Problem Set 8 (due May 11, 2014)

Some notation:

$$\begin{aligned} L_a{}^b &= \exp(iS^{\mu\nu}(L)_a{}^b\omega_{\mu\nu}/2) \\ R_{\dot{a}}{}^{\dot{b}} &= \exp(iS^{\mu\nu}(R)_{\dot{a}}{}^{\dot{b}}\omega_{\mu\nu}/2) \\ \Lambda^\sigma{}_\lambda &= \exp(i(M^{\mu\nu})^\sigma{}_\lambda\omega_{\mu\nu}/2) \end{aligned} \tag{1}$$

$$\begin{aligned} S^{\mu\nu}(L) &= \begin{cases} \varepsilon^{ijk}\sigma^k/2 & \text{for } S^{ij} \\ -i\sigma^k/2 & \text{for } S^{0k} \end{cases} \\ S^{\mu\nu}(R) &= \begin{cases} -\varepsilon^{ijk}\sigma^{*k}/2 & \text{for } S^{ij} \\ -i\sigma^{*k}/2 & \text{for } S^{0k} \end{cases} \\ (M^{\mu\nu})^\sigma{}_\lambda &= \frac{1}{i}(g^{\mu\sigma}\delta^\nu_\tau - g^{\nu\sigma}\delta^\mu_\tau) \end{aligned} \tag{2}$$

1. **Skew symmetric metric**

   (a) We argued that the antisymmetric subspace of the product of the representations $(2,1)\times(2,1)$ forms a singlet (one dimensional) representation $(1,1)$. Explicitly show that,

   $$L_a{}^c L_b{}^d \varepsilon_{cd} = \varepsilon_{ab}\ . \tag{3}$$

   That is, you do not get an extra factor on the right hand side.

   (b) We expect $[\sigma^\mu]_{a\dot{b}}[\sigma_\mu]_{c\dot{d}}$ to be a scalar (μ is contracted). Show that

   $$[\sigma^\mu]_{a\dot{b}}[\sigma_\mu]_{c\dot{d}} = -2\varepsilon_{ac}\varepsilon_{\dot{b}\dot{d}} \tag{4}$$

   (c) We expect $\varepsilon^{ac}\varepsilon^{\dot{b}\dot{d}}[\sigma^\mu]_{a\dot{b}}[\sigma^\nu]_{c\dot{d}}$ to be a invariant under Lorentz transformations (all spinor indices are contracted). Show that

   $$\varepsilon^{ac}\varepsilon^{\dot{b}\dot{d}}[\sigma^\mu]_{a\dot{b}}[\sigma^\nu]_{c\dot{d}} = -2g^{\mu\nu} \tag{5}$$

   (Hint: Write it as a trace of an anti-commutator)

2. **($(2,2)$ is a Vector representation)**

   (a) Show explicitly that $\sigma^\mu_{a\dot{b}} = (1,\vec{\sigma})$ transforms like a Lorentz vector under the action of L and R. Convince yourself that this implies that the combination

   $$\sigma^\mu_{a\dot{b}}\chi^{\dagger\dot{b}}\zeta^a \tag{6}$$

   transforms as a vector if the spinors χ and ζ are transformed.

(b) Define $[\bar{\sigma}^\mu]^{\dot{b}a} = \varepsilon^{ac}\varepsilon^{\dot{b}\dot{d}}\sigma^\mu_{c\dot{d}}$. Show that $[\bar{\sigma}^\mu] = (1, -\vec{\sigma})$

3. **Symmetric and antisymmetric combinations of σ**

(a) Show that $S^{\mu\nu}$ can be written in a Lorentz covariant form

$$\begin{aligned} S^{\mu\nu}(L)_a{}^b &= \frac{i}{4}([\sigma^\mu]_{a\dot{c}}[\bar{\sigma}^\nu]^{\dot{c}b} - [\sigma^\nu]_{a\dot{c}}[\bar{\sigma}^\mu]^{\dot{c}b}) \\ S^{\mu\nu}(R)^{\dot{a}}{}_{\dot{c}} &= -\frac{i}{4}([\bar{\sigma}^\mu]^{\dot{a}b}[\sigma^\nu]_{b\dot{c}} - [\bar{\sigma}^\nu]^{\dot{a}b}[\sigma^\mu]_{b\dot{c}}) \end{aligned} \tag{7}$$

Show that the two relations can be combined in Dirac notation by defining the transformation of the Dirac spinor

$$\Psi(x) \to \exp(iS^{\mu\nu}\omega_{\mu\nu}/2)\Psi(\Lambda^{-1}x) \tag{8}$$

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]\,. \tag{9}$$

(Hint: The Dirac spinor is defined as

$$\Psi = \begin{pmatrix} \chi_a \\ \psi^{\dot{a}\dagger} \end{pmatrix} \tag{10}$$

and

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\,. \tag{11}$$

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(b) Now consider the symmetric combination of σ^μ, $\bar{\sigma}^\nu$. Show that

$$\begin{aligned} ([\sigma^\mu]_{a\dot{c}}[\bar{\sigma}^\nu]^{\dot{c}b} + [\sigma^\nu]_{a\dot{c}}[\bar{\sigma}^\mu]^{\dot{c}b}) &= -2g^{\mu\nu}\delta_a{}^b \\ ([\bar{\sigma}^\mu]^{\dot{a}b}[\sigma^\nu]_{b\dot{c}} + [\bar{\sigma}^\nu]^{\dot{a}b}[\sigma^\mu]_{b\dot{c}}) &= -2g^{\mu\nu}\delta^{\dot{a}}{}_{\dot{c}} \end{aligned} \tag{12}$$

Show that the two relations can be combined in Dirac notation

$$-2g^{\mu\nu} = \{\gamma^\mu, \gamma^\nu\} \tag{13}$$

4. **Gamma matrix relations**

(a) Based on your experience with σ^μ, write down the Lorentz transformation for the quantity

$$\bar{\Psi}(x)_1\gamma^\mu\Psi(y)_2 \tag{14}$$

where Ψs are Dirac spinors

(b) Consider the charge conjugation matrix

$$\mathcal{C} = \begin{pmatrix} \varepsilon_{ac} & 0 \\ 0 & \varepsilon_{\dot{a}\dot{c}} \end{pmatrix} \tag{15}$$

Show that

$$\mathcal{C}^{-1}\gamma^{\mu}\mathcal{C} = -(\gamma^{\mu})^{T} \tag{16}$$

5. **Spinor Lagrangians** Read chapter 36 from Srednicki. In particular, look at the derivation of Eq. 36.28 and Eq. 36.41, which connect the Weyl notation for the Lagrangian to the Dirac notation

6. **The Clifford algebra** The 4, 4×4 dimensional matrices, γ^{μ} generate a Clifford algebra. For our purposes, it means that any hermitian 4×4 tensor can be written as a linear combination of the following products.

$$\begin{aligned} &1 \\ &\gamma^{\mu} \\ \Gamma^{\mu\nu} &= \frac{i}{2!}\varepsilon^{\mu\nu\sigma\lambda}\gamma_{\mu}\gamma_{\nu} \text{ or equivalently by } \sigma^{\mu\nu} = (i/2)[\gamma^{\mu}, \gamma^{\nu}] \\ \Gamma^{\lambda} &= \frac{1}{3!}\varepsilon^{\mu\nu\sigma\lambda}\gamma_{\mu}\gamma_{\nu}\gamma_{\sigma} \text{ or equivalently by } \gamma^{\lambda}\gamma^{5} \\ \gamma^{5} &= -\frac{i}{4!}\varepsilon_{\mu\nu\sigma\lambda}\gamma^{\mu}\gamma^{\nu}\gamma^{\sigma}\gamma^{\lambda} \end{aligned} \tag{17}$$

Each of the matrices is an antisymmetric product of various distinct γ^{μ} (the symmetric combination gives $g^{\mu\nu}$ as in Problem 2, and gives a product of a smaller number [by 2] of γ^{μ}s.)

(a) Show that the sixteen matrices defined above are linearly independent (Hint: Multiply and take the trace)

(b) Show that in the representation of γ^{μ} that we are using,

$$\gamma^{5} = \begin{pmatrix} -1_{2\times 2} & 0 \\ 0 & 1_{2\times 2} \end{pmatrix} \tag{18}$$

(This means that $(1+\gamma^{5})/2$ $[(1-\gamma^{5})/2]$ projects the right handed (left handed) part of the Dirac spinor.)

(c) Normalize the 16 Γ^{A} such that $\mathrm{tr}[\Gamma^{\mathrm{A}}\Gamma^{\mathrm{B}}] = 4\delta^{\mathrm{AB}}$ by multiplying by appropriate powers of i. Then for Dirac spinors Ψ_i

$$\bar{\Psi}_1\Gamma^{A}\Psi_2\bar{\Psi}_3\Gamma^{B}\Psi_4 = C^{AB}_{CD}\bar{\Psi}_1\Gamma^{C}\Psi_4\bar{\Psi}_3\Gamma^{D}\Psi_2 \tag{19}$$

where $C^{AB}_{CD} = \frac{1}{16}\mathrm{tr}[\Gamma^{C}\Gamma^{A}\Gamma^{D}\Gamma^{B}]$ (This can be used to prove Fierz relations in Dirac notation.)

7. **Fierz relations in Weyl spinor notation** Question 36.3 in Srednicki

1 Asides

1. **For students with extra time** (Not for credit.) The Weyl notation that we learnt (using L and R spinors) is a convenient formalism to study supersymmetry. For eg. see Problem 3.5 in Peskin (Hint: In Srednicki's notation, take χ to be a Left handed Weyl spinor, and $i\sigma^2 = \varepsilon^{ab}$.)