Advanced Quantum Mechanics

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Quantum Dynamics

Lecture #2

Recap of Last Class

- Schrodinger and Heisenberg Picture
- Time Evolution operator/ Propagator : Retarded and Advanced Propagator.
- Time Evolution for time dependent Hamiltonians : Time Ordered Exponentials
- Breaking up the Hamiltonian: Interaction Picture
- Driven Harmonic Oscillator and coherent states

Time Dependent Perturbation Theory

$$\hat{H}(t) = \hat{H}_0 - f(t)\hat{A} \; = H_0 + H_1(t)$$

Control Field

Operator to which
the field couples

We will assume H_1 to be small and expand in it. Formally, we can add a factor λin front of H_1, expand in powers of λ upto a certain order, and then set λ=1 at the end.

Interaction Rep.: $i\partial_t|\psi_I(t)\rangle = \hat{H}_{1I}(t)|\psi_I(t)\rangle$ $\qquad$ $|\psi_I(t)\rangle = U_I(t)|\psi_I(0)\rangle$

Comparing with the case for generic time-dependent Hamiltonian, the time evolution operator in the interaction representation is given by

$$\hat{U}_I(t) = T[e^{-i\int_0^t dt' H_{1I}(t')}] = 1 - i\int_0^t dt_1 \hat{H}_{1I}(t_1) - \int_0^t dt_1 \int_0^{t_1} dt_2 \hat{H}_{1I}(t_1)\hat{H}_{1I}(t_2) + ...$$

Using $|\psi_I(t)\rangle = e^{i\hat{H}_0 t}|\psi_S(t)\rangle$ and $|\psi_I(0)\rangle = |\psi_S(0)\rangle$

$$U(t,0) = e^{-i\hat{H}_0 t}[1 - i\int_0^t dt_1 \hat{H}_{1I}(t_1) - \int_0^t dt_1 \int_0^{t_1} dt_2 \hat{H}_{1I}(t_1)\hat{H}_{1I}(t_2) + ...]e^{i\hat{H}_0 t}$$

Linear Response Theory

R. Kubo, Journal of Physical Society of Japan,
Vol 6 Page 570 (1957)

In experiments, in order to study the properties of a system, we couple a probe to the system. e.g. we may turn on an electric or a magnetic field, we may change the pressure etc.

The effect of the probe is modelled as an additional piece in the Hamiltonian of the system, which is turned on at t=0.

$$\hat{H}(t) = \hat{H}_0 - f(t)\hat{A}$$

Operator to which the field couples

Control Field

Eg: Transport ---- Electric vector potential -- couples to current density
Susceptibility ----- magnetic field --- couples to spin density

We usually measure some quantity represented by the operator B after turning on the probe. In case B has finite expectation in absence of probe, we are interested in change of B due to the probe, δB; i.e. we wish to study how the system responds to the probe stimulus.

Eg: Transport ---- electric current
Susceptibility ----- magnetization

Linear Response Theory: The response <B(t)> or <δB(t)> is calculated to linear order in f(t)

The basic assumption is that the probe disturbs the system gently, so that perturbation theory can be used to calculate the response.

Linear Response Theory

Work in the interaction representation

$$B(t) = \langle \psi_I(t)|B_I(t)|\psi_I(t)\rangle = \langle \psi(0)|\hat{U}_I^\dagger(t)\hat{B}_I(t)\hat{U}_I(t)|\psi(0)\rangle$$

Using the expansion of U_I $U_I(t) = 1 - i\int_0^t dt_1 \hat{H}_{1I}(t_1) - \int_0^t dt_1 \int_0^{t_1} dt_2 \hat{H}_{1I}(t_1)\hat{H}_{1I}(t_2) + ...$

and collecting terms linear in H_1

$$B(t) = \langle \psi(0)|e^{i\hat{H}_0 t}\hat{B}e^{-i\hat{H}_0 t}|\psi(0)\rangle + i\int_0^t dt_1[\hat{H}_{1I}(t_1)\hat{B}_I(t) - \hat{B}_I(t)\hat{H}_{1I}(t_1)] + ..$$

Using H_1=-f(t)A

$$\delta B(t) = -i\int_0^t dt_1 f(t_1)[\hat{A}_I(t_1)\hat{B}_I(t) - \hat{B}_I(t)\hat{A}_I(t_1)] + ..$$

It is useful to write

$$\boxed{\delta B(t) = \int dt' f(t')\chi_{BA}(t,t')}$$

$$\boxed{\chi_{BA}(t,t') = i\Theta(t-t')\langle \psi(0)|[\hat{B}_I(t), \hat{A}_I(t')]|\psi(0)\rangle}$$

Lin. Response Function (also called Kubo formula)

Linear Response Function

$$\boxed{\delta B(t) = \int dt^{'} f(t')\chi_{BA}(t,t')}$$

$$\boxed{\chi_{BA}(t,t') = i\Theta(t-t')\langle\psi(0)|[\hat{B}_I(t),\hat{A}_I(t')]|\psi(0)\rangle}$$

- ◉ The response function is a property of the unperturbed state
- ◉ Response is causal, i.e. perturbation at time t′ can only affect response after t′. Theta fn. ensures that.

Time translation invariance of unperturbed system: $\chi_{BA}(t,t') = \chi_{BA}(t-t')$

Fourier Transform: Convolution —> Multiplication

$$\boxed{\delta B(t) = \int dt^{'} f(t')\chi_{BA}(t-t') \Rightarrow \delta B(\omega) = f(\omega)\chi_{BA}(\omega)}$$

Characteristic of Lin. Response: The system responds only at the frequency at which it is modulated

Spectral Decomposition

Assume that the system is initially in its ground state , or at least an eigenstate of H_0

$$\chi_{BA}(t-t') = i\Theta(t-t')\langle\psi(0)|e^{i\hat{H}_0 t}\hat{B}e^{-i\hat{H}_0(t-t')}\hat{A}e^{-i\hat{H}_0 t'} - e^{i\hat{H}_0 t'}\hat{A}e^{+i\hat{H}_0(t-t')}\hat{B}e^{-i\hat{H}_0 t}|\psi(0)\rangle$$

$$\sum_n |n\rangle\langle n| = 1 \qquad\qquad \sum_n |n\rangle\langle n| = 1$$

$$= i\Theta(t-t')\sum_n e^{-i(\omega_n-\omega_0)(t-t')}A_{n0}B_{0n} - e^{i(\omega_n-\omega_0)(t-t')}B_{n0}A_{0n}$$

where $A_{n0} = \langle n|\hat{A}|0\rangle$ $\qquad \hat{H}_0|n\rangle = \omega_n|n\rangle$

We will use retarded response

$$\chi_{BA}(\omega) = \int_0^\infty dt e^{i\omega t}\chi_{BA}(t)$$

limit from theta fn

where

$\eta \to 0^+$

$\omega_{n0} = \omega_n - \omega_0$

and

$$\int_0^\infty dt e^{i(\omega-\omega_{n0})t} = \int_0^\infty dt e^{i(\omega+i\eta-\omega_{n0})t} = \frac{-1}{i(\omega+i\eta-\omega_{n0})}$$

$$\boxed{\chi_{BA}(\omega) = \sum_n \frac{A_{0n}B_{n0}}{\omega+i\eta+\omega_{n0}} - \frac{A_{n0}B_{0n}}{\omega+i\eta-\omega_{n0}}}$$

$$\boxed{\chi_{AA}(\omega) = \sum_n \frac{2\omega_{n0}|A_{0n}|^2}{(\omega+i\eta)^2-\omega_{n0}^2}}$$

We will assume A and B to be Hermitian operators

Real and Imaginary parts

$$\delta B(\omega) = f(\omega)\chi_{BA}(\omega) \longleftarrow$$ Acts like an inverse of impedance

- Real part of χ controls the modulation of B in phase with external perturbation.
- Imaginary part of χ controls the modulation of B out of phase with external perturbation.

$$\chi_{BA}(\omega + i\eta) = \chi'_{BA}(\omega) + i\chi''_{BA}(\omega)$$

Using $$\frac{1}{\omega - \omega_0 + i\eta} = \mathcal{P}\left(\frac{1}{\omega - \omega_0}\right) - i\pi\delta(\omega - \omega_0)$$

$$\boxed{\chi''_{BA}(\omega) = \pi \sum_n A_{n0}B_{0n}\delta(\omega - \omega_{n0}) - A_{0n}B_{n0}\delta(\omega + \omega_{n0})}$$

It is clear from above formula that $$\chi''_{BA}(\omega) = -\chi''_{AB}(-\omega)$$

$$\chi''_{AA}(\omega) = -\chi''_{AA}(-\omega)$$ odd function of frequency

Similarly it can be shown that $$\chi'_{AA}(\omega) = \chi'_{AA}(-\omega)$$ even function of frequency

Real and Imaginary parts

Relating $\chi''_{AA}(\omega)$ to energy dissipated in the process

Infinitesimal work done on the system by external perturbation in changing A to A+ dA

$$dW = -f(t)d\langle\hat{A}(t)\rangle$$

Generalization of force

Generalization of displacement

Instantaneous Power

$$\frac{dW}{dt} = -f(t)\partial_t\langle\hat{A}(t)\rangle = -f(t)\partial_t\int_{-\infty}^{\infty} dt'\chi_{AA}(t-t')f(t')$$

$$= -f(t)\partial_t\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty} dt'\chi_{AA}(\omega)e^{-i\omega(t-t')}f(t')$$

$$= f(t)\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}(i\omega)\chi_{AA}(\omega)e^{-i\omega t}\int_{-\infty}^{\infty} dt' e^{i\omega t'}f(t')$$

$$= f(t)\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}(i\omega)\chi_{AA}(\omega)e^{-i\omega t}f(\omega)$$

As long as f(t) --> 0 as t --> ∞, one can choose a T large enough to do this

Average Power

$$P = \frac{1}{T}\int_{-T/2}^{T/2}\frac{dW}{dt} = \int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}f(\omega')\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}(i\omega)\chi_{AA}(\omega)\left[\overbrace{\frac{1}{T}\int_{-T/2}^{T/2} dt e^{-i(\omega+\omega')t}}^{\text{Replace by delta fn.}}\right]$$

Real and Imaginary parts

$$P \sim \int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega f(\omega') f(\omega)(i\omega)\chi_{AA}(\omega)\delta(\omega+\omega') \; = \int_{-\infty}^{\infty} d\omega f(\omega) f(-\omega)(i\omega)\chi_{AA}(\omega)$$

$$= -\int_{-\infty}^{\infty} d\omega f(\omega) f(-\omega)\omega\chi_{AA}^{''}(\omega)$$

Since the real part is even in freq,
its contribution to the integral vanishes

For real f(t) $f(-\omega) = f^*(\omega)$

$$P \sim -\int_{-\infty}^{\infty} d\omega |f(\omega)|^2 \omega\chi_{AA}^{''}(\omega)$$

For harmonic perturbation (which has single frequency component with amplitude f_0)

$$P \sim -|f_0|^2 \omega\chi_{AA}^{''}(\omega)$$

For system in eqbm.

$$\omega\chi_{AA}^{''}(\omega) \geq 0$$

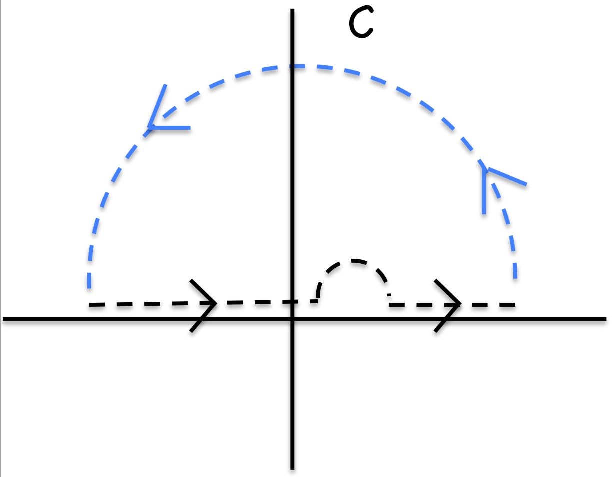
Kramers-Kronig Relations

The real and imaginary part of a retarded linear response function are not independent quantities. They are related to each other by Kramers-Kronig relations.

$$\boxed{\chi''_{BA}(\omega) = -\mathcal{P}\int_{-\infty}^{\infty}\frac{d\omega'}{\pi}\frac{\chi'_{BA}(\omega')}{\omega'-\omega}} \qquad \boxed{\chi'_{BA}(\omega) = \mathcal{P}\int_{-\infty}^{\infty}\frac{d\omega'}{\pi}\frac{\chi''_{BA}(\omega')}{\omega'-\omega}}$$

If either the real or the imaginary part is known (for all frequencies), the other part can be obtained. This is often used in experiments e.g. absorption of light is related to imaginary part of optical conductivity, but can be used to obtain the full response function.

The retarded response function is analytic in the upper half-plane (similar to retrded propagators)



$$\oint_C dz\frac{\chi_{BA}(z)}{z-\omega} = 0 \quad = \int_{-\infty}^{\infty} d\omega'\frac{\chi_{BA}(\omega'+i\eta)}{\omega'+i\eta-\omega}$$

$$= \int_{-\infty}^{\infty} d\omega'\left[\chi'_{BA}(\omega') + i\chi''_{BA}(\omega')\right]\left[\mathcal{P}\frac{1}{\omega'-\omega} - i\pi\delta(\omega'-\omega)\right]$$

Equating real and Im parts

$$\boxed{\chi''_{BA}(\omega) = -\mathcal{P}\int_{-\infty}^{\infty}\frac{d\omega'}{\pi}\frac{\chi'_{BA}(\omega')}{\omega'-\omega}} \qquad \boxed{\chi'_{BA}(\omega) = \mathcal{P}\int_{-\infty}^{\infty}\frac{d\omega'}{\pi}\frac{\chi''_{BA}(\omega')}{\omega'-\omega}}$$

Sum Rules

$$\chi''_{BA}(\omega) = \pi \sum_n A_{n0} B_{0n} \delta(\omega - \omega_{n0}) - A_{0n} B_{n0} \delta(\omega + \omega_{n0})$$

Need to know solution of the full problem

$$\frac{1}{\pi} \int d\omega \chi''_{BA}(\omega) = \sum_n B_{0n} A_{n0} - A_{0n} B_{n0} = \langle \psi(0) | [B, A] | \psi(0) \rangle$$

Ground State Property

$$\frac{1}{\pi} \int d\omega \omega \chi''_{BA}(\omega) = \sum_n \omega_{n0} (B_{0n} A_{n0} + A_{0n} B_{n0})$$

$$= \sum_n \omega_{n0} [\langle \psi(0) | B | n \rangle \langle n | A | \psi(0) \rangle + \langle \psi(0) | A | n \rangle \langle n | B | \psi(0) \rangle$$

$$= \sum_n [\langle \psi(0) | BH - HB | n \rangle \langle n | A | \psi(0) \rangle + \langle \psi(0) | A | n \rangle \langle n | HB - BH | \psi(0) \rangle$$

$$= \sum_n [\langle \psi(0) | [B, H] | n \rangle \langle n | A | \psi(0) \rangle - \langle \psi(0) | A | n \rangle \langle n | [B, H] | \psi(0) \rangle$$

$$= \langle \psi(0) | [B, H] A - A [B, H] | \psi(0) \rangle \quad = \langle \psi(0) | [[B, H], A] | \psi(0) \rangle$$

Moments of Imaginary part of response function in terms of ground state properties

Example: Driven Harmonic Oscillator

$$H_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 x^2 = \omega_0(a^\dagger a + 1/2) \qquad H_1 = -f(t)x = -f(t)(a^\dagger + a)$$

Suppose we start the system in its ground state and want to measure the average position

$$\langle x(t)\rangle = \int_0^t dt' f(t')\chi_{xx}(t-t')$$

Spectral Decomposition

$$\boxed{\chi_{AA}(\omega) = \sum_n \frac{2\omega_{n0}|A_{0n}|^2}{(\omega+i\eta)^2 - \omega_{n0}^2}}$$

$$\langle n|\hat{x}|0\rangle = \langle n|a^\dagger + a|0\rangle = \delta_{n,1}$$

$$\chi_{xx}(\omega) = \frac{2\omega_0}{\omega^2 - \omega_0^2} = \frac{1}{\omega - \omega_0} - \frac{1}{\omega + \omega_0}$$

$$\chi_{xx}(t-t') = i\Theta(t-t')[e^{i\omega_0(t-t')} - e^{-i\omega_0(t-t')}]$$

$$\langle x(t)\rangle = -i\int_0^t dt'[f(t')(e^{-i\omega_0(t-t')} - e^{i\omega_0(t-t')}) = \zeta(t)e^{i\omega_0 t} + \zeta^* e^{-i\omega_0 t}$$

Compare with the exact solution $U(t) = e^{i\beta(t)}D[\zeta(t)e^{-i\omega_0 t}]$

The drive creates coherent states and avg of x is just the real part of the complex number denoting the coherent state.

Example: Time Dependent Anharmonic Perturbation

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 x^2 = \omega_0(a^\dagger a + 1/2) \qquad H_1 = -f(t)x^4 = -f(t)[a^\dagger + a]^4$$

Suppose we start the system in its ground state and want to measure the average position

$$\langle x(t)\rangle = \int_0^t dt' f(t')\chi_{xx^4}(t-t')$$

Spectral Decomposition

$$\boxed{\chi_{BA}(\omega) = \sum_n \frac{A_{0n}B_{n0}}{\omega+i\eta+\omega_{n0}} - \frac{A_{n0}B_{0n}}{\omega+i\eta-\omega_{n0}}}$$

$$\chi_{xx^4}(\omega) = 0$$

The full time dependent H is inversion symmetric

How about measuring < x^2> ?

$$\hat{x}^2_{n0} = \sqrt{2}\delta_{n,2} + \delta_{n,0} \qquad \hat{x}^4_{n0} = 2\sqrt{6}\delta_{n,4} + 6\sqrt{2}\delta_{n,2} + 3\delta_{n,0}$$

Check from explicit expansion

$$\chi_{x^2x^4}(\omega) = 12\left(\frac{1}{\omega+i\eta+2\omega_0} - \frac{1}{\omega+i\eta-2\omega_0}\right)$$

Transition Probability

Consider the old problem of absorption of light which falls on an atom.

System Hamiltonian H₀ ⟶ the energy levels of the atom.

We assume that the atom is initially in one of the eigenstates, i, where the time dependent perturbation H₁(t) (in this case the light-matter interaction term) is switched on.

We want to know the probability of finding it in other states, i.e. we want to know what is the probability that a particular transition will occur.

$$|\psi(t)\rangle = \sum_n |n\rangle\langle n|U(t)|i\rangle = \sum_n c_n(t)|n\rangle$$

Transition Probability: $P_{ni}(t) = |c_n(t)|^2 = |\langle n|U(t)|i\rangle|^2$

$$c_n(t) = |\langle n|e^{-iH_0t}U_I(t)e^{iH_0t}|i\rangle = e^{-i\omega_{n0}t}\langle n|U_I(t)|i\rangle$$

$$U_I(t) = 1 - i\int_0^t dt_1 \hat{H}_{1I}(t_1) - \int_0^t dt_1 \int_0^{t_1} dt_2 \hat{H}_{1I}(t_1)\hat{H}_{1I}(t_2) + ...$$

Pert. Expn. for U(t)

Dyson Series

$$c_n^{(0)}(t) = \delta_{ni}$$

$$c_n(t) = c_n^{(0)}(t) + c_n^{(1)}(t) + c_n^{(2)}(t)$$

$$c_n^{(1)}(t) = i\int_0^t dt' f(t')\langle n|\hat{A}_I(t')|i\rangle$$

$$c_n^{(2)}(t) = -\int_0^t dt_1 f(t_1) \int_0^{t_1} dt_2 f(t_2)\langle n|\hat{A}_I(t_1)\hat{A}_I(t_2)|i\rangle$$

Transition Probability

Since we are interested in transition probabilities, we are thinking of $n \neq i$

Then, upto second order in perturbation theory, $P_{ni}(t) = |c_n^{(1)}(t)|^2$

$$c_n^{(1)}(t) = i \int_0^t dt' f(t') \langle n | \hat{A}_I(t') | i \rangle$$

$$c_n^{(1)}(t) = i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f(\omega) \int_0^t dt' e^{i(\omega_{ni}-\omega)t'} A_{ni} \; = 2iA_{ni} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f(\omega) e^{i(\omega_{ni}-\omega)t/2} \frac{\sin[(\omega-\omega_{ni})t/2]}{\omega-\omega_{ni}}$$

$$P_{ni}(t) = 4|A_{ni}|^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} f(\omega) f(\omega') \cos[(\omega'-\omega)t/2] \frac{\sin[(\omega-\omega_{ni})t/2]}{\omega-\omega_{ni}} \frac{\sin[(\omega'-\omega_{ni})t/2]}{\omega'-\omega_{ni}}$$

Harmonic Perturbation: $f(t) = 2\cos(\Omega t) \Rightarrow f(\omega) = 2\pi[\delta(\omega-\Omega) + \delta(\omega+\Omega)]$

$$P_{ni}(t) = 4|A_{ni}|^2 \left[\frac{\sin^2[(\Omega-\omega_{ni})t/2]}{(\Omega-\omega_{ni})^2} + \frac{\sin^2[(\Omega+\omega_{ni})t/2]}{(\Omega+\omega_{ni})^2} + \cos[\Omega t] \frac{\sin[(\Omega-\omega_{ni})t/2] \sin[(\Omega+\omega_{ni})t/2]}{\omega^2-\omega_{ni}^2} \right]$$

Fermi's Golden Rule

$$P_{ni}(t) = 4|A_{ni}|^2 \left[\frac{\sin^2[(\Omega - \omega_{ni})t/2]}{(\Omega - \omega_{ni})^2} + \frac{\sin^2[(\Omega + \omega_{ni})t/2]}{(\Omega + \omega_{ni})^2} + \cos[\Omega t] \frac{\sin[(\Omega - \omega_{ni})t/2]\sin[(\Omega + \omega_{ni})t/2]}{\omega^2 - \omega_{ni}^2} \right]$$

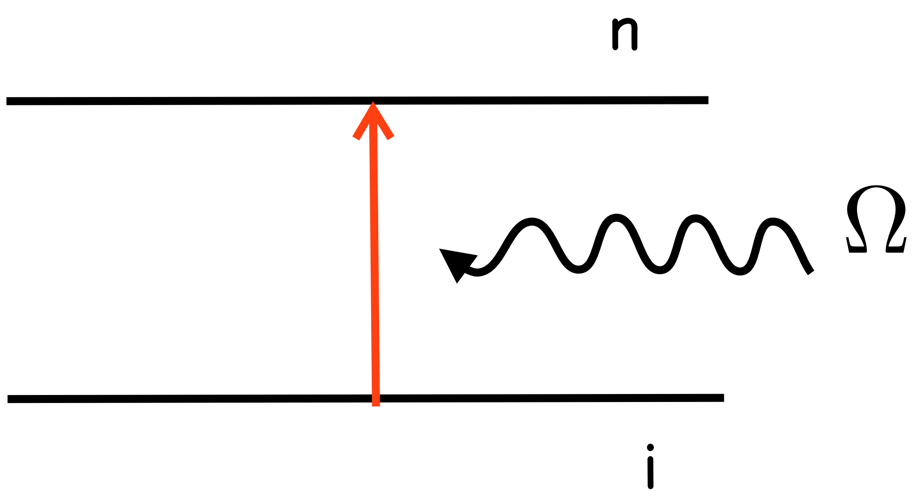
Usually we are interested in the transition probabilities when $\Omega \sim \omega_{ni}$ or $\Omega \sim -\omega_{ni}$

Then, considering a long timescale t >> Ω^{-1} , either the first term or the 2nd term survives.

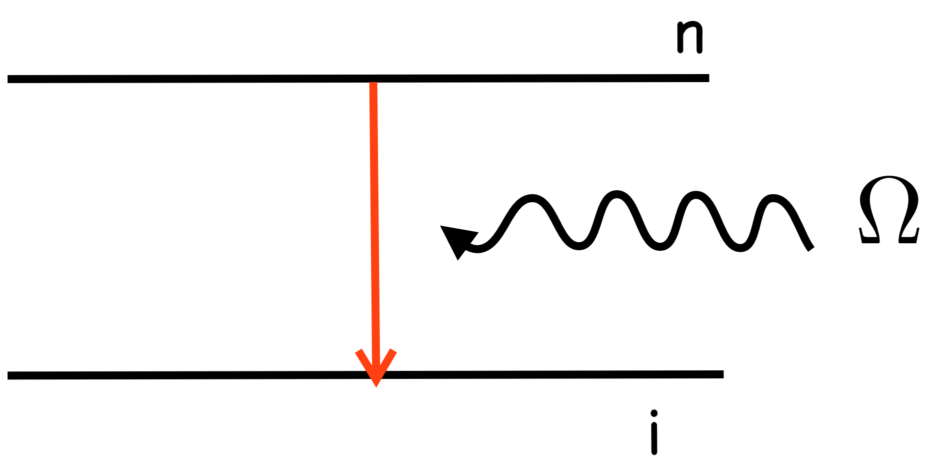
Using $Lim_{a\to\infty} \frac{\sin^2(ax)}{ax^2} = \pi\delta(x)$ $$P_{ni}(t) = 2\pi|A_{ni}|^2 t\delta(\Omega \pm \omega_{ni})$$

Define Transition rate $$\Gamma_{ni} = P_{ni}(t)/t = 2\pi|A_{ni}|^2\delta(\Omega \pm \omega_{ni})$$

Matrix Elements ——> Symmetry Considerations



Absorption of energy from perturbation



Stimulated Emission due to perturbation
(happens when initial state is excited state)

Radiation Coupling to Matter

The interaction of charged particles with E-M fields can be incorporated through the term

$$H = \frac{(\vec{p} - e\vec{A})^2}{2m} + V(r)$$

A is the vector potential

Assuming weak fields, so that A^2 terms can be neglected, we get

$$H \simeq \frac{\vec{p}^2}{2m} + V(r) - \frac{e}{2m}(\vec{A}\cdot\vec{p} + \vec{p}\cdot\vec{A}) = \frac{\vec{p}^2}{2m} + V(r) - \frac{e}{m}\vec{A}\cdot\vec{p}$$

$$= H_0 - \frac{e}{mc}\vec{A}(\vec{r},t)\cdot\vec{\hat{p}}$$

Use

$$\nabla \cdot \vec{A} = 0$$

For E-M waves, $\vec{A}(\vec{r},t) = 2\vec{A}_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$

So the perturbation term $H_1(t) = -\frac{e}{mc}A_0(e^{i\vec{k}\cdot\vec{r} - \omega t} + h.c)\hat{\varepsilon}\cdot\vec{\hat{p}}$

Dipole Approximation

The size of the atom is typically much smaller than the wavelength of light

So, $e^{i\vec{k}\cdot\vec{r}} \simeq 1$ This is called the dipole approximation

Not a good approximation for Rydberg atoms/ electrons in very high radial quantum no. states

$$H_1(t) = -\frac{e}{mc}A_0(e^{-i\omega t} + h.c)\hat{\varepsilon}\cdot\vec{\hat{p}}$$

For a Hamiltonian $H_0 = \frac{\vec{p}^2}{2m} + V(r)$

$$[H_0, \vec{r}] = \frac{\vec{p}}{im} \qquad \vec{p} = im[H_0, \vec{r}]$$

Since we are interested in the absorption of the light, we will be interested in matrix element of the perturbation operator

$$\langle\phi_n|\hat{\varepsilon}\cdot\vec{p}|\phi_0\rangle = im\hat{\varepsilon}\langle\phi_n|\cdot[H_0,\vec{r}]|\phi_0\rangle \;\; = im(E_n - E_0)\langle\phi_n|\hat{\varepsilon}\cdot\vec{r}|\phi_0\rangle$$

Dipole Matrix Element