Problem Set 3 (due Feb 26, 2014)

1. **Correlation function for time-like and space-like separations** (Feel free to use *Mathematica*. Another useful resource is *Gradsteyn and Ryzhik*.) We define

$$D(x-y) = \langle\Omega|\phi(x)\phi(y)|\Omega\rangle. \tag{1}$$

Show that in non-interacting scalar field theory

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x-y)} \quad , \tag{2}$$

where p^μ is on shell. We saw that $D(x-y) = D(\Lambda(x-y))$. Let us restrict to orthochronous lorentz transformations Λ, which do not change the sign of $y^0 - x^0$. Consider three cases.

   (a) First consider $(x-y)^2 = t^2 > 0$, with $(y^0 - x^0) > 0$. Then we can choose Λ to go to a frame where $\Lambda^\mu{}_\nu(y-x)^\nu = (t, \mathbf{0})$, with $t > 0$. Express the value of D in terms of Bessel functions. Plot the real and imaginary part of the D as a function of t from $m = 1$MeV.

   (b) Now consider $(x-y)^2 = t^2 > 0$, with $(y^0 - x^0) < 0$. Then we can choose Λ to go to a frame where $\Lambda^\mu{}_\nu(y-x)^\nu = (t, \mathbf{0})$, with $t < 0$. Plot the real and imaginary part of the D as a function of t for $m = 1$MeV.

   (c) Now consider space-like displacements $(x-y)^2 = -r^2 < 0$. Peskin, Problem 2.3. Plot the real and imaginary part of the D as a function of r for $m = 1$MeV.

2. **Green's functions for Feynman and Retarded propagators.**

   (a) Show that the Feynman propagator

$$D_F(x-y) \equiv \langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle \tag{3}$$

and the retarded propagator

$$D_R(x-y) \equiv \langle\Omega|[\phi(x), \phi(y)]|\Omega\rangle\theta(x^0 - y^0) \tag{4}$$

are Green's functions for the operator $(-\partial_\mu\partial^\mu - m^2)$ satisfying

$$(-\partial_\mu\partial^\mu - m^2)D_{F/R}(x-y) = i\delta(x^\mu - y^\mu) \tag{5}$$

(b) Show that for the free field theory

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} = D_F(x-y) \ , \tag{6}$$

with $\epsilon \to 0^+$ and where p^μ in the four momentum integral is not necessarily onshell.

(c) Rewrite

$$\begin{aligned} &\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{ip\cdot(y-x)} = \\ &\int \frac{d^4p}{(2\pi)^4} \frac{i}{(p^0 - \sqrt{\mathbf{p}^2 + m^2} + i\epsilon'\text{sign}(\text{p}^0))(p^0 + \sqrt{\mathbf{p}^2 + m^2} + i\epsilon'\text{sign}(\text{p}^0))} e^{-ip\cdot(x-y)} = \\ &\int \frac{d^4p}{(2\pi)^4} \frac{i}{(p^0(i + i\epsilon'') - \sqrt{\mathbf{p}^2 + m^2})(p^0(1 + i\epsilon'') + \sqrt{\mathbf{p}^2 + m^2})} e^{-ip\cdot(x-y)} \end{aligned} \tag{7}$$

with $\epsilon,\ \epsilon',\ \epsilon'' \to 0^+$. The last form illustrates that the Feynman propagator can be obtained by integrating p^0 over a contour in the complex plane at a small positive angle, ie. from $(-\infty(1+i\epsilon), \infty(1+i\epsilon))$ rather than from $(-\infty, \infty)$.

(d) Choose a $i\epsilon$ prescription in momentum space representation of propagator to obtain the retarded propagator. (Hint: Try modifying the signs of $i\epsilon'$ on the right hand side of Eq. 7).

(e) Now consider Eq. 2. Write $D(x-y)$ in terms of the real part of the Feynman propagator. (Hint: $\frac{1}{x \pm i\epsilon} = \mp i\pi\delta(x) + \mathcal{P}[\frac{1}{x}]$.)