Advanced Quantum Mechanics

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Lecture #22

Path Integrals and QM

Recap of Last Class

- Statistical Mechanics and path integrals in imaginary time

- Imaginary time or Euclidean action and partition function

- Partition fn. for free particle and harmonic oscillator

- Real time correlators from imaginary time correlators

- Probability: moments and cumulants ... Gaussian distribution

Gaussian Distribution : Moments and Cumulants

$$P(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Moment Generating Function:

$$F[\alpha] = \int_{-\infty}^{\infty} e^{\alpha x} P(x) dx = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{\alpha x} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Cumulant Generating Function:

$$F[\alpha] = e^{\mu\alpha + \frac{1}{2}\alpha^2\sigma^2}$$

$$G[\alpha] = \mu\alpha + \frac{1}{2}\alpha^2\sigma^2$$

It is evident that all cumulants beyond n=2 vanishes in this case.

If we stick to $\mu = 0$, then the only relevant parameter is the second moment σ^2.

All odd order moments are zero.

Even order moments: $$\mu_{2n} = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx x^{2n} e^{-\frac{x^2}{2\sigma^2}} = \frac{\sigma^{2n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx x^{2n} e^{-\frac{x^2}{2}}$$

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}} 2^{n+1/2} \Gamma[n+1/2] = (2n+1)\sigma^2 \mu_{2n-2}$$

All moments are products of the second moment

Gaussian Integrals and Wick's Theorem

This is equivalent to the moment generating fn.

$$F[J] = \int dx_1 \int dx_2 .. \int dx_n e^{-\frac{1}{2}\sum_{ij} x_i \mathcal{G}^{-1}_{ij} x_j - \sum_i J_i x_i} = \frac{(2\pi)^{N/2}}{\sqrt{Det[\mathcal{G}^{-1}]}} e^{\frac{1}{2}\sum_{ij} J_i \mathcal{G}_{ij} J_j}$$

Note : we have not normalized the Gaussian, hence the appearance of 2π and Det[G^{-1}] factors

$$\int dx_1 \int dx_2 .. \int dx_n x_1 x_2 .. x_p e^{-\frac{1}{2}\sum_{ij} x_i \mathcal{G}^{-1}_{ij} x_j} = \left. \frac{\partial^p F}{\partial J_1 .. \partial J_p} \right|_{\vec{J}=0}$$

$$\int dx_1 \int dx_2 .. \int dx_n x_i e^{-\frac{1}{2}\sum_{ij} x_i \mathcal{G}^{-1}_{ij} x_j} = \left. \frac{\partial^p F}{\partial J_i} \right|_{\vec{J}=0} = \left. \frac{(2\pi)^{N/2}}{\sqrt{Det[\mathcal{G}^{-1}]}} \frac{1}{2} \sum_j \mathcal{G}_{ij} J_j e^{\frac{1}{2}\sum_{ij} J_i \mathcal{G}_{ij} J_j} \right|_{\vec{J}=0} = 0$$

Similarly $$\int dx_1 \int dx_2 .. \int dx_n x_i x_j x_k e^{-\frac{1}{2}\sum_{ij} x_i \mathcal{G}^{-1}_{ij} x_j} = 0$$ and so on

Once again odd moments are 0

Even Moments

$$\int dx_1 \int dx_2 .. \int dx_n x_i x_j e^{-\frac{1}{2}\sum_{ij} x_i \mathcal{G}^{-1}_{ij} x_j} = \frac{(2\pi)^{N/2}}{\sqrt{Det[\mathcal{G}^{-1}]}} \mathcal{G}_{ij}$$

Gaussian Integrals and Wick's Theorem

$$F[J] = \int dx_1 \int dx_2 .. \int dx_n e^{-\frac{1}{2}\sum_{ij} x_i \mathcal{G}^{-1}_{ij} x_j - \sum_i J_i x_i} = \frac{(2\pi)^{N/2}}{\sqrt{Det[\mathcal{G}^{-1}]}} e^{\frac{1}{2}\sum_{ij} J_i \mathcal{G}_{ij} J_j}$$

Even Moments
$$\int dx_1 \int dx_2 .. \int dx_n x_i x_j e^{-\frac{1}{2}\sum_{ij} x_i \mathcal{G}^{-1}_{ij} x_j} = \frac{(2\pi)^{N/2}}{\sqrt{Det[\mathcal{G}^{-1}]}} \mathcal{G}_{ij}$$

$$\int dx_1 \int dx_2 .. \int dx_n x_i x_j x_k x_l e^{-\frac{1}{2}\sum_{ij} x_i \mathcal{G}^{-1}_{ij} x_j} = \left. \frac{\partial^4 F}{\partial J_i \partial J_j \partial J_k \partial J_l} \right|_{\vec{J}=0}$$

The first derivative brings down a term G_{lm} J_m It is clear that one of the other derivatives must get rid of this term by acting on J_m, i.e. m has to be one of i,j,k. Otherwise this term is zero when we set J=0.

The non-zero terms come when all the x s are paired up with others

So, the final answer is a sum over all possible pairings (contractions) with each pairing giving a factor of G.

$$\int dx_1 \int dx_2 .. \int dx_n x_i x_j x_k x_l e^{-\frac{1}{2}\sum_{ij} x_i \mathcal{G}^{-1}_{ij} x_j} = \frac{(2\pi)^{N/2}}{\sqrt{Det[\mathcal{G}^{-1}]}} [\mathcal{G}_{ij}\mathcal{G}_{kl} + \mathcal{G}_{ik}\mathcal{G}_{jl} + \mathcal{G}_{il}\mathcal{G}_{jk}]$$

Gaussian Integrals and Wick's Theorem

Let us first work in the imaginary time formalism

The thermal expectation value for 2n-point correlator in a quadratic Lagrangian

$$\langle T[x(\tau_1)x(\tau_2)..x(\tau_n)]\rangle_{th} = \frac{\int_{x_0=x}^{x_\beta=x} \mathcal{D}[x(\tau)]x(\tau_1)x(\tau_2)..x(\tau_{2n})e^{-S_E[x(\tau)]}}{\int_{x_0=x}^{x_\beta=x} \mathcal{D}[x(\tau)]e^{-S_E[x(\tau)]}}$$

$$= \mathcal{G}(\tau_1,\tau_2)\mathcal{G}(\tau_3,\tau_4)..\mathcal{G}(\tau_{2n-1},\tau_{2n})$$ + all other contractions

where $\mathcal{G}(\tau_i,\tau_j) = \langle T[x(\tau_i)x(\tau_j)]\rangle_{th}$

Note: dividing by Z takes care of multiplicative constants

One can now use the previous dictionary to go back to real time if required.

So for a quadratic Lagrangian, the base quantity is the 2-point function.

This is going to be the fundamental quantity, in terms of which everything is written.

Beyond Quadratic Lagrangians : Perturbation Theory

How do we deal with Lagrangians which are not quadratic in x?

E.g. : A particle moving under the potential $V(x) = \frac{1}{2} m\omega_0^2 x^2 + g x^4$

No exact way to deal with this ——> Approximation Methods

If the quartic term is parametrically small ——> Perturbation theory ... Expansion around a quadratic Lagrangian in a dimensionless small number.

In case of H.O.: natural energy scale ω_0 natural length scale $(m\omega_0)^{-1/2}$

natural units for the coupling g $m^2\omega_0^3$

dimensionless coupling $\lambda = \frac{g}{m^2\omega_0^3} \ll 1$ for validity of perturbation theory

Expansion in powers of λ

Perturbation Theory with x^4 potential

$$U(x_f, t_f; x_i, t_i) = \left(\frac{m}{2\pi i\epsilon}\right)^{1/2} \int \mathcal{D}[x(t)] \; e^{i \int_{t_i}^{t_f} dt \frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2 x^2 - \frac{g}{4}x^4}$$

$$S_0 = \int_{t_i}^{t_f} dt \frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2 x^2 \qquad S_1 = -\int_{t_i}^{t_f} dt \frac{g}{4} x^4$$

$$U(x_f, t_f; x_i, t_i) = \left(\frac{m}{2\pi i\epsilon}\right)^{1/2} \int \mathcal{D}[x(t)] \; e^{i \int_{t_i}^{t_f} dt \frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2 x^2} \left[1 - i\frac{g}{4}\int_{t_i}^{t_f} dt x^4(t) \right.$$

$$- \frac{1}{2!}\left(\frac{g}{4}\right)^2 \int_{t_i}^{t_f} dt_1 \int_{t_i}^{t_f} dt_2 x^4(t_1) x^4(t_2)$$

$$\left. + \frac{i}{3!}\left(\frac{g}{4}\right)^3 \int_{t_i}^{t_f} dt_1 \int_{t_i}^{t_f} dt_2 \int_{t_i}^{t_f} dt_3 x^4(t_1) x^4(t_2) x^4(t_3) + ... \right]$$

We immediately see that we need to evaluate expectations of powers of x, or in general n-point correlation functions in the quadratic action

——> work in the imaginary time representation and apply Wick's Theorem. Rotate back to real time.

Perturbation Theory with x^4 potential

$$Z(\beta) = \left(\frac{m}{2\pi i\epsilon}\right)^{1/2} \int \mathcal{D}[x(\tau)] \;\; e^{-\int_0^\beta d\tau \frac{m}{2}\dot{x}^2 + \frac{m}{2}\omega^2 x^2 + \frac{g}{4}x^4}$$

It is easier to take the ratio Z/Z_0, where Z_0 is the partition fn. for the quadratic Lagrangian and develop a perturbation expansion for it.

Dividing by Z_0 takes care of the constant factors. Note that we have already calculated Z_0 before.

$$\frac{Z(\beta)}{Z_0(\beta)} = \frac{\left(\frac{m}{2\pi i\epsilon}\right)^{1/2} \int \mathcal{D}[x(\tau)] \;\; e^{-\int_0^\beta d\tau \frac{m}{2}\dot{x}^2 + \frac{m}{2}\omega^2 x^2 + \frac{g}{4!}x^4}}{\left(\frac{m}{2\pi i\epsilon}\right)^{1/2} \int \mathcal{D}[x(\tau)] \;\; e^{-\int_0^\beta d\tau \frac{m}{2}\dot{x}^2}}$$

Perturbation Expansion :

$$Z(\beta) = \left(\frac{m}{2\pi\epsilon}\right)^{1/2} \int \mathcal{D}[x(\tau)] \;\; e^{-\int_0^\beta d\tau \frac{m}{2}\dot{x}^2 + \frac{m}{2}\omega^2 x^2} \Bigg[1 - \frac{g}{4}\int_0^\beta d\tau_1 x^4(\tau_1)$$

$$+\frac{1}{2!}\left(\frac{g}{4}\right)^2 \int_{\tau_i}^{\tau_f} d\tau_1 \int_{\tau_i}^{\tau_f} d\tau_2 x^4(\tau_1) x^4(\tau_2)$$

$$-\frac{1}{3!}\left(\frac{g}{4}\right)^3 \int_{\tau_i}^{\tau_f} d\tau_1 \int_{\tau_i}^{\tau_f} d\tau_2 \int_{\tau_i}^{\tau_f} d\tau_3 x^4(\tau_1) x^4(\tau_2) x^4(\tau_3) + \ldots.\Bigg]$$

Perturbation Theory with x^4 potential

First Order Correction

$$\frac{Z^{(1)}}{Z_0} = \frac{-\frac{g}{4}\left(\frac{m}{2\pi\epsilon}\right)^{1/2}\int \mathcal{D}[x(\tau)]\ \ e^{-\int_0^\beta d\tau \frac{m}{2}\dot{x}^2+\frac{m}{2}\omega^2 x^2}\int_0^\beta d\tau_1 x^4(\tau_1)}{\left(\frac{m}{2\pi\epsilon}\right)^{1/2}\int \mathcal{D}[x(\tau)]\ \ e^{-\int_0^\beta d\tau \frac{m}{2}\dot{x}^2+\frac{m}{2}\omega^2 x^2}}$$

Now we can use Wick's rule to write each of the perturbation term as a product of two-point functions or Green's functions. Division by Z_0 gets rid of the multiplicative Determinant.

Note that for $\tau \neq \tau_1$, the integrals are the same as for the quadratic Lagrangian. For $\tau = \tau_1$, the fourth moment of the Gaussian can be related to product of second moments and we can write this as a product of Green's functions

$$x^4(\tau_1) = x(\tau_1)x(\tau_1)x(\tau_1)x(\tau_1) \qquad x^4(\tau_1) = x(\tau_1)x(\tau_1)x(\tau_1)x(\tau_1) \qquad x^4(\tau_1) = x(\tau_1)x(\tau_1)x(\tau_1)x(\tau_1)$$

Three possible contractions

Each term gives $-\frac{g}{4}\int_0^\beta d\tau_1 \mathcal{G}^2(\tau_1,\tau_1)$

Total correction $-3\frac{g}{4}\int_0^\beta d\tau_1 \mathcal{G}^2(\tau_1,\tau_1)$

Perturbation Theory with x^4 potential

2nd Order Correction

$$\frac{Z^{(2)}}{Z_0} = \frac{\frac{1}{2}\left(\frac{g}{4}\right)^2 \left(\frac{m}{2\pi\epsilon}\right)^{1/2} \int \mathcal{D}[x(\tau)] \;\; e^{-\int_0^\beta d\tau \frac{m}{2}\dot{x}^2 + \frac{m}{2}\omega^2 x^2} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 x^4(\tau_1) x^4(\tau_2)}{\left(\frac{m}{2\pi\epsilon}\right)^{1/2} \int \mathcal{D}[x(\tau)] \;\; e^{-\int_0^\beta d\tau \frac{m}{2}\dot{x}^2 + \frac{m}{2}\omega^2 x^2}}$$

$$x^4(\tau_1)x^4(\tau_2) = x(\tau_1)x(\tau_1)x(\tau_1)x(\tau_1)x(\tau_2)x(\tau_2)x(\tau_2)x(\tau_2)$$

How do we pair them up?

a) Pair up the 4 $x(\tau_1)$ s among themselves and the 4 $x(\tau_2)$ s among themselves

9 possible terms

$$x^4(\tau_1)x^4(\tau_2) = x(\tau_1)x(\tau_1)x(\tau_1)x(\tau_1)x(\tau_2)x(\tau_2)x(\tau_2)x(\tau_2)$$

$$\frac{Z^{(2)}}{Z_0} = \frac{1}{2}\left(3\frac{g}{4}\int_0^\beta d\tau_1 \mathcal{G}^2(\tau_1,\tau_1)\right)^2 + ..$$

b) Pair up each $x(\tau_1)$ with a $x(\tau_2)$

4! =24 terms

$$x^4(\tau_1)x^4(\tau_2) = x(\tau_1)x(\tau_1)x(\tau_1)x(\tau_1)x(\tau_2)x(\tau_2)x(\tau_2)x(\tau_2)$$

$$\frac{Z^{(2)}}{Z_0} = \frac{1}{2}24\left(\frac{g}{4}\right)^2 \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \mathcal{G}^4(\tau_1,\tau_2) + ..$$

Perturbation Theory with x^4 potential

2nd Order Correction
$$\frac{Z^{(2)}}{Z_0} = \frac{\frac{1}{2}\left(\frac{g}{4}\right)^2\left(\frac{m}{2\pi\epsilon}\right)^{1/2}\int \mathcal{D}[x(\tau)]\;\; e^{-\int_0^\beta d\tau \frac{m}{2}\dot{x}^2+\frac{m}{2}\omega^2x^2}\int_0^\beta d\tau_1\int_0^\beta d\tau_2 x^4(\tau_1)x^4(\tau_2)}{\left(\frac{m}{2\pi\epsilon}\right)^{1/2}\int \mathcal{D}[x(\tau)]\;\; e^{-\int_0^\beta d\tau \frac{m}{2}\dot{x}^2+\frac{m}{2}\omega^2x^2}}$$

$$x^4(\tau_1)x^4(\tau_2) = x(\tau_1)x(\tau_1)x(\tau_1)x(\tau_1)x(\tau_2)x(\tau_2)x(\tau_2)x(\tau_2)$$

How do we pair them up?

c) Pair up the 2 $x(\tau_1)$ s among themselves and 2 $x(\tau_2)$ s among themselves. For the rest, pair up a $x(\tau_1)$ with a $x(\tau_2)$

$$x^4(\tau_1)x^4(\tau_2) = x(\tau_1)x(\tau_1)x(\tau_1)x(\tau_1)x(\tau_2)x(\tau_2)x(\tau_2)x(\tau_2)$$

6 ways to choose the $x(\tau_1)$ s to be paired amongst themselves

6 ways to choose the $x(\tau_2)$ s to be paired amongst themselves

2 ways to pair up the rest

total 72 terms

$$\frac{Z^{(2)}}{Z_0} = \frac{1}{2}72\left(\frac{g}{4}\right)^2\int_0^\beta d\tau_1\int_0^\beta d\tau_2\mathcal{G}(\tau_1,\tau_1)\mathcal{G}^2(\tau_1,\tau_2)\mathcal{G}(\tau_2,\tau_2) + ..$$

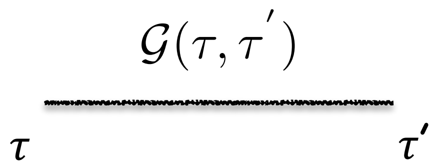
In all,

$$\frac{Z^{(2)}}{Z_0} = \frac{1}{2}\left(\frac{g}{4}\right)^2\int_0^\beta d\tau_1\int_0^\beta d\tau_2[72\mathcal{G}(\tau_1,\tau_1)\mathcal{G}^2(\tau_1,\tau_2)\mathcal{G}(\tau_2,\tau_2) + 24\mathcal{G}^4(\tau_1,\tau_2) + 9\mathcal{G}^2(\tau_1,\tau_1)\mathcal{G}^2(\tau_2,\tau_2)]$$

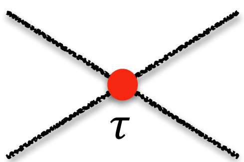
Feynman Diagrams

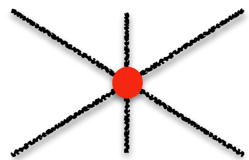
It is apparent that we need a better book keeping device for these integrals, as they proliferate with the order in perturbation theory. These are the Feynman diagrams, which visually represent different kinds of integrals that we need to evaluate.

#1) Represent a two-point function by a line with the end-points of the line indexed by the time points of the 2 pt. fn.



#2) Represent the x^4 potential by a dot, with 4 lines coming out of the dot. The 4 lines represent the 4 factors of x for the x^4 potential. For a x^6 potential, 6 lines should be coming out of it.

-g/4 

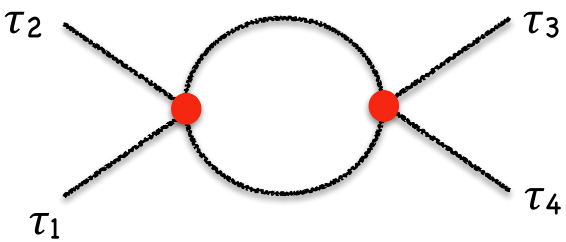
vertex for x^6 potential 

Since the factors of x are absorbed in the 2 pt. fns the dot would contribute -g/4

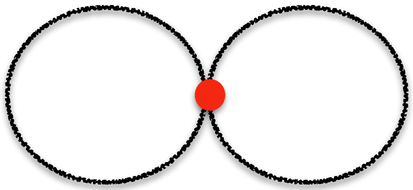
#3) Connect the lines coming out of the interaction vertex with other lines to form 2 pt fns. Different connections correspond to different diagrams.

Feynman Diagrams

#4) The kind of diagram to be drawn depends on the quantity being calculated. E.g. if we are calculating a perturbation series for a 2n point function, the fn. has 2n variables, and the diagrams should have 2n lines hanging out. For Z, the diagrams should have no lines hanging out.

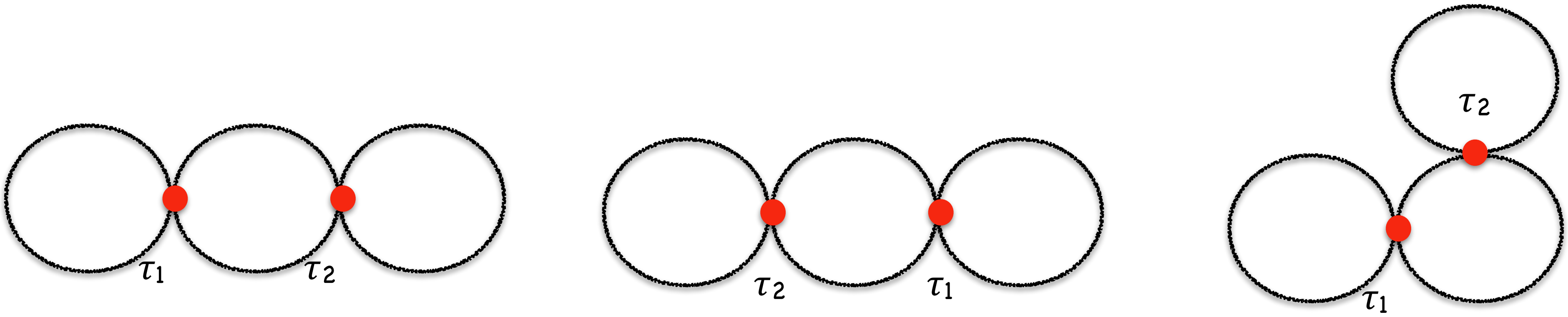


A diagram for a 4 pt fn



A diagram for Z

#5) Draw only topologically distinct diagrams, i.e. diagrams which cannot be converted to each other without tearing them apart.



These diagrams are topologically equivalent.

Feynman Diagrams

#5) Multiply all the propagators corresponding to a diagram (with corr. τ index)

Multiply a factor of (-g/4) for each interaction vertex.

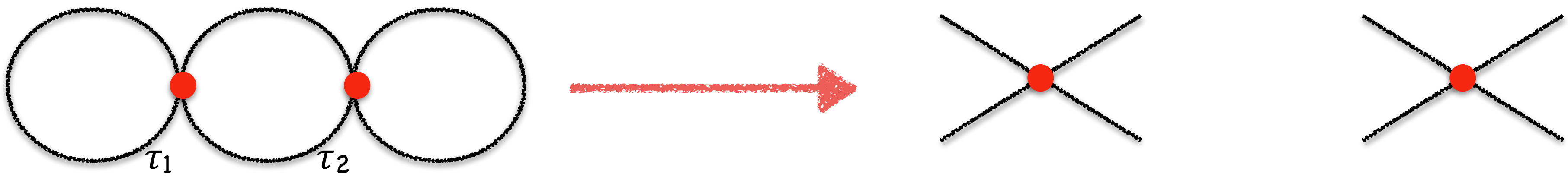
Divide by n! where n is no. of interaction vertex.

Multiply by the "multiplicity" of the diagram.

Integrate over all internal co-ordinates (leaving out the hanging ones)

Multiplicity of a diagram:

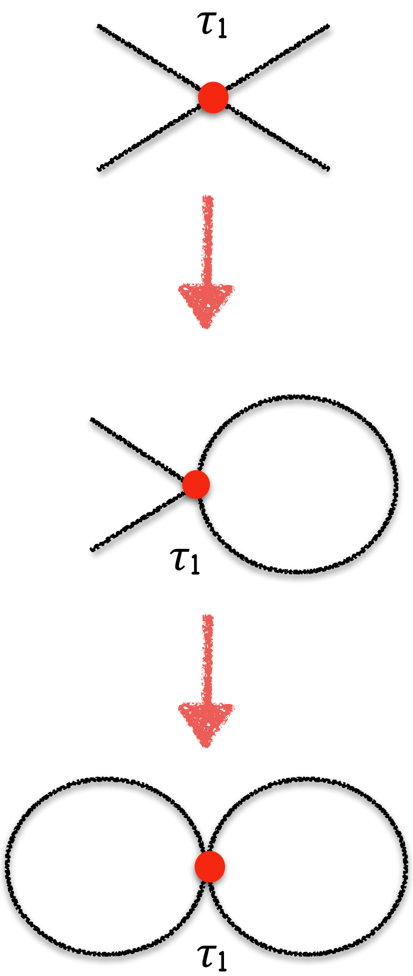
How many ways can the lines be connected to get the diagram?



You can choose to connect any 2 of 4 lines from the left to connect with any 2 of 4 lines from the right i.e. 6 X 6 = 36 choices. Having chosen the left and right ones to be connected, there are 2 ways to connect them. Multiplicity of the diagram is 72.

Perturbation Theory for Z with x^4 potential

First Order Correction through Feynman Diagrams



Since no line can be hanging out, two of them will connect with each other and so would the other 2.

The first line can connect with any of the other 3.

Once this connection is made the other 2 have to connect as no lines can be hanging out. Multiplicity of the diagram is 3.

$$\frac{Z^{(1)}}{Z_0} = -3\frac{g}{4}\int_0^\beta d\tau_1 \mathcal{G}^2(\tau_1, \tau_1)$$

Multiplicity

Integration over intermediate time

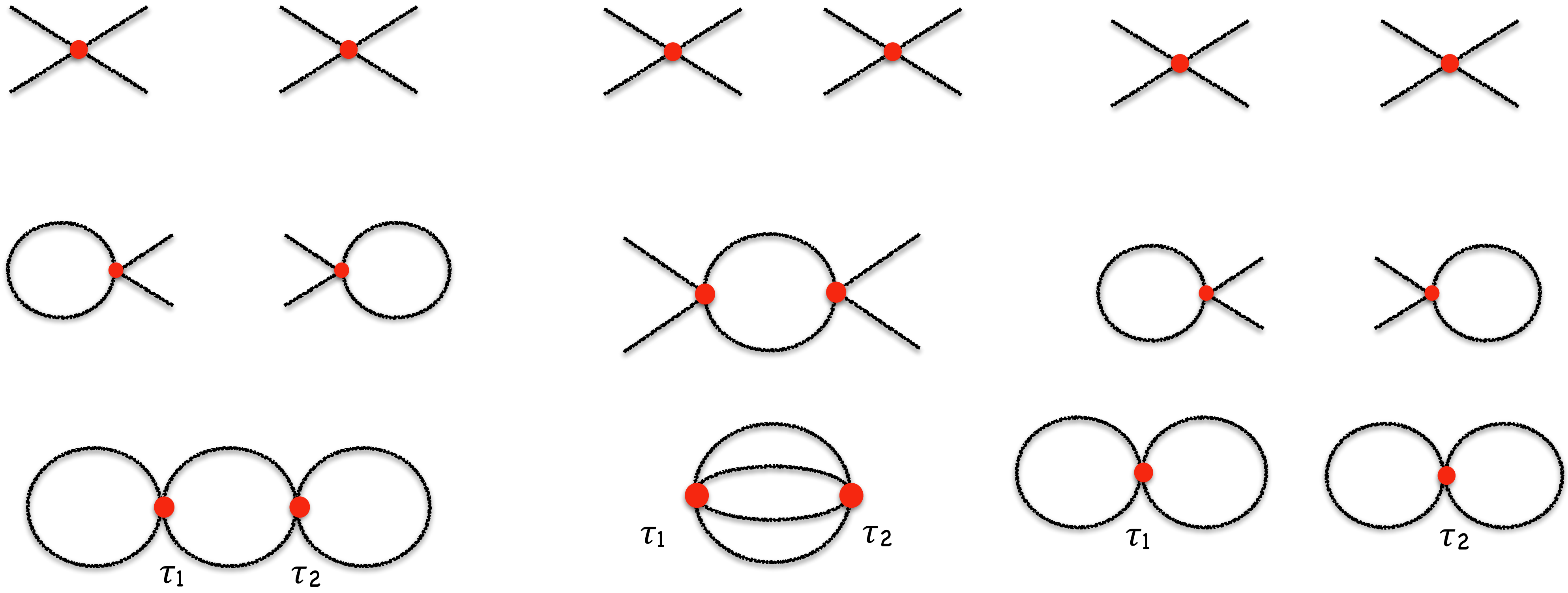
One factor of coupling

Two factors of G with correct time indices

This is the only 1st order diagram for Z

Perturbation Theory for Z with x^4 potential

Second Order Correction through Feynman Diagrams



Multiplicity: 72

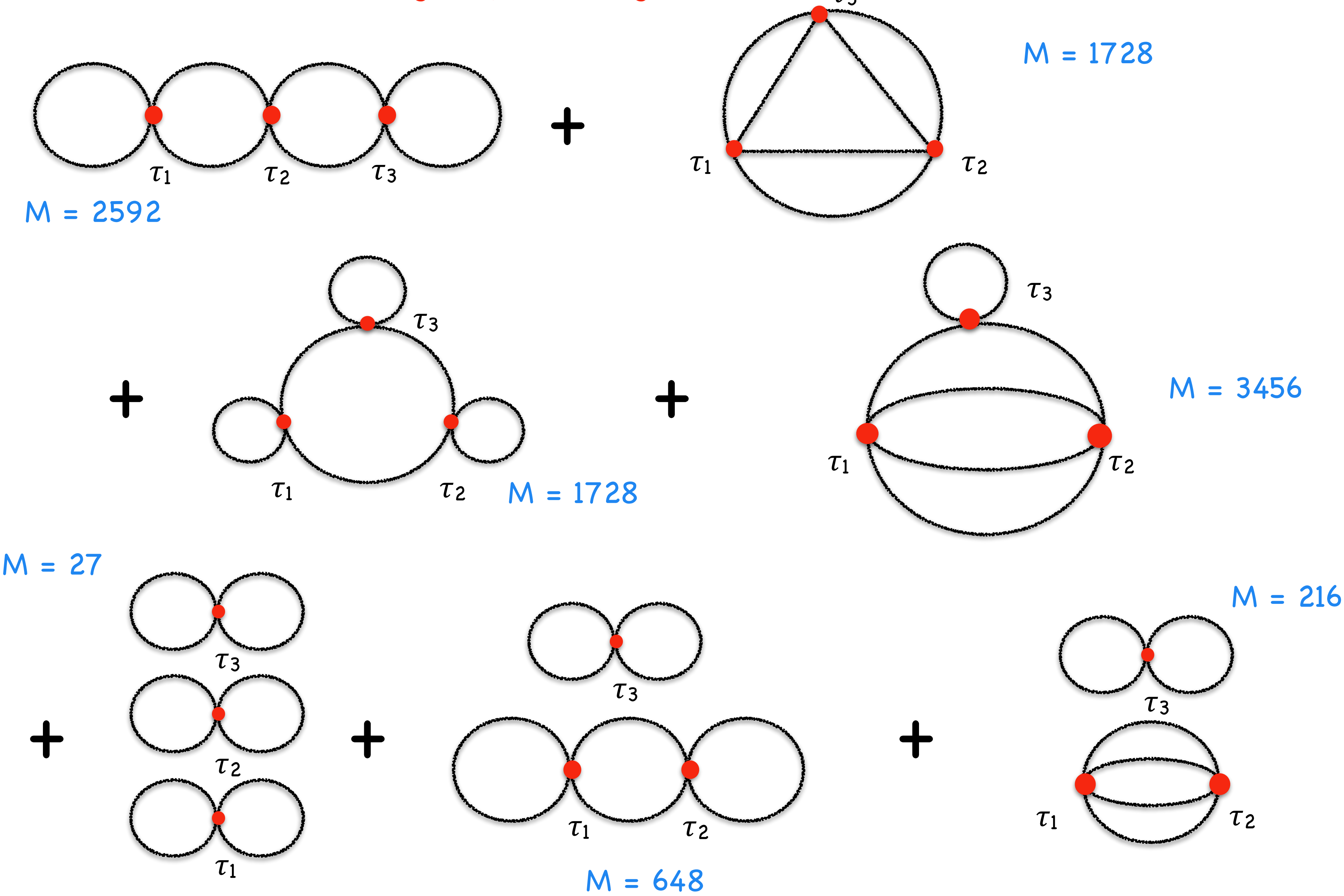
Multiplicity: 24

Multiplicity: 9

$$\frac{Z^{(2)}}{Z_0} = \frac{1}{2}\left(\frac{g}{4}\right)^2 \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 [72\mathcal{G}(\tau_1,\tau_1)\mathcal{G}^2(\tau_1,\tau_2)\mathcal{G}(\tau_2,\tau_2) + 24\mathcal{G}^4(\tau_1,\tau_2) + 9\mathcal{G}^2(\tau_1,\tau_1)\mathcal{G}^2(\tau_2,\tau_2)]$$

Perturbation Theory for Z with x^4 potential

Third Order Correction through Feynman Diagrams

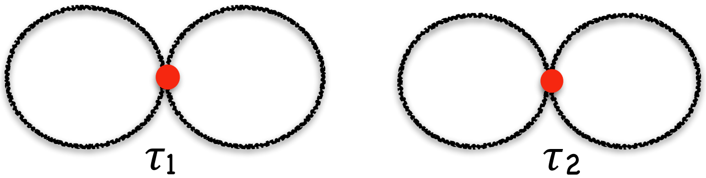


Perturbation Theory and connected Diagrams

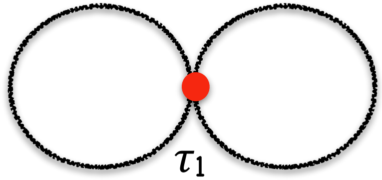
We have seen that the perturbation theory for Z in higher orders include product of terms which we already obtained in lower orders.

These are represented by disconnected diagrams

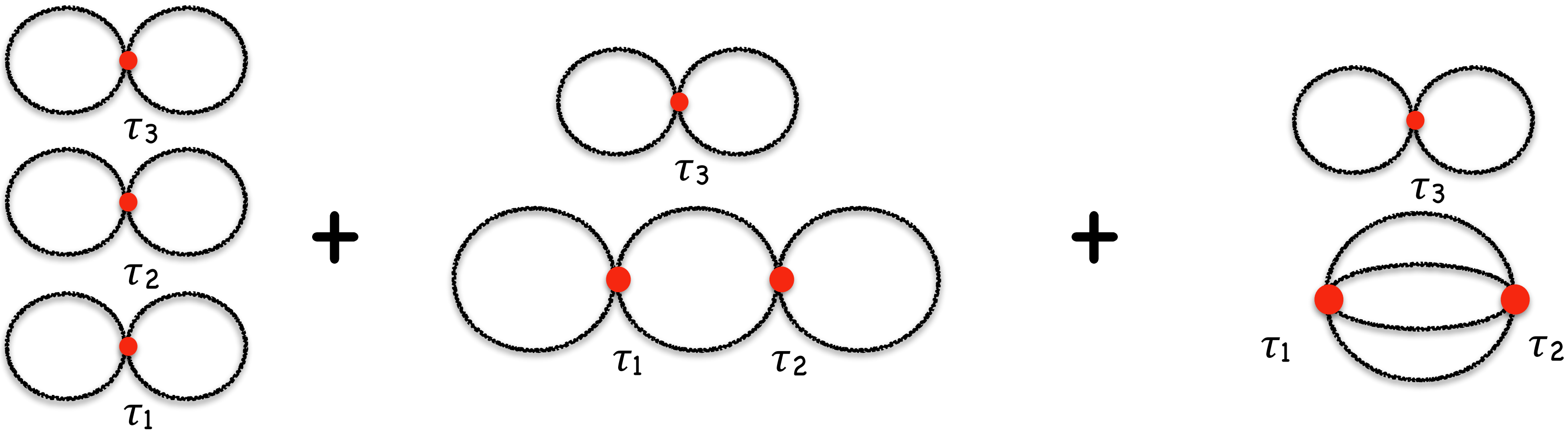
Eg: In 2nd order



obtained from



In 3rd order



As we go to higher orders these type of terms proliferate very fast.

Is there a way to construct a perturbation expansion which gets rid of this terms and focus on the new connected diagrams obtained in each order?

Perturbation Theory for Free Energy F

Let us work with the Free Energy, which is defined by $Z = e^{-\beta F}$

$$F - F_0 = \frac{-1}{\beta} \ln \frac{Z}{Z_0} \quad = \frac{-1}{\beta} \ln \left[1 + \frac{Z^{(1)}}{Z_0} + \frac{Z^{(2)}}{Z_0} + \frac{Z^{(3)}}{Z_0} + ... \right]$$

$$\ln \left[1 + \frac{Z^{(1)}}{Z_0} + \frac{Z^{(2)}}{Z_0} + \frac{Z^{(3)}}{Z_0} + ... \right] = \left(\frac{Z^{(1)}}{Z_0} + \frac{Z^{(2)}}{Z_0} + \frac{Z^{(3)}}{Z_0} + ... \right) - \frac{1}{2} \left(\frac{Z^{(1)}}{Z_0} + \frac{Z^{(2)}}{Z_0} + \frac{Z^{(3)}}{Z_0} + ... \right)^2$$

$$+ \frac{1}{3} \left(\frac{Z^{(1)}}{Z_0} + \frac{Z^{(2)}}{Z_0} + \frac{Z^{(3)}}{Z_0} + ... \right)^3 + ...$$

Pull all the terms which have the same powers of the coupling

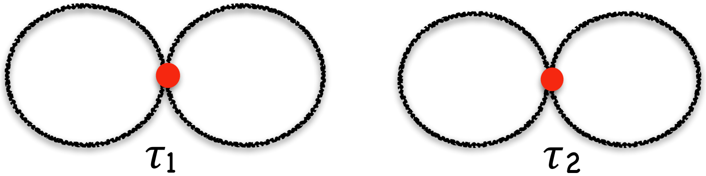
$$= \left[\frac{Z^{(1)}}{Z_0} \right] + \left[\frac{Z^{(2)}}{Z_0} - \frac{1}{2} \left(\frac{Z^{(1)}}{Z_0} \right)^2 \right] + \left[\frac{Z^{(3)}}{Z_0} - \frac{Z^{(1)}}{Z_0} \frac{Z^{(2)}}{Z_0} + \frac{1}{3} \left(\frac{Z^{(1)}}{Z_0} \right)^3 \right] + ...$$

Let us now look at this series closely

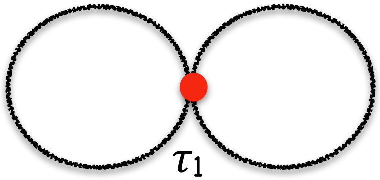
Perturbation Theory for Free Energy F

$$\left[\frac{Z^{(1)}}{Z_0}\right] + \left[\frac{Z^{(2)}}{Z_0} - \frac{1}{2}\left(\frac{Z^{(1)}}{Z_0}\right)^2\right] + \left[\frac{Z^{(3)}}{Z_0} - \frac{Z^{(1)}}{Z_0}\frac{Z^{(2)}}{Z_0} + \frac{1}{3}\left(\frac{Z^{(1)}}{Z_0}\right)^3\right] + \ldots$$

In the second order the subtraction $\left(\frac{Z^{(1)}}{Z_0}\right)^2$ has the same structure as the disconnected diagram



$$\sim \left(\frac{Z^{(1)}}{Z_0}\right)^2 \quad = \frac{1}{2}\left(\frac{Z^{(1)}}{Z_0}\right)^2$$



$$\left(\frac{Z^{(1)}}{Z_0}\right)$$

What about the factor of 1/2?

Go back to Feynman rules ——> n^{th} order diagram has a division by n!

The subtraction exactly cancels out the contribution of the disconnected diagram !