Advanced Quantum Mechanics

Rajdeep Sensarma

sensarma@theory.tifr.res.in

Lecture #27

QM of Relativistic Particles

Recap of Last Class

Inertial Frames, Galilean Transformation and Lorentz Transformation

(Im) Proper (Non) Orthochronous Lorentz Transformation

Lorentz Group (each possible LT being a group element) ⟶ Lie Group SO(3,1)

Generators of Lorentz Group ⟶ Lie Algebra of SO(3,1)

The Angular Momentum and Boost Operators.

Function Space and Infinite Dim. Generators

Function Space and Lorentz Transformations

Let us study the action of LT on functions of (t,x,y,z). i.e. ϕ(t,x,y,z)

i.e. Let the same quantity be described in 2 frames by ϕ(t,x,y,z) and ϕ'(t',x',y',z')

Using T[G_a] ϕ = ϕ' = ϕ(T^{-1}[G_a] (x,y,z,t))

$$\phi^{'}(x') = \phi(\Lambda^{-1}x')$$

Let us consider infinitesimal transformations

$$(\Lambda^{-1}x')^{\mu} = (\Lambda^{-1})^{\mu}_{\nu}(x')^{\nu} = (\delta^{\mu}_{\nu} + \epsilon^{\mu}_{\nu})(x')^{\nu} = (x')^{\mu} + \epsilon^{\mu}_{\nu}(x')^{\nu} = (x')^{\mu} + g^{\mu\rho}\epsilon_{\rho\nu}(x')^{\nu}$$

$$\phi^{'}(x') = \phi(\Lambda^{-1}x') = \phi(x') + g^{\mu\rho}\epsilon_{\rho\nu}(x')^{\nu}\partial_{\mu}\phi(x')$$

By defn. of generators, $\phi^{'}(x') = [1 - i\epsilon_{\rho\nu}M^{\rho\nu}]\phi(x)$

Since ε is antisymmetric, we should compare the antisymmetric part of its co-eff

$$M^{\rho\nu} = i[g^{\mu\rho}x^{\nu}\partial_{\mu} - g^{\mu\nu}x^{\rho}\partial_{\mu}] = -i[x^{\rho}\partial^{\nu} - x^{\nu}\partial^{\rho}]$$

This is a infinite dimensional form for the generators, but is it the most general form?

Function Space and Lorentz Transformations

For a more general form of the generator, we should consider more than one function, ⟶
e.g. a multicomponent function (e.g. Vector potential which is a four component function, a two component spinor etc.)

Then the the functions can additionally transform to a linear combination of the individual components as well (a matrix operation in the space of component fn.s).

If we choose the vector of functions to transform according to a specific, finite dim. irrep of the Lorentz group, then the matrices would correspond to corresponding irreps.

The most general form of the Lorentz generator is then

$$L^{\mu\nu} = M^{\mu\nu} + S^{\mu\nu}$$

where S commutes with M and follows the same Lie Algebra relations as M

S acts in the space of component indices of the functions.

e.g. for spatial components (i,j =1,2,3), M would be the orbital angular momentum while S is the spin operator.

(Lorentz) scalars, vectors, tensors

Recall the definition of scalar, vectors tensors etc. wrt. rotation

Scalars (they could be numbers, functions, operators etc.) which are invariant.

Vectors are a set of entities which transform among themselves in the same way as co-ordinates.

Cartesian Tensors of Rank n are set of entities which transform in the same way as products of n co-ordinates.

Lorentz Transformation is a co-ord transformation in 4D space.

Lorentz Scalars are entities (they could be numbers, functions, operators etc.) which are invariant under LT.

For fn.s, invariance is to be understood as "the value of the fn. at points related by LT is same".

Example: s^2 is a Lorentz scalar, Norm of any 4 vector is a lorentz scalar, scalar fields etc.

Scalar Operator: $U(\Lambda)\hat{A}U^{-1}(\Lambda) = \hat{A}$

$$[\hat{A}, L^{\mu\nu}] = 0$$

For Infinitesimal Transforms, $U(\Lambda) = 1 - i\epsilon_{\mu\nu}L^{\mu\nu}$

Scalar Operators commute with all generators

(Lorentz) scalars, vectors, tensors

Lorentz Vectors are set of 4 entities (they could be numbers, functions, operators etc.) which transform among themselves (on their linear combinations) in the same way as the co-ordinates transform under LT.

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} \qquad x'_{\mu} = g_{\mu\nu} x'^{\nu} = g_{\mu\nu}\Lambda^{\nu}_{\rho} x^{\rho} = g_{\mu\nu}\Lambda^{\nu}_{\rho} g^{\rho\sigma} x_{\sigma} = \Lambda_{\mu}^{\ \sigma} x_{\sigma}$$

contravariant vector — covariant vector

$$A^{\mu} \to \Lambda^{\mu}_{\nu} A^{\nu} \qquad A_{\mu} \to \Lambda_{\mu}^{\ \nu} A_{\nu}$$

with the same Λ which transforms the co-ord.s

Example: co-ordinates (t,x,y,z) is a contravariant vector. energy-momentum (-E, p_x,p_y,p_z) is a covariant vector.

$$\frac{\partial}{\partial x^{\mu}} = \partial_{\mu}$$

is a Lorentz covariant vector operator

$$\frac{\partial}{\partial x^{\mu}} = \frac{\partial}{\partial x'^{\nu}} \frac{\partial x'^{\nu}}{\partial x^{\mu}} = \Lambda^{\nu}_{\mu} \frac{\partial}{\partial x'^{\nu}} = \Lambda^{\nu}_{\mu} \partial'_{\nu} \qquad \partial'_{\mu} = [\Lambda^{\mu}_{\nu}]^{-1} \partial_{\nu} = \Lambda_{\mu}^{\ \nu} \partial_{\nu}$$

Derivatives of scalar functions transform as covariant vectors

(Lorentz) scalars, vectors, tensors

Vector operators are set of 4 operators, which transform like vector quantities

$$\hat{A}^\mu \to U(\Lambda)\hat{A}^\mu U^{-1}(\Lambda) = \Lambda^\mu_\nu \hat{A}^\nu \qquad \hat{A}_\mu \to U(\Lambda)\hat{A}_\mu U^{-1}(\Lambda) = \Lambda_\mu^{\ \nu} \hat{A}_\nu$$

For Infinitesimal Transforms $\quad U(\Lambda) = 1 - i\epsilon_{\mu\nu}L^{\mu\nu}$

$$U(\Lambda)\hat{A}_\mu U^{-1}(\Lambda) = [1 - i\epsilon_{\rho\sigma}L^{\rho\sigma}]\hat{A}_\mu[1 + i\epsilon_{\alpha\beta}L^{\alpha\beta}]$$

$$\simeq \hat{A}_\mu - i\epsilon_{\rho\sigma}[L^{\rho\sigma}, \hat{A}_\mu]$$

Now $\quad \Lambda_\mu^{\ \nu} = \delta_\mu^{\ \nu} + \epsilon_\mu^{\ \nu} = \delta_\mu^{\ \nu} + g^{\nu\sigma}\epsilon_{\mu\sigma}$ $\qquad$ So $\quad \Lambda_\mu^{\ \nu}\hat{A}_\nu = \hat{A}_\mu + g^{\nu\sigma}\epsilon_{\mu\sigma}\hat{A}_\nu$

Comparing Antisymmetric part of co-eff. of $\varepsilon_{\mu\nu}$ and noting $L^{\mu\nu} = -L^{\nu\mu}$

$$[L^{\rho\sigma}, \hat{A}_\mu] = i[\hat{A}^\sigma g^\rho_\mu - \hat{A}^\rho g^\sigma_\mu]$$

Four vectors transform according to the first non-trivial faithful irrep (1/2,1/2)

(Lorentz) scalars, vectors, tensors

Similarly one can define rank 2 tensors $B^{\mu\nu}$ as set of 16 quantities which transform in the same way as $x^\mu x^\nu$

$$B^{\mu\nu} \to \Lambda^\mu_\rho \Lambda^\nu_\sigma B^{\rho\sigma}$$

metric is a Rank 2 tensor

It can be easily shown that contracting upper and lower indices reduces the rank of the tensor.

$A_\mu B^\mu$ $\quad\partial_\mu A^\mu$ $\quad F_{\mu\nu}F^{\mu\nu}$ are scalars

$\partial_\mu F^{\mu\nu}$ is a contravariant vector

(A, B are vectors, F is rank 2 tensor)

$\partial^\mu F_{\mu\nu}$ is a covariant vector

Translations and Poincare Group

We had defined LT as set of linear transformations on (t,x,y,z) which keeps $s^2=t^2-x^2-y^2-z^2$ invariant

To this let us add translations of space and time and define a set of linear transformations on (t,x,y,z) which keeps $ds^2= dt^2-dx^2-dy^2-dz^2= g_{\mu\nu}dx^{\mu}dx^{\nu}$ invariant.

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}$$

This set forms a group called the Poincare group

$$x^{\mu} \to \Lambda^{\mu}_{\nu}(1) x^{\nu} + a^{\mu}(1) \to \Lambda^{\mu}_{\rho}(2)\Lambda^{\rho}_{\nu}(1) x^{\nu} + \Lambda^{\mu}_{\rho}(2) a^{\rho}(1) + a^{\mu}(2)$$

Note that a^{μ} transforms like a four vector

Additional Generators of Translations: $P_{\mu} = -i\partial_{\mu}$

$$[P_{\mu}, P_{\nu}] = 0 \qquad [L_{\rho\sigma}, \hat{P}_{\mu}] = i[\hat{P}_{\sigma} g_{\rho\mu} - \hat{P}_{\rho} g_{\sigma\mu}]$$

This, together with the Lie Algebra of the Lorentz Generators, form the Lie Algebra of the Poincare group, which has 10 generators.

If we want to describe a free particle (with nothing else in the world), the description must be Poincare invariant. The Casimir operators would provide good Lorentz invariant quantities (quantum numbers) to describe the system

Casimir operators of Poincare group

These have to be bilinears of the generators, which should commute with all the 10 generators

The translation generators P_μ transform as 4-vector. So its norm $P_\mu P^\mu$ is a Lorentz scalar.

i.e. $P_\mu P^\mu$ commutes with all the Lorentz generators.

Since $[P_\mu, P_\nu] = 0$, the norm also commutes with all the translation generators.

Writing $P_\mu P^\mu = E^2 - p_x^2 - p_y^2 - p_z^2 = m^2$, we see that mass is a good quantity to identify particles.

In fact norm of any four-vector which commutes with all P will do the job

Casimir operators of Poincare group

Norm of any four-vector which commutes with all P

Pauli-Lubanski vector: $$W^{\mu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_{\nu} L_{\rho\sigma}$$

$\varepsilon^{\mu\nu\rho\sigma}$ is the completely antisymmetric Levi-Civita symbol in 4D with $\varepsilon^{0123} = 1$

$$[W^{\mu}, P^{\rho}] = 0$$

Show This !!

W^{μ} is by construction a four vector. So $W_{\mu} W^{\mu}$ is the other Casimir operator.

So for the Poincare Group, $P_{\mu} = -i\partial_{\mu}$ $\quad L^{\mu\nu} = M^{\mu\nu} + S^{\mu\nu}$ $\quad W^{\mu} = \frac{-i}{2}\epsilon^{\mu\nu\rho\sigma} S_{\rho\sigma}\partial_{\nu}$

1) $m^2 > 0$ Eigenvalue of $W_{\mu} W^{\mu}$ is $-m^2 s(s+1)$, where s is the spin. Representation labelled by m, s.

   quantum numbers: m, s, s^z and momentum (which is continuous)

2) $m^2 = 0$ Eigenvalue of $W_{\mu} W^{\mu}$ is 0. Also, $P^{\mu}W_{\mu} = 0$. P and W are proportional. Proportionality constant is called helicity. It has eigenvalues ± s, s= 0, 1/2, 1, ... is the spin. Example: photon with spin 1 and helicity ± 1.

QM and Special Relativity

Let us go back to the beginning of QM

- An isolated quantum system at time t: $|\psi(t)\rangle$, a vector in an abstract Hilbert space over complex no.s.

- Any observable (measurable quantity): a linear Hermitian operator in this Hilbert space. With a choice of basis they can be rep. by matrices.

- A special operator, the Hamiltonian, generates the time evolution of the state through the Schrodinger eqn.

$$i\partial_t|\psi(t)\rangle = H|\psi(t)\rangle$$

For the moment, stick to free particles. With p —> -i ∇, the Schrodinger Equation is

$$i\partial_t\psi(\vec{r},t) = -\frac{\nabla^2}{2m}\psi(x,t)$$

First order in time derivative, 2nd order in spatial derivative.
Not invariant under Lorentz transformations.

Need to find Alternatives

QM and Special Relativity and Locality

First Pass:

- Work with $i\partial_t|\psi(t)\rangle = H|\psi(t)\rangle$, but use relativistic dispersion

$$E = \sqrt{p^2c^2 + m^2c^4} = \sqrt{p^2 + m^2}$$

For p << m $\quad E \simeq m + \frac{p^2}{2m} - \frac{1}{8}\frac{p^4}{m^3} + ..$

With p —> -i ∇, the "Schrodinger" Equation has all order spatial derivatives in it.

This violates locality, i.e. changes in the wavefn. at a point should be affected only by what is happening at or near that point. Higher order derivatives imply rate of change of wfn at a point is affected by what is happening at points arbitrarily far from it.

Work with the square: $\quad E^2 = p^2 + m^2 \quad$ and use E -> $i\partial_t$ $\quad$ p —> -i ∇

$$[\partial_t^2 - \nabla^2 + m^2]\phi(x,t) = 0$$

Klein Gordon Equation

Solution: $\phi(x,t) \sim \phi_1 e^{i(p.x - \omega_p t)}$

$$\omega_p = \pm\sqrt{p^2 + m^2}$$

Problems:

Spectrum not bounded from below

ϕ(x,t) cannot be interpreted as wfn (its norm is not conserved)

QM and Special Relativity

What are we doing?

Trying to find a differential eqn., different from Schrodinger Eqn. which is Lorentz invariant

Dirac fixed the issue of norms by finding a first order equation for 4 component fields.

The Dirac Equation still has spectrum which is not bounded from below.

Filled Fermi sea of -ve energy states ——> a theory of many particles —> works for Fermions

The problem with this approach is that this is not a systematic way of constructing possible Lorentz invariant theories.

At this point, we will skip this historical narrative and jump right to the heart of the matter.

i.e. what is the main problem in making QM compatible with STR?

The Unequal Status of Space and Time in QM

QM fundamentally treats space and time on diff. footing

Co-ordinates are observables and are represented by operators

Time, on the other hand is simply a parameter. There is no operator corr. to it.

STR fundamentally treats space and time on equal footing

Lorentz transformations are co-ordinate transformations in t-x space.

2 Options:

1) Raise time to the status of operator

2) Lower spatial coord to status of parameter

We have already seen a formalism of QM which achieves the second option.

The Field Theory formalism used to treat QM of many particles.

Remember that, while the field $\phi(x,t)$ is an operator valued variable, x is simply a parameter which labels the field. In this sense, its status is similar to time.

Possibility: Relativistic Quantum Theory of single particle is similar to Field Theories for Non-relativistic many-particle systems. They of course have different symmetries.

Fields and Lagrangians as fundamental entities

The basic entities of description of a relativistic quantum particle is a field (an operator valued function of space-time) corresponding to the particle.

The dynamics of the system is obtained from the action of the system which is a function of the fields and their space-time derivatives. In particular transition amplitudes can be written as a functional integral over field config. with each config contributing e^{iS}

The field equations (equivalent of "Schrodinger" Eqns) are obtained as the "Euler Lagrange equations" corresponding to the action. These can be justified in terms of saddle point equations.

In particular, for bosonic fields, there is a straightforward classical limit that can be taken. For fermionic fields, which are Grassmann numbers, the classical limit can only be thought of in terms of bilinears of the fields.

The Action, which gives rise to the equation of motion, should be a Lorentz scalar, so that the field equations are Lorentz invariant. For free particles, it should be Poincare Invariant

The standard techniques like perturbation theory and stationary phase approximation can be used in this formalism to calculate transition amplitudes, scattering cross-sections, partition functions, correlation functions (including time dependent ones).