Advanced Quantum Mechanics

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Lecture #16

Quantum Mechanics of Many Particles

Recap of Last Class

Bogoliubov Theory of Weakly Interacting Bose gas

Validity of Bogoliubov Approximation.

BEC and Broken Symmetry

Hamiltonian for interacting Bose gas

$$H = \sum_k \frac{k^2}{2m} a_k^\dagger a_k + \frac{1}{2}\sum_{xx'} V(x-x') a_x^\dagger a_{x'}^\dagger a_{x'} a_x \quad = \sum_x a_x^\dagger \frac{-\nabla_x^2}{2m} a_x + \frac{1}{2}\sum_{xx'} V(x-x') a_x^\dagger a_{x'}^\dagger a_{x'} a_x$$

Consider the following global U(1) transformation $a_x^\dagger \to a_x^\dagger e^{i\chi}, \qquad a_x \to a_x e^{-i\chi}$

The Hamiltonian is invariant under this transformation ⟶ Global U(1) symmetry

However, the ground state of BEC is not invariant under this transformation

To see this: $a_0 \to a_0 e^{-i\chi}$ i.e. it does not transform according to identity irrep of U(1)

Since GS has finite expectation value of a_0, the GS is not invariant under the U(1) transformation

Example of a system where the ground state has lower symmetry than H

spontaneous symmetry breaking

Note: under U(1), $\phi \longrightarrow \phi\, e^{i\chi}$. Since the Hamiltonian is invariant under this, G.S. energy is invariant under this. Hence we can choose arbitrary phase for ϕ. We choose it to be zero.

Excited States and Bogoliubov Spectrum

$$H = \frac{1}{2}\sum_k E_k(\gamma_k^\dagger \gamma_k + \gamma_{-k}^\dagger \gamma_{-k}) + \frac{1}{2}\sum_k E_k - k^2/2m - g\rho$$

Excited states are obtained by populating the Bogoliubov QP modes.

1 particle Excitations : $\gamma_k^\dagger |\psi_G\rangle$ $\gamma_k^\dagger = u_k a_k^\dagger + v_k a_{-k}$

Does not conserve number

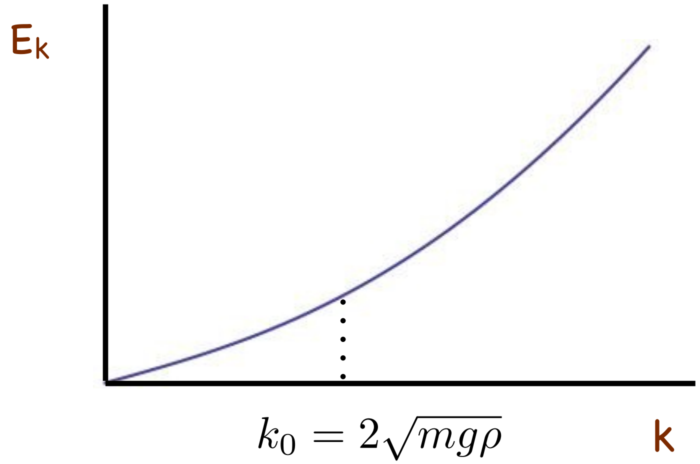
linear combination of original boson creation and annihilation operator

Energy of excitations: E_k

linear dispersion

For small k $$E_k = \sqrt{\left(\frac{k^2}{2m}\right)^2 + \frac{g\rho}{m}k^2} \simeq \sqrt{\frac{g\rho}{m}}k = ck$$

$$c = \sqrt{\frac{g\rho}{m}}$$ Speed of sound



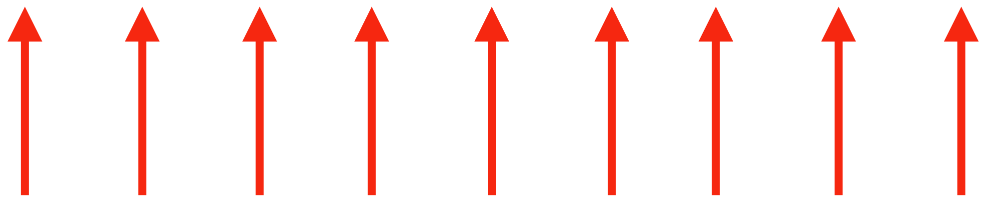
For large k $$E_k = \sqrt{\left(\frac{k^2}{2m}\right)^2 + \frac{g\rho}{m}k^2} \simeq \frac{k^2}{2m}$$ free particle dispersion

Interactions completely change the character of low energy excitations

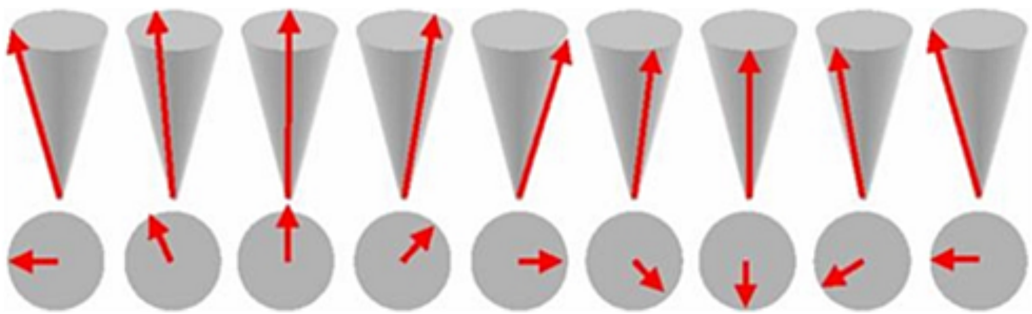
Sound Waves as Goldstone Modes

Goldstone's Theorem:

In a system with short range interactions, if the ground state spontaneously breaks a continuous symmetry of the Hamiltonian, there exists excitation modes whose energy ⟶ 0 in the long wavelength limit.



Eg. ferromagnets
and magnons (spin waves)

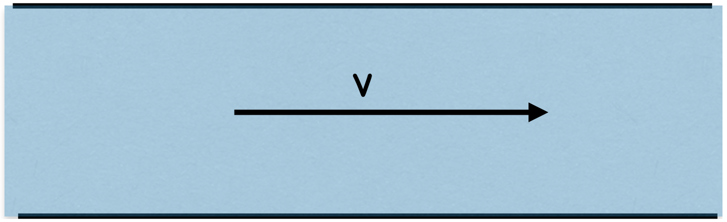


For BEC the broken symmetry is manifested by the phase of the condensate being fixed.

Corresponding Goldstone excitations would be long wavelength variations of the phase in space.

These are the phonons/ Bogoliubov excitations, which have a gapless spectrum.

Landau Criterion and Superfluidity



Work in a co-ordinate co-moving with the fluid.

The walls are now moving with velocity -v

For viscous fluids, the wall drags the fluid along

This friction shows up as viscosity of the fluid.

Landau's argument: In the co-moving frame, if the liquid is dragged by the walls, excitations carrying momentum should appear in the fluid.

Consider a single excitation with momentum p and energy E_p appearing in this fluid. (Note that in the co-moving frame the fluid is at rest, so Bogoliubov theory applies).

In co-ordinates fixed to the walls, $P = p + Mv$

$$E = E_p + p \cdot v + 1/2\, Mv^2$$

$1/2\, Mv^2$ is the K.E. of the fluid So, energy change due to excitations $E = E_p + p \cdot v$

Landau Criterion and Superfluidity

Now energy change should be negative for the liquid to generate the excitation as it moves

So, $E = E_p + p \cdot v < 0$ for excitations and hence for viscosity to appear.

Min $[E_p + p \cdot v]$ occurs when p and v are anti-parallel, and given by $E_p - pv$

Hence $v > E_p/p$ for some values of p for excitations to appear.

So for $v < \min[E_p/p]$ no excitations can occur ⟶ the system is superfluid

For Bogoliubov theory, $\min[E_p/p] = c$

the system is superfluid till the flow velocity exceeds speed of sound