Problem Set 3 (due Feb 23, 2014)

1. **Correlation function for time-like and space-like separations** (Feel free to use *Mathematica*. Another useful resource is *Gradsteyn and Ryzhik*.) We define

$$D(x-y) = \langle \Omega | \phi(x)\phi(y) | \Omega \rangle. \tag{1}$$

Show that in non-interacting scalar field theory

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)} \quad , \tag{2}$$

where p^μ is on shell. We saw that $D(x-y) = D(\Lambda(x-y))$. Let us restrict to orthochronous lorentz transformations Λ, which do not change the sign of $y^0 - x^0$ for time-like $y-x$. (For space-like $y-x$ the sign may or may not change.) Consider three cases.

   (a) First consider $(x-y)^2 = t^2 > 0$, with $(y^0 - x^0) > 0$. Then we can choose Λ to go to a frame where $\Lambda^\mu{}_\nu (y-x)^\nu = (t, \mathbf{0})$, with $t > 0$. Express the value of D in terms of Bessel functions. Plot the real and imaginary part of the D as a function of t for $m = 1\text{MeV}$. How is the plot modified if $m \neq 1$.

   (b) Now consider $(x-y)^2 = t^2 > 0$, with $(y^0 - x^0) < 0$. Then we can choose Λ to go to a frame where $\Lambda^\mu{}_\nu (y-x)^\nu = (t, \mathbf{0})$, with $t < 0$. Plot the real and imaginary part of the D as a function of t for $m = 1\text{MeV}$.

   (c) Now consider space-like displacements $(x-y)^2 = -r^2 < 0$. Peskin, Problem 2.3. Plot the real and imaginary part of the D as a function of r for $m = 1\text{MeV}$.

2. **Green's functions for Feynman and Retarded propagators.**

   (a) Show that the Feynman propagator

$$D_F(x-y) \equiv \langle \Omega | T\{\phi(x)\phi(y)\} | \Omega \rangle \tag{3}$$

and the retarded propagator

$$D_R(x-y) \equiv \langle \Omega | [\phi(x), \phi(y)] | \Omega \rangle \theta(x^0 - y^0) \tag{4}$$

are Green's functions for the operator $(-\partial_\mu \partial^\mu - m^2)$ satisfying

$$(-\partial_\mu \partial^\mu - m^2) D_{F/R}(x-y) = i\delta(x^\mu - y^\mu) \tag{5}$$

(b) Show that for the free field theory

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} = D_F(x-y) \,, \tag{6}$$

with $\epsilon \to 0^+$ and where p^μ in the four momentum integral is not necessarily onshell.

(c) Rewrite

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} =$$
$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{(p^0 - \sqrt{\mathbf{p}^2 + m^2} + i\epsilon' \mathrm{sign}(\mathrm{p}^0))(p^0 + \sqrt{\mathbf{p}^2 + m^2} + i\epsilon' \mathrm{sign}(\mathrm{p}^0))} e^{-ip\cdot(x-y)} =$$
$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{(p^0(1+i\epsilon'') - \sqrt{\mathbf{p}^2 + m^2})(p^0(1+i\epsilon'') + \sqrt{\mathbf{p}^2 + m^2})} e^{-ip\cdot(x-y)} \tag{7}$$

with $\epsilon,\ \epsilon',\ \epsilon'' \to 0^+$. The last form illustrates that the Feynman propagator can be obtained by integrating p^0 over a contour in the complex plane at a small positive angle, ie. from $(-\infty(1+i\epsilon), \infty(1+i\epsilon))$ rather than from $(-\infty, \infty)$.

(d) Choose a $i\epsilon$ prescription in momentum space representation of propagator to obtain the retarded propagator.

3. **Correlation functions for the complex scalar field**

(a) Using the expression for the quantized complex scalar field compute the following retarded correlation functions

$$\begin{aligned} &\langle \Omega | [\Phi(x), \Phi(y)] | \Omega \rangle \Theta(x^0 - y^0) \\ &\langle \Omega | [\Phi^\dagger(x), \Phi^\dagger(y)] | \Omega \rangle \Theta(x^0 - y^0) \\ &\langle \Omega | [\Phi(x), \Phi^\dagger(y)] | \Omega \rangle \Theta(x^0 - y^0) \end{aligned} \tag{8}$$

(b) Now consider $\langle \Omega | [\Phi(x), \Phi^\dagger(y)] | \Omega \rangle \Theta(x^0 - y^0)$. Suppose the masses of the particle and antiparticle are different. Is the retarded correlation function sensible?

4. **Correlation functions in a non-relativistic theory**

(a) Write down the Lagrangian for a non-interacting non-relativistic complex field.

(b) Write down the retarded correlation function $\langle \Omega | [\Psi(x), \Psi^\dagger(y)] | \Omega \rangle \Theta(x^0 - y^0)$ as an integral over four momentum. Choose the $i\epsilon$ prescription so that the correlation function is zero for $x^0 < y^0$. Is the retarded correlation function causal?