Advanced Quantum Mechanics

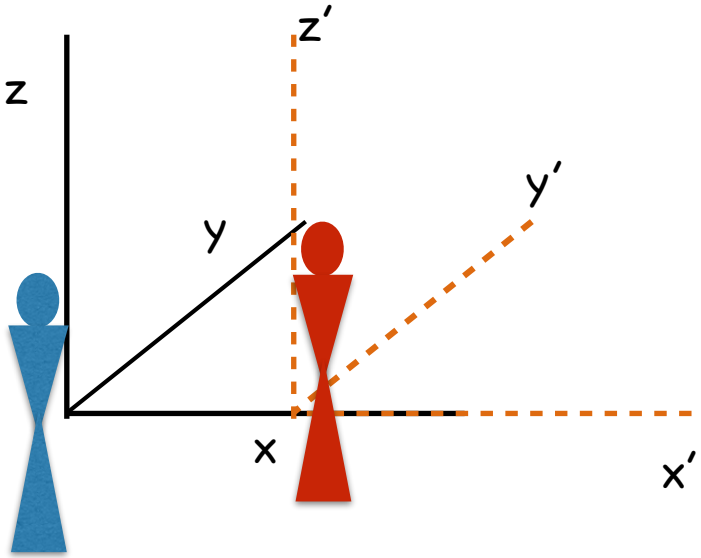
Rajdeep Sensarma

sensarma@theory.tifr.res.in

Lecture #26

QM of Relativistic Particles

Inertial Frames and Newton's Laws.



Two observers travelling with a velocity v wrt each other should write the same equation of motion for a given system

Let the co-ordinates used by the 2 observers be

A ⟶ (x,y,z,t) B ⟶ (x′,y′,z′,t′)

Galilean Transformation $x' = x - vt$ $y' = y$ $z' = z$ $t' = t$

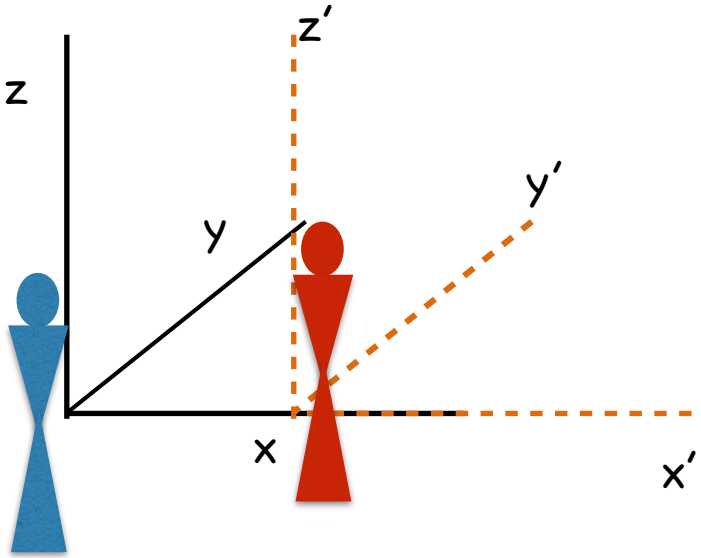
The co-ord are not invariant in the two frames.

Newton's Law: $$m\ddot{\vec{r}} = \vec{F}(\vec{r}, t) \rightarrow m\ddot{\vec{r'}} = \vec{F'}(\vec{r'}, t')$$

If Newton's law is to look the same , F′(r′,t′) = F(r,t) at all points connected by Galilean Transformation. Note that the fn F can change to another fn F′.

Equivalent statement: The force is same in all inertial frames.

Inertial Frames and Lagrangian



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Lagrangian:

$$\mathcal{L} = \frac{m}{2}\dot{\vec{r}}^2 - V(\vec{r})$$

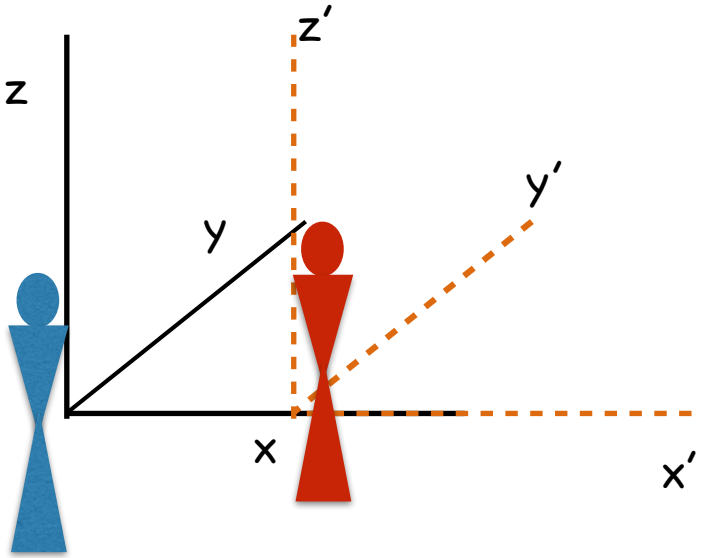
$$= \frac{m}{2}([\dot{\vec{r}'}]^2 + 2\dot{\vec{r}'}\cdot\vec{v} + \vec{v}^2) - V(\vec{r}' + \vec{v}t)$$

Note that as long as v is constant in time, the Euler Lagrange Equations look the same in both frames, as long as V(r)=V′(r′+vt′) for all points connected by Galilean transformations.

Once again note that functional form of both L and V changes, but eqn. of motion looks the same

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{\partial\mathcal{L}}{\partial x} = 0 \rightarrow \frac{d}{dt'}\frac{\partial\mathcal{L}'}{\partial\dot{x}'} - \frac{\partial\mathcal{L}'}{\partial x'} = 0$$

Constant Speed of light and Lorentz Transformation



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Galilean Transformations are incompatible with the observed phenomenon that speed of light in vacuum is same in all inertial frames and is equal to c= 3 X 10^{10} m/s

Lorentz Transformations $x' = \gamma(x - vt)$ $y' = y$ $z' = z$ $ct' = \gamma(ct - \beta x)$

$\beta = v/c$

$\gamma = 1/\sqrt{(1-\beta^2)}$

Check that : If dx/dt = c in one frame, dx′/dt′ =c in all frames connected by LT

Use c =1 to set units $\hbar = 1, \quad c = 1$ E = M = T^{-1} = L^{-1}

Lorentz Boost

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\eta \\ -\eta & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

Rapidity

$\eta = \beta\gamma$

$\gamma^2 - \eta^2 = 1$

$$\begin{aligned} & t'^2 - x'^2 \\ &= (\gamma t - \eta x)^2 - (\eta t - \gamma x)^2 \\ &= (\gamma^2 - \eta^2)(t^2 - x^2) \\ &= t^2 - x^2 \end{aligned}$$

$t^2 - x^2 - y^2 - z^2$ is invariant under LT

Lorentz Transformations

Generalize: Set of Linear transformations of (t,x,y,z) which keeps $s^2=t^2-x^2-y^2-z^2$ invariant

The Lorentz transformations form a continuous non-compact group O(3,1)

Four Vector: $$x^\mu = (x^0, x^i) = (t, \vec{x})$$

$$s^2 = x^\mu x^\nu g_{\mu\nu}$$

Metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Some Notations: $x_\mu = g_{\mu\nu} x^\nu$

Lowering / Raising an index multiplies spatial components by -1 $x_\mu = (t, -\vec{x})$

$$g^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Any set of four quantities which transform like the co-ordinates under LT will be called Lorentz vector.

E.g.: $(-E, p_x, p_y, p_z)$ is the energy-momentum vector.

Lorentz Transformations

Lorentz Transform: Set of Linear transformations of (t,x,y,z) which keeps $s^2=t^2-x^2-y^2-z^2$ inavriant

Linear Transforms: $x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$

Lorentz Invariance of s^2 : $g_{\mu\nu} x'^{\mu} x'^{\nu} = g_{\mu\nu} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} x^{\rho} x^{\sigma} = g_{\rho\sigma} x^{\rho} x^{\sigma}$

$$\boxed{g_{\rho\sigma} = g_{\mu\nu} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma}}$$

Matrix Form: $s^2 = \mathbf{x}^T \mathbf{g} \mathbf{x}$ $\quad \mathbf{x}' = \mathbf{L}\mathbf{x}$

$$\boxed{\mathbf{g} = \mathbf{L}^T \mathbf{g} \mathbf{L}}$$

$$g_{\rho\sigma} = g_{\mu\nu} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} = \Lambda_{\nu\rho} \Lambda^{\nu} \sigma$$

Using $g^{\rho}_{\sigma} = \delta^{\rho}_{\sigma}$ $\quad \delta^{\rho}_{\sigma} = \Lambda_{\nu}{}^{\rho} \Lambda^{\nu} \sigma$ $\quad$ So $(\Lambda^{-1})^{\rho}_{\nu} = \Lambda_{\nu}{}^{\rho}$

Taking Det: Det g =(Det L)^2 Det g $\quad$ Det L= ± 1

Det L= + 1 corresponds to the group SO(3,1). These are called proper Lorentz Transforms, while the transforms with Det L = -1 are the improper Lorentz transforms

Lorentz Transformations

$$g_{\rho\sigma} = g_{\mu\nu}\Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma}$$

Considering g_{00} $\quad 1 = \Lambda^{\rho}_{0} g_{\rho\sigma}\Lambda^{\sigma}_{0} = (\Lambda^{0}_{0})^2 - (\Lambda^{i}_{i})^2 \qquad |\Lambda^{0}_{0}| \geq 1$

	Det L	$\Lambda^0{}_0$
Proper Orthochronous	+1	≥ 1
Improper Orthochronous	-1	≥ 1
Proper Non-Orthochronous	+1	≤ -1
Improper Non-Orthochronous	-1	≤ -1

Example:

Proper Orthochronous

Rotations: $\mathbf{L} = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$ → 3 X 3 Rotation Matrix

Improper Rotations: $\mathbf{L} = \begin{pmatrix} 1 & 0 \\ 0 & RW \end{pmatrix} \qquad W = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Improper Orthochronous

Lorentz Transformations

Example: Lorentz Boost $$\mathbf{L} = \begin{pmatrix} \cosh\eta - \sinh\eta\sigma^x & 0 \\ 0 & 1 \end{pmatrix}$$ Proper Orthochronous

While Rotations form a group by themselves, Lorentz Boosts do not form a group by themselves

Time Reversal $$\mathbf{L} = \begin{pmatrix} -1 & 0 \\ 0_3 & 1_3 \end{pmatrix}$$ Improper Non-Orthochronous

IO =PO X Parity In 4D $$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

IN =PO X Time Reversal In 4D $$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

PN =PO X Time Reversal X Parity

Will focus on proper orthochronous Lorentz Transformations from now on

Group SO(3,1)

Generators of Lorentz Group

Infinitesimal Lorentz Transformations: $\Lambda^\mu_\nu = \delta^\mu_\nu + \epsilon^\mu_\nu$

$$g_{\rho\sigma} = g_{\mu\nu}\Lambda^\mu_\rho \Lambda^\nu_\sigma$$

$$g_{\rho\sigma} = g_{\mu\nu}(\delta^\mu_\rho + \epsilon^\mu_\rho)(\delta^\nu_\sigma + \epsilon^\nu_\sigma) = g_{\rho\sigma} + g_{\mu\sigma}\epsilon^\mu_\rho + g_{\rho\nu}\epsilon^\nu_\sigma$$
$$= g_{\rho\sigma} + \epsilon_{\sigma\rho} + \epsilon_{\rho\sigma}$$

$\varepsilon_{\mu\nu}$ is a 4D antisymmetric tensor

6 indep, parameters ⟶ 6 generators of Lorentz Group

Parametrization of Infinitesimal Transforms:

$$\epsilon = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = a \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{-i\,M^{01}} + b \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{-i\,M^{02}} + c \underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}}_{-i\,M^{03}}$$

$$U(\Lambda) = 1 - i\epsilon_{\mu\nu}M^{\mu\nu}$$

$$M^{\mu\nu} = -M^{\nu\mu}$$

$$+d \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{-i\,M^{12}} + e \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}}_{-i\,M^{13}} + f \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}}_{-i\,M^{23}}$$

Generators of Lorentz Group

By Group Properties $$U^{-1}(\Lambda)U(\Lambda')U(\Lambda) = U(\Lambda^{-1}\Lambda'\Lambda)$$

Use $$\Lambda'^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \epsilon^{\mu}_{\nu} \qquad U(\Lambda') = 1 - i\epsilon_{\mu\nu}M^{\mu\nu}$$

LHS $$U^{-1}(\Lambda)U(\Lambda')U(\Lambda) = 1 - i\epsilon_{\mu\nu}U^{-1}(\Lambda)M^{\mu\nu}U(\Lambda)$$

RHS $$(\Lambda^{-1}\Lambda'\Lambda)_{\rho\sigma} = 1 + (\Lambda^{-1})_{\rho}{}^{\mu}\epsilon_{\mu\nu}\Lambda^{\nu}_{\sigma} = 1 + \Lambda^{\mu}{}_{\rho}\epsilon_{\mu\nu}\Lambda^{\nu}_{\sigma}$$

$$U(\Lambda^{-1}\Lambda'\Lambda) = 1 - i\epsilon_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}_{\sigma}M^{\rho\sigma}$$

Comparing coeff. $$U^{-1}(\Lambda)M^{\mu\nu}U(\Lambda) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}_{\sigma}M^{\rho\sigma}$$

Generators themselves transform as rank 2 Tensors

Transformation Law for vectors $$U^{-1}(\Lambda)P^{\mu}U(\Lambda) = \Lambda^{\mu}_{\nu}P^{\nu}$$

Generators of Lorentz Group

$$U^{-1}(\Lambda)M^{\mu\nu}U(\Lambda) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}_{\sigma}M^{\rho\sigma}$$

Now take Λ to be an infinitesimal transformation.

Expand and compare co-efficients on both sides to get the Lie Brackets

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\,[g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma}] - \rho \leftrightarrow \sigma$$

$$\epsilon = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = a\underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{-i\,M^{01}=iK_1} + b\underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{-i\,M^{02}=iK_2} + c\underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}}_{-i\,M^{03}=iK_3}$$

Identify:

Boost Generators

$K_i \equiv M^{i0}$

Rotation Generators

$J_i \equiv (1/2)\ \varepsilon_{ijk}\ M^{jk}$

$$+d\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{-i\,M^{12}=-2i\,J_3} + e\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}}_{-i\,M^{13}=2i\,J_2} + f\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}}_{-i\,M^{23}=-2i\,J_1}$$

Lie Algebra of Lorentz Group

You can explicitly show that $[J_i, J_j] = i\epsilon_{ijk} J_k$

Easiest way to see this: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & d & 0 \\ 0 & -d & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is infinitesimal form of $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos d & \sin d & 0 \\ 0 & -\sin d & \cos d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

which represents rotation about z axis

The transforms corresponding to J_i is a spatial rotation about axis i.
J_i are the Angular Momentum Operators.

The generators K_i correspond to infinitesimal Lorentz Boosts with velocity in i direction

Easy to show that

$$[J_i, K_j] = i\epsilon_{ijk} K_k$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k$$

SO(3,1) and SU(2) X SU(2)

$$[J_i, J_j] = i\epsilon_{ijk}J_k \qquad [J_i, K_j] = i\epsilon_{ijk}K_k$$

$$[K_i, K_j] = -i\epsilon_{ijk}J_k$$

Consider the operators

$$N_i = \frac{1}{2}(J_i + iK_i) \qquad N_i^\dagger = \frac{1}{2}(J_i - iK_i)$$

Note: for SO(4) you used (1/2) (J+ K)

$$[N_i, N_j] = \frac{1}{4}[(J_i + iK_i), (J_j + iK_j)] \quad = \frac{1}{4}([J_i, J_j] - [K_i, K_j] + i[K_i, J_j] + i[J_i, K_j])$$

$$= \frac{1}{4}(i\epsilon_{ijk}J_k + i\epsilon_{ijk}J_k + \epsilon_{jik}K_k - \epsilon_{ijk}K_k) \; = \frac{1}{2}i\epsilon_{ijk}(J_k + iK_k)$$

$$= i\epsilon_{ijk}N_k$$

$$[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk}N_k^\dagger$$

$$[N_i, N_j^\dagger] = \frac{1}{4}[(J_i + iK_i), (J_j - iK_j)] \; = \frac{1}{4}([J_i, J_j] + [K_i, K_j] + i[K_i, J_j] - i[J_i, K_j])$$

$$= 0$$

So, N_i and $N^\dagger_i$ independently follow the Lie Algebra of SU(2)

Irreps of Lorentz Group

N_i and $N^{\dagger}_i$ independently follow the Lie Algebra of SU(2)

E.g. $N_i N_i$ and $N^{\dagger}_i N^{\dagger}_i$ are operators with eigenvalue m(m+1) and n(n+1) for half-integer m and n

All finite dimensional irreps of the Lorentz Group can be written as (m/2, n/2)

where m/2 is the m+1 dimensional irrep of SU(2)

We will later see a description of Lorentz invariant systems in terms of fields which transform according to different finite dimensional irreps of the Lorentz group

E.g. (0,0) is corresponds to scalars,
(1/2,0) and (0,1/2) corresponds to (left and right handed) Weyl Spinors
(1/2,0) $\otimes$ (0,1/2) = (1/2,1/2) gives a spin 1 representation in 4D (vector fields)

Function Space and Lorentz Transformations

Let us study the action of LT on functions of (t,x,y,z). i.e. ϕ(t,x,y,z)

i.e. Let the same quantity be described in 2 frames by ϕ(t,x,y,z) and ϕ'(t′,x′,y′,z′)

Using T[G_a] $\phi = \phi' = \phi(T^{-1}[G_a]$ (x,y,z,t))

$$\phi^{'}(x') = \phi(\Lambda^{-1}x')$$

Let us consider infinitesimal transformations

$$(\Lambda^{-1}x')^{\mu} = (\Lambda^{-1})^{\mu}_{\nu}(x')^{\nu} = (\delta^{\mu}_{\nu} + \epsilon^{\mu}_{\nu})(x')^{\nu} = (x')^{\mu} + \epsilon^{\mu}_{\nu}(x')^{\nu} = (x')^{\mu} + g^{\mu\rho}\epsilon_{\rho\nu}(x')^{\nu}$$

$$\phi^{'}(x') = \phi(\Lambda^{-1}x') = \phi(x') + g^{\mu\rho}\epsilon_{\rho\nu}(x')^{\nu}\partial_{\mu}\phi(x')$$

By defn. of generators, $\phi^{'}(x') = [1 - i\epsilon_{\rho\nu}M^{\rho\nu}]\phi(x)$

Since ε is antisymmetric, we should compare the antisymmetric part of its co-eff

$$M^{\rho\nu} = i[g^{\mu\rho}x^{\nu}\partial_{\mu} - g^{\mu\nu}x^{\rho}\partial_{\mu}] = -i[x^{\rho}\partial^{\nu} - x^{\nu}\partial^{\rho}]$$

This is a infinite dimensional form for the generators, but is it the most general form?