Advanced Quantum Mechanics

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Lecture #17

Quantum Mechanics of Many Particles

Recap of Last Class

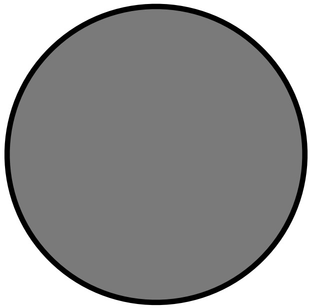
BEC and Broken Symmetry

Goldstone's Theorem and Goldstone modes

Landau Criterion for Superfluidity

Free Fermions

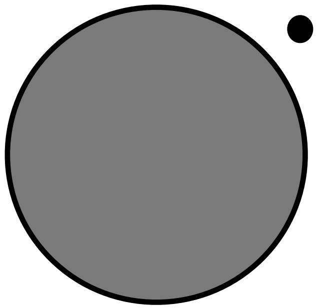
$$H = \sum_{k\sigma} \frac{k^2}{2m} c^\dagger_{k\sigma} c_{k\sigma}$$



Ground State of N non-interacting free Fermions

Filled Fermi Sea $\quad |\psi\rangle = \prod_{\sigma, |k|<k_F} c^\dagger_{k\sigma}|0\rangle \quad\quad k_F^3 = 3\pi^2 \frac{N}{V}$

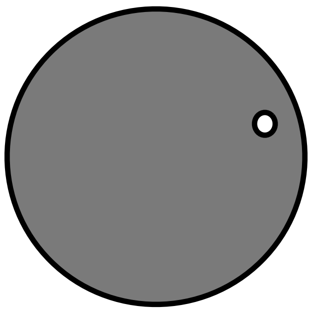
Fermi Surface: Surface separating the filled states from empty states



1-particle Excitations:

Additional particle outside the filled FS at mom. k (N+1 part. state)

Energy (measured from FS) $\quad \epsilon_k = \frac{k^2}{2m} - \frac{k_F^2}{2m} \simeq v_F(k - k_F) \quad\quad v_F = \frac{k_F}{m}$



Particle inside the filled FS taken out at mom. k (N+1 part. state)

Energy of the hole (measured from FS) $\quad \epsilon_k = \frac{k^2}{2m} - \frac{k_F^2}{2m} \simeq v_F(k - k_F)$

F.S. is the locus of zero energy 1-particle excitations in the k space

Interacting Fermions

$$H = \sum_{k\sigma} (\epsilon_k - E_F) c^\dagger_{k\sigma} c_{k\sigma} + \sum_{xx'\{\alpha\}} V^{\alpha\beta}_{\gamma\delta}(x - x') c^\dagger_{x\alpha} c^\dagger_{x'\beta} c_{x'\delta} c_{x\gamma}$$

$$H = \sum_{k\sigma} (\epsilon_k - E_F) c^\dagger_{k\sigma} c_{k\sigma} + \sum_{kk'q\{\alpha\}} V^{\alpha\beta}_{\gamma\delta}(q) c^\dagger_{k\alpha} c^\dagger_{k'\beta} c_{k'+q\delta} c_{k-q\gamma}$$

Beyond this we will work with specific models of V

A) Electrons in a metal: Coulomb interactions

In real metals, electrons have complicated band dispersions and hence complicated FS.

We will ignore these complications and work with a spherical FS

B) Ultracold Fermi gases interacting through Van-Der Waal's potential

Low energy phenomenon —> s-wave scattering

Replace actual potential by a delta fn (in space), tune parameters to get correct scattering length

Electrons in Metal:Coulomb Interaction

$$H = \sum_{k\sigma} (\epsilon_k - E_F) c^\dagger_{k\sigma} c_{k\sigma} + \frac{e^2}{4\pi\epsilon_0} \sum_{xx'\sigma\sigma'} \frac{1}{|x - x'|} c^\dagger_{x\sigma} c^\dagger_{x'\sigma'} c_{x'\sigma'} c_{x\sigma}$$

3D F.T.

$$H = \sum_{k\sigma} (\epsilon_k - E_F) c^\dagger_{k\sigma} c_{k\sigma} + \frac{e^2}{\epsilon_0} \sum_{kk'q\sigma\sigma'} \frac{1}{q^2} c^\dagger_{k\sigma} c^\dagger_{k'\sigma'} c_{k'+q\sigma'} c_{k-q\sigma}$$

Dimensionless parameter characterizing ratio of interaction energy to KE $r_s = \frac{e^2}{\epsilon_0 v_F}$

Measure k in units of k_F, q in units of k_F, energy in units of E_F

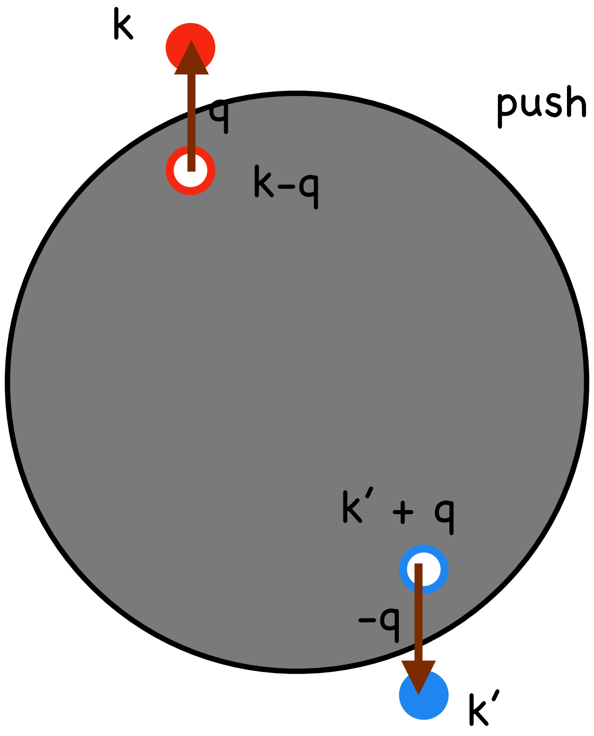
$$H = \sum_{k\sigma} (k^2 - 1) c^\dagger_{k\sigma} c_{k\sigma} + r_s \sum_{kk'q\sigma\sigma'} \frac{1}{q^2} c^\dagger_{k\sigma} c^\dagger_{k'\sigma'} c_{k'+q\sigma'} c_{k-q\sigma}$$

High Density –> large v_F –> KE dominated (weak interaction)

Low Density –> small v_F –> Interaction dominated (strong interaction)

We will work in the high density limit and treat interactions perturbatively.

Particle-Hole Excitations and Interactions



Number conserving excitations:
push a Fermion from inside the FS to a momentum state outside FS

A particle with momentum k and a hole at k-q

State carries a net momentum q

Energy of the state (measured from ground state energy)

$$E = \epsilon_k - \epsilon_{k-q}$$

Effect of the Interaction term on the Fermi Sea

$$c^\dagger_{k\sigma} c^\dagger_{k'\sigma'} c_{k'+q\sigma'} c_{k-q\sigma}$$

Note: k, k′ outside FS; k-q, k′+q inside FS

2 pairs of particle-hole excitations of spin σ and σ'

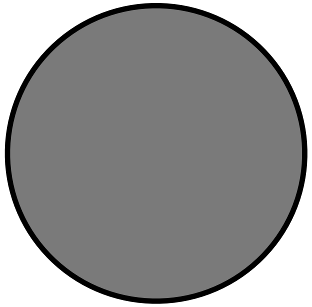
State carries zero momentum

Energy of the state (measured from ground state energy) $E = \epsilon_k + \epsilon_{k'} - \epsilon_{k-q} - \epsilon_{k'+q}$

Coulomb Gas: 1st Order Perturbation Theory

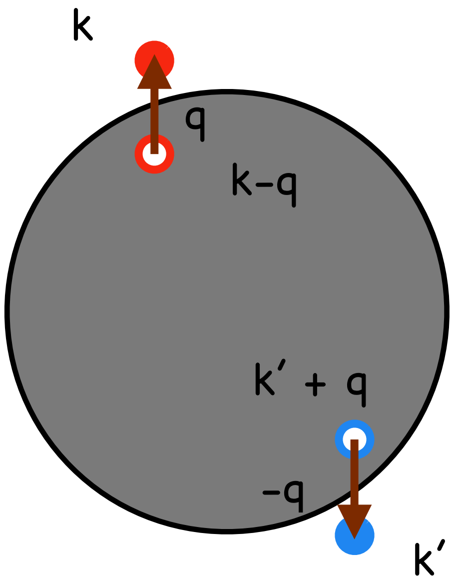
$$H_0 = \sum_{k\sigma} (k^2 - 1) c^\dagger_{k\sigma} c_{k\sigma} \qquad H_1 = r_s \sum_{kk'q\sigma\sigma'} \frac{1}{q^2} c^\dagger_{k\sigma} c^\dagger_{k'\sigma'} c_{k'+q\sigma'} c_{k-q\sigma}$$

Ground State under H_0 $\qquad |\psi\rangle = \prod_{\sigma, |k|<k_F} c^\dagger_{k\sigma} |0\rangle$



Calculate the energy change of the GS in 1st order Perturbation Theory

$$\Delta E^{(1)} = \langle \psi | H_1 | \psi \rangle = r_s \sum_{kk'q\sigma\sigma'} \frac{1}{q^2} \langle \psi | c^\dagger_{k\sigma} c^\dagger_{k'\sigma'} c_{k'+q\sigma'} c_{k-q\sigma} | \psi \rangle$$

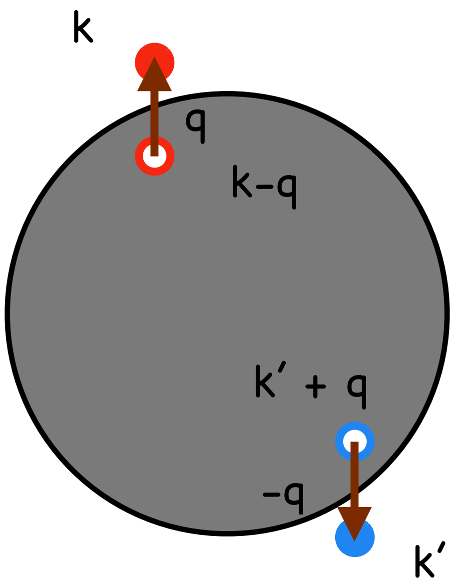


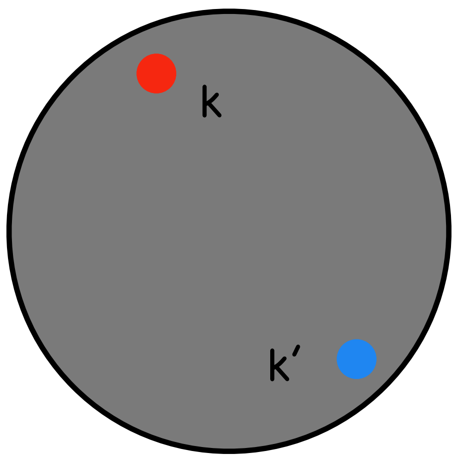
Coulomb Gas: 1st Order Perturbation Theory

$$\Delta E^{(1)} = \langle \psi | H_1 | \psi \rangle = r_s \sum_{kk'q\sigma\sigma'} \frac{1}{q^2} \langle \psi | c^\dagger_{k\sigma} c^\dagger_{k'\sigma'} c_{k'+q\sigma'} c_{k-q\sigma} | \psi \rangle$$

For contribution to 1st order pert theory the state after applying the Int. operator should be the FS

The Direct or Hartree Contribution (q=0)





$$\Delta E^{(1)} = V(q=0) \sum_{kk'} n_k n_{k'}$$

$$n_k = \sum_\sigma \langle \psi | c^\dagger_{k\sigma} c_{k\sigma} | \psi \rangle$$

Momentum Distribution

$$\Delta E^{(1)} = V(q=0)\rho^2$$

Problem with Coulomb Interaction $V(q) \sim 1/q^2$

Coulomb Gas: 1st Order Perturbation Theory

No Hartree Contribution in Metals

We have only looked at the repulsive interaction between electrons till now.

There is an additional attractive interaction with the nuclei/ions left behind.

The q=0 mode couples to the avg. density. It should actually couple to the difference between the avg. density of electrons and the avg. density of ions.

Since the metal is overall charge neutral, the avg. density of ions=avg. density of electrons

Thus the Hartree contribution vanishes in metals.

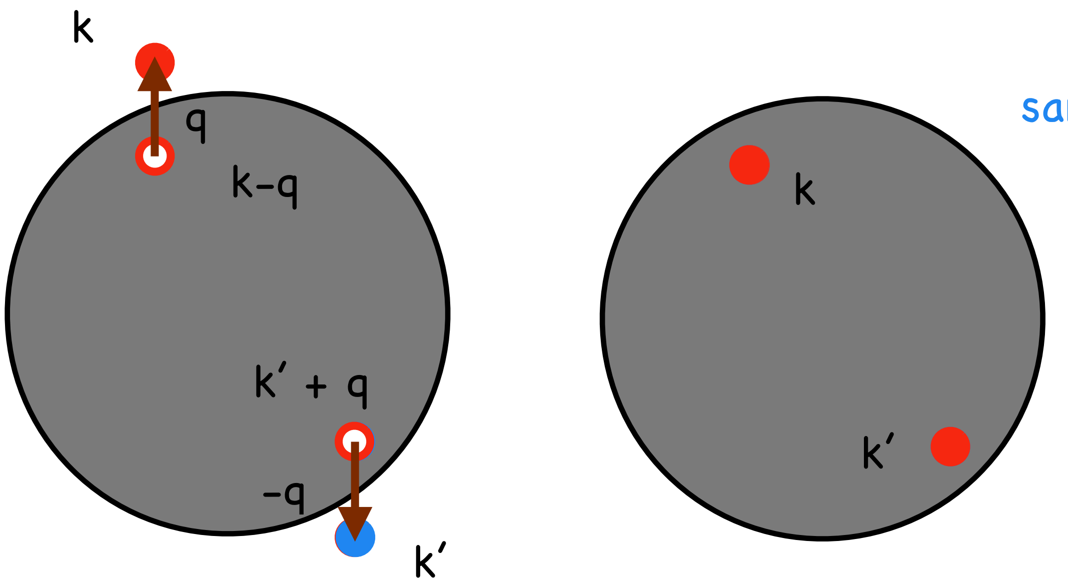
Coulomb Gas: 1st Order Perturbation Theory

$$\Delta E^{(1)} = \langle\psi|H_1|\psi\rangle = r_s \sum_{kk'q\sigma\sigma'} \frac{1}{q^2}\langle\psi|c^\dagger_{k\sigma}c^\dagger_{k'\sigma'}c_{k'+q\sigma'}c_{k-q\sigma}|\psi\rangle$$

For contribution to 1st order pert theory the state after applying the Int. operator should be the FS

The Exchange or Fock Contribution

k=k'+q $\longrightarrow$ q = k-k' $\qquad \sigma = \sigma'$



Interaction between electrons having
same spin component (either both ↑ or both ↓)

$$\langle\psi|c^\dagger_{k\sigma}c^\dagger_{k'\sigma}c_{k\sigma}c_{k'\sigma}|\psi\rangle = \langle\psi|c^\dagger_{k\sigma}[\delta_{kk'} - c_{k\sigma}c^\dagger_{k'\sigma}]c_{k'\sigma}|\psi\rangle$$

$$= n_{k\sigma}\delta_{kk'} - n_{k\sigma}n_{k'\sigma}$$

Coulomb Gas: 1st Order Perturbation Theory

The Exchange or Fock Contribution

$$\langle\psi|c^\dagger_{k\sigma}c^\dagger_{k'\sigma}c_{k\sigma}c_{k'\sigma}|\psi\rangle = \langle\psi|c^\dagger_{k\sigma}[\delta_{kk'} - c_{k\sigma}c^\dagger_{k'\sigma}]c_{k'\sigma}|\psi\rangle$$

$$= n_{k\sigma}\delta_{kk'} - n_{k\sigma}n_{k'\sigma}$$

Fermions $\longrightarrow$ $n^2_{k\sigma} = n_{k\sigma}$, so the k=k′ term vanishes

$$\Delta E^{(1)} = \langle\psi|H_1|\psi\rangle = r_s \sum_{kk'q\sigma\sigma'} \frac{1}{q^2}\langle\psi|c^\dagger_{k\sigma}c^\dagger_{k'\sigma'}c_{k'+q\sigma'}c_{k-q\sigma}|\psi\rangle \quad = -r_s \sum_{k,q\neq 0,\sigma} \frac{1}{q^2} n_{k\sigma}n_{k+q\sigma}$$

Exchange contribution to energy is negative (Fermion anticommutation crucial for - sign)

Attractive interaction between electrons of same spin
Drives Ferromagnetism in itinerant Ferromagnets

Coulomb Gas: 1st Order Perturbation Theory

The Exchange or Fock Contribution

$$\Delta E^{(1)} = \langle\psi|H_1|\psi\rangle = r_s \sum_{kk'q\sigma\sigma'} \frac{1}{q^2}\langle\psi|c^\dagger_{k\sigma}c^\dagger_{k'\,\sigma'}c_{k'+q\sigma'}c_{k-q\sigma}|\psi\rangle \quad = -r_s \sum_{k,q\neq 0,\sigma} \frac{1}{q^2} n_{k\sigma}n_{k+q\sigma}$$

$$= -2r_s \sum_{k\neq k'} n_{k\sigma}n_{k'\sigma}\frac{1}{|k-k'|^2} \qquad = \sum_k \Sigma(k) n_{k\sigma}$$

q limit from momentum dist.

$$\Sigma(k) = 2r_s \sum_q \frac{1}{|k-q|^2} n_{q\sigma} \qquad = \frac{2r_s}{8\pi^3}(2\pi)\int_0^1 q^2 dq \int_0^\pi \sin\theta d\theta \frac{1}{k^2+q^2-2kq\cos\theta}$$

$$= \frac{r_s}{2\pi^2}\int_0^1 q^2 dq \int_{-1}^1 du \frac{1}{k^2+q^2-2kqu}$$

$$= -\frac{r_s}{2\pi^2}\int_0^1 dq \frac{q^2}{2kq}\ln\left[\frac{(k+q)^2}{(k-q)^2}\right]$$

$$= -\frac{r_s}{4\pi^2 k}\int_0^1 dq q \ln\left[\frac{(k+q)^2}{(k-q)^2}\right]$$

$$= -\frac{r_s}{4\pi^2 k}\left[\frac{q^2}{2}\ln\left[\frac{(k+q)^2}{(k-q)^2}\right]\Bigg|_0^1 - \int_0^1 dq q^2\left(\frac{1}{k+q}+\frac{1}{k-q}\right)\right]$$

Coulomb Gas: 1st Order Perturbation Theory

The Exchange or Fock Contribution

$$\Delta E^{(1)} = \langle\psi|H_1|\psi\rangle = r_s \sum_{kk'q\sigma\sigma'} \frac{1}{q^2} \langle\psi|c^\dagger_{k\sigma} c^\dagger_{k'\sigma'} c_{k'+q\sigma'} c_{k-q\sigma}|\psi\rangle \quad = -r_s \sum_{k,q\neq 0,\sigma} \frac{1}{q^2} n_{k\sigma} n_{k+q\sigma}$$

$$= -2r_s \sum_{k\neq k'} n_{k\sigma} n_{k'\sigma} \frac{1}{|k-k'|^2} \qquad = \sum_k \Sigma(k) n_{k\sigma}$$

$$\Sigma(k) = 2r_s \sum_q \frac{1}{|k-q|^2} n_{q\sigma} \quad = -\frac{r_s}{4\pi^2 k}\left[\frac{q^2}{2}\ln\left[\frac{(k+q)^2}{(k-q)^2}\right]\Big|_0^1 - \int_0^1 dq q^2 \left(\frac{1}{k+q} + \frac{1}{k-q}\right)\right]$$

$$= -\frac{r_s}{4\pi^2 k}\left[\ln\left[\frac{|1+k|}{|1-k|}\right] - 2k\int_0^1 dq \frac{q^2}{k^2-q^2}\right] \quad = -\frac{r_s}{\pi^2}\underbrace{\left[\frac{1}{2} + \frac{1-k^2}{4k}\ln\left(\frac{|1+k|}{|1-k|}\right)\right]}_{u(k)}$$

$$E = \sum_{k\sigma} E_k n_{k\sigma}$$

$$E_k = k^2 - 1 - \frac{r_s}{\pi^2}\left[\frac{1}{2} + \frac{1-k^2}{4k}\ln\left(\frac{|1+k|}{|1-k|}\right)\right]$$

Effective Dispersion