsstrahlung from rigid sphere** Problem 15.2 Jackson

6. **Bremsstrahlung from rigid sphere** Problem 15.3 Jackson