Advanced Quantum Mechanics

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Lecture #29

QM of Relativistic Particles

Recap of Last Class

Scalar Fields and Lorentz invariant actions

Complex Scalar Field and Charge conjugation

Spin 1/2 representations and Weyl Spinors

Transformation properties of Weyl spinors

Rules for Possible Actions

The dynamics of relativistic particles is obtained from the action of the system, which is a function of the fields (operator valued functions of space-time) and their space-time derivatives.

In particular transition amplitudes can be written as a functional integral over field config. with each config contributing e^{iS}

Choose fields transforming according to particular irreps of the Lorentz group.

The transformation properties of the fields are frame independent, e.g. a scalar field remains a scalar field in all frames.

Construct possible Lorentz invariant quantities (Lorentz scalars) out of fields and their derivatives. These provide possible Lagrangian Densities for describing the system.

For free particle, the action should be quadratic and Poincare invariant. It should involve atmost 2nd order derivatives.

Our Aim: Construct possible Lorentz scalars out of Weyl Spinors

Weyl Fields and Grassmann Numbers

Since the (LH) Weyl Field transforms according to (1/2,0) irrep, we should be able to construct a Lorentz Scalar by taking antisymmetric products of 2 LH Weyl Fields

SU(2) decomposition rule 1/2 X 1/2 = 0 + 1

For this we would like to consider two Weyl spinors $\psi_L(x,t) \;\; and \;\; \chi_L(x,t)$

and consider their products. We would like to construct products which are Lorentz invariant.

Under LT: $\chi_L^T \sigma^2 \psi_L \to \chi_L^T U_L^T(\Lambda) \sigma^2 U_L(\Lambda) \psi_L \quad = \chi_L^T \sigma^2 \psi_L$

Since $U_L^T(\Lambda)\sigma^2 U_L(\Lambda) = \sigma^2$

Since the scalar is obtained by antisymmetrizing the product (look up exchange symmetry and SU(2)), we should get 0 if χ is same as ψ

$$\psi_L^T \sigma^2 \psi_L = (\psi_{L1}, \psi_{L2}) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \psi_{L1} \\ \psi_{L2} \end{pmatrix} = -i\psi_{L1}\psi_{L2} + i\psi_{L2}\psi_{L1}$$

= 0 if ψ is a complex no.

However, if the ψ field is a Grassmann number (anti-commuting number), there can be a non-trivial scalar representation formed out of the spin 1/2 field.

Grassmann Numbers

Representing numbers:

A binary representation of an integer

Start with basic objects (generators) 2^n for all non-negative integer n

Define combination rules: a) $2^m\ 2^n = 2^{m+n}$ b) The objects commute with each other $2^m\ 2^n = 2^n\ 2^m$

Arbitrary linear combinations of these generators with co-efficients (0,1) represent integers

Complex Numbers:

Start with basic objects (generators) 1 and i

Define combination rules: 1 * i = i * 1 = i   i * i = -1   1* 1 =1

A complex no. can be represented as arbitrary linear combination of { 1,i} with real co-efficients

Grassmann Numbers:

Start with set of 2n anti-commuting generators $\zeta_\alpha \zeta_\beta + \zeta_\beta \zeta_\alpha = 0$ $\zeta_\alpha^2 = 0$

A Grassmann number is an arbitrary linear comb. of $\{ 1, \zeta_{\alpha 1}, \ldots \zeta_{\alpha 2n}, \zeta_{\alpha 1} \zeta_{\alpha 2} \ldots, \zeta_{\alpha 1} \zeta_{\alpha 2} \ldots \zeta_{\alpha 2n} \}$ with complex co-eff.

E.g. with 2 generators ζ_1 and ζ_2

$$\xi = a_0 + a_1 \zeta_1 + a_2 \zeta_2 + a_{12} \zeta_1 \zeta_2$$

Grassmann Numbers

Adding Grassmann Numbers:

Think of { 1, $\zeta_{\alpha 1}$, ... $\zeta_{\alpha 2n}$, $\zeta_{\alpha 1} \zeta_{\alpha 2}$...., $\zeta_{\alpha 1} \zeta_{\alpha 2}$... $\zeta_{\alpha 2n}$ } as independent "unit vectors" and add component-wise coefficients

Multiplying Grassmann Numbers:

Multiply each component with every other component, and keep track of

$$\zeta_\alpha \zeta_\beta + \zeta_\beta \zeta_\alpha = 0 \qquad \zeta_\alpha^2 = 0$$

Complex conjugation of Grassmann Numbers:

Select a set of n generators and associate a conjugate generator to each

$$(\zeta_\alpha)^* = \zeta_\alpha^* \qquad (\lambda \zeta_\alpha)^* = \lambda^* \zeta_\alpha^* \qquad (\zeta_\alpha \zeta_\beta)^* = \zeta_\beta^* \zeta_\alpha^* = -\zeta_\alpha^* \zeta_\beta^*$$

The generators commute with complex numbers $\lambda \zeta_\alpha = \zeta_\alpha \lambda$

Define conjugation as an operation which conjugates both generators and co-efficients.

Grassmann Numbers

Stick to 2 generators ζ and ζ^*

Functions of Grassmann numbers

Any analytic fn. of ζ^* $\quad f(\zeta^*) = f_0 + f_1\zeta^*$

Any analytic fn. of ζ $\quad g(\zeta) = g_0 + g_1\zeta$

Any analytic fn. of ζ and ζ^* $\quad A(\zeta^*, \zeta) = a_0 + a_1\zeta + \bar{a}_1\zeta^* + a_{12}\zeta^*\zeta$

Grassmann derivatives: Identical to complex deriv., BUT, for the derivative to act on the variable, the variable has to be anticommuted till it is adjacent to the derivative operator.

E.g.:

$$\frac{\partial}{\zeta}(\zeta^*\zeta) = -\frac{\partial}{\zeta}(\zeta\zeta^*) = -\zeta^* \qquad \frac{\partial}{\zeta^*}\frac{\partial}{\zeta}A(\zeta^*, \zeta) = -a_{12} = -\frac{\partial}{\zeta}\frac{\partial}{\zeta^*}A(\zeta^*, \zeta)$$

Grassmann derivatives anticommute

Grassmann Integrals:

Define

$$\int d\zeta 1 = \int d\zeta^* 1 = 0 \qquad \int d\zeta\zeta = \int d\zeta^*\zeta^* = 1$$

Like derivatives, the variable has to be anticommuted till it sits adjacent to integral operator

E.g.:

$$\int d\zeta^*(\zeta^*) = f_1 \qquad \int d\zeta^* d\zeta A(\zeta^*, \zeta) = -a_{12} = \int d\zeta d\zeta^* A(\zeta^*, \zeta)$$

Scalar product of Grassmann Fn.s : $\quad g(\zeta) = g_0 + g_1\zeta \qquad h(\zeta) = h_0 + h_1\zeta$

$$\langle h|g\rangle = \int d\zeta^* d\zeta\ e^{-\zeta^*\zeta} h^*(\zeta) g(\zeta^*) \ = \int d\zeta^* d\zeta\ (1 - \zeta^*\zeta)(h_0^* + h_1^*\zeta)(g_0 + g_1\zeta^*) \ = h_0^* g_0 + h_1^* g_1$$

Weyl Fields and Lorentz Scalars

If Weyl fields are represented by Grassmann numbers, we have seen that

$\psi_L^T \sigma^2 \psi_L$ and $\psi_R^T \sigma^2 \psi_R$ are quadratic Lorentz scalars that can be formed.

These would indicate the possibility of writing down "mass" terms for Weyl Fermions.

However,

Consider the transformation properties of $\sigma^2 \psi_L^*$ where ψ_L is a left-handed Weyl field

$$\sigma^2 \psi_L^* \to \sigma^2 U_L^*(\Lambda) \psi_L^* = \sigma^2 U_L^*(\Lambda) \sigma^2 \sigma^2 \psi_L^* = U_R(\Lambda) \sigma^2 \psi_L^* \qquad \sigma^2 U_L(\Lambda) \sigma^2 = U_R^*(\Lambda)$$

Thus $\sigma^2 \psi^*_L$ transforms like a right handed Weyl Field

Taking h.c. of above equation $\psi_L^T \sigma^2 \to \psi_L^T \sigma^2 U_R^\dagger(\Lambda)$ $\quad \psi_L^T \sigma^2 \to \psi_R^\dagger$ $\quad \psi_R^T \sigma^2 \to \psi_L^\dagger$

Thus the possible "mass" term has the form $m \psi_R^\dagger \psi_L$ or $m \psi_L^\dagger \psi_R$

The mass term mixes left-handed and right handed Weyl fields. Let us first describe massless Fermions with Weyl fields and we will come back to massive fields later.

Lorentz vectors from Weyl Fields

It is possible to construct four-vectors (transf. acc. to (1/2,1/2) irrep) from products of Weyl Fields.

If we can construct Lorentz vectors, we can take its norm, contract with other Lorentz vectors like ∂_μ etc. to form possible Lorentz scalars

Consider the transformation properties of $\psi_L^\dagger \psi_L$

Under rotations, represented by Unitary operators, this is a scalar.

Under Boosts, $U_L(\Lambda) = e^{-i\frac{\vec{\sigma}}{2}\cdot(\vec{\omega}-i\vec{\eta})}$ $\quad \psi_L \to e^{-\frac{1}{2}\vec{\sigma}\cdot\vec{\eta}}\psi_L \quad \psi_L^\dagger \to \psi_L^\dagger e^{-\frac{1}{2}\vec{\sigma}\cdot\vec{\eta}}$

$$\psi_L^\dagger \psi_L \to \psi_L^\dagger e^{-\vec{\sigma}\cdot\vec{\eta}}\psi_L \quad \simeq \psi_L^\dagger \psi_L - \vec{\eta}\cdot\psi_L^\dagger \vec{\sigma}\psi_L = \psi_L^\dagger \psi_L - \eta_i \psi_L^\dagger \sigma^i \psi_L$$

For Infinitesimal Boosts

$$\psi_L^\dagger \sigma^i \psi_L \to \psi_L^\dagger e^{-\frac{1}{2}\vec{\sigma}\cdot\vec{\eta}}\sigma^i e^{-\frac{1}{2}\vec{\sigma}\cdot\vec{\eta}}\psi_L \quad \simeq \psi_L^\dagger \sigma^i \psi_L - \frac{1}{2}\eta_j \psi_L^\dagger (\sigma^j \sigma^i + \sigma^i \sigma^j)\psi_L$$

$$= \psi_L^\dagger \sigma^i \psi_L - \eta_i \psi_L^\dagger \psi_L \qquad \{\sigma^i, \sigma^j\} = 2\delta^{ij}$$

Lorentz vectors from Weyl Fields

For Infinitesimal Boosts

$$\psi_L^\dagger \psi_L \to \psi_L^\dagger e^{-\vec{\sigma}\cdot\vec{\eta}}\psi_L \simeq \psi_L^\dagger\psi_L - \vec{\eta}\cdot\psi_L^\dagger\vec{\sigma}\psi_L = \psi_L^\dagger\psi_L - \eta_i\psi_L^\dagger\sigma^i\psi_L$$

$$\psi_L^\dagger\sigma^i\psi_L \to \psi_L^\dagger e^{-\frac{1}{2}\vec{\sigma}\cdot\vec{\eta}}\sigma^i e^{-\frac{1}{2}\vec{\sigma}\cdot\vec{\eta}}\psi_L \simeq \psi_L^\dagger\sigma^i\psi_L - \frac{1}{2}\eta_j\psi_L^\dagger(\sigma^j\sigma^i + \sigma^i\sigma^j)\psi_L \;\; = \psi_L^\dagger\sigma^i\psi_L - \eta_i\psi_L^\dagger\psi_L$$

Consider $\psi_L^\dagger\sigma^\mu\psi_L$ with σ^0 being 2 X 2 identity matrix

Under Boosts, this behaves like a 4-vector $\delta\psi_L^\dagger\sigma^\mu\psi_L = \epsilon^\mu_\nu\psi_L^\dagger\sigma^\nu\psi_L$ with $\varepsilon^{0i} = \eta_i$

Further, $\psi_L^\dagger\sigma^i\psi_L$ behaves as a 3-vector under rotations

So in all $i\psi_L^\dagger\sigma^\mu\psi_L = i(\psi_L^\dagger\psi_L, \psi_L^\dagger\sigma^i\psi_L)$ is a Lorentz 4-vector

Similarly $i\psi_R^\dagger\bar{\sigma}^\mu\psi_R = i(\psi_R^\dagger\psi_R, -\psi_R^\dagger\sigma^i\psi_R)$ is a Lorentz 4-vector

Since $(\psi_R^\dagger\psi_R)^* = -\psi_R^\dagger\psi_R$ the above Grassmann bilinears are real
(in the sense of conjugation of G No.s)

Lorentz Scalars and Weyl Action

The secret behind Dirac's magic of 1st order Lorentz invariant eqn.

The simplest Lorentz scalar is formed by contracting the LV with ∂_μ

$$\partial_\mu(\psi_L^\dagger)\sigma^\mu\psi_L \qquad \text{or} \qquad \psi_L^\dagger\sigma^\mu\partial_\mu(\psi_L)$$

Real Linear combination $\frac{1}{2}\left[\psi_L^\dagger\sigma^\mu\partial_\mu(\psi_L) - \partial_\mu(\psi_L^\dagger)\sigma^\mu\psi_L\right] \equiv \frac{1}{2}\psi_L^\dagger\sigma^\mu\vec{\partial}_\mu\psi_L$

Quadratic Action for Left Handed Weyl Spinors

$$S = \int d^4x \frac{1}{2}\left[\psi_L^\dagger\sigma^\mu\partial_\mu(\psi_L) - \partial_\mu(\psi_L^\dagger)\sigma^\mu\psi_L\right] \equiv \int d^4x \frac{1}{2}\psi_L^\dagger\sigma^\mu\vec{\partial}_\mu\psi_L$$

Similarly for Right Handed Weyl spinors

$$\partial_\mu(\psi_R^\dagger)\bar{\sigma}^\mu\psi_R \quad \text{or} \quad \psi_R^\dagger\bar{\sigma}^\mu\partial_\mu(\psi_R) \quad \text{or} \quad \frac{1}{2}\left[\psi_R^\dagger\bar{\sigma}^\mu\partial_\mu(\psi_R) - \partial_\mu(\psi_R^\dagger)\bar{\sigma}^\mu\psi_R\right] \equiv \frac{1}{2}\psi_R^\dagger\bar{\sigma}^\mu\vec{\partial}_\mu\psi_R$$

Quadratic Action for Right Handed Weyl Spinors

$$S = \int d^4x \frac{1}{2}\left[\psi_R^\dagger\bar{\sigma}^\mu\partial_\mu(\psi_R) - \partial_\mu(\psi_R^\dagger)\bar{\sigma}^\mu\psi_R\right] \equiv \int d^4x \frac{1}{2}\psi_R^\dagger\bar{\sigma}^\mu\vec{\partial}_\mu\psi_R$$

Gradient Terms ⟶ Poincare Invariant

Field Equations for Weyl Fields

Quadratic Action for Left Handed Weyl Spinors

$$S = \int d^4x \frac{1}{2}\psi_L^\dagger \sigma^\mu \overleftrightarrow{\partial}_\mu \psi_L \quad = \int d^4x \psi_L^\dagger \sigma^\mu \partial_\mu \psi_L$$

upto boundary terms

Field Equation: $\sigma^\mu \partial_\mu \psi_L = 0$

$$(\sigma^0 i\partial_t + \sigma^i i\partial_i)\psi_L = 0$$

Multiplying by i,

$$[i\partial_t - \vec{\sigma}\cdot\vec{p}]\psi_L = 0$$

Similarly for RH Weyl Fermions

$$[i\partial_t + \vec{\sigma}\cdot\vec{p}]\psi_R = 0$$

Since the Fermions are massless $i\partial_t \longrightarrow E = |p|$

$$[1 - \frac{\vec{\sigma}\cdot\vec{p}}{p}]\psi_L = 0 \qquad [1 + \frac{\vec{\sigma}\cdot\vec{p}}{p}]\psi_R = 0$$

The Solution of Weyl Equation are eigenstates of the helicity operator given by $\frac{2\vec{s}\cdot\vec{p}}{p}$

where s is the spin operator, with eigenvalues ± 1

Pauli Lubanski vector $W^0 = -\vec{s}\cdot\vec{p}$ related to helicity in this case

Parity and Weyl Fermions

The helicity operator is a pseudoscalar under O(3), i.e. it changes sign under a parity transformation.

So a LH Weyl spinor with helicity +1 will transform to a RH Weyl spinor with helicity -1 under parity transformation.

A parity invariant theory (even for massless Fermions) requires both LH and RH Weyl fields.

Massless Dirac Fermions [(1/2,0) + (0,1/2)

We have seen that parity mixes the left and right handed Weyl Fermions. Let us keep both fields and construct a 4 component spinor. We want to write a parity invariant theory

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

This transforms as (1/2,0) ⊕ (0,1/2)

This is called a Dirac spinor in the chiral basis

How does parity act on the Dirac spinor?

$$P\psi = P \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \gamma_0 \psi$$

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

1 is 2 X 2 identity matrix

$$\gamma_0^2 = 1$$

1 is 4 X 4 identity matrix

Let us now write the theory of a left handed and a right handed Weyl Fermion together

$$S = \int d^4x \frac{1}{2} [\psi_L^\dagger \sigma^\mu \vec{\partial}_\mu \psi_L + \psi_R^\dagger \bar{\sigma}^\mu \vec{\partial}_\mu \psi_R] = \int d^4x \frac{1}{2} [\psi^\dagger \gamma_0 \gamma^\mu \vec{\partial}_\mu \psi] = \int d^4x \frac{1}{2} [\bar{\psi} \gamma^\mu \vec{\partial}_\mu \psi]$$

where $\gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$

$$\bar{\psi} = \psi^\dagger \gamma_0$$

Pauli conjugate

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

Clifford Algebra

Massless Dirac Fermions [(1/2,0) + (0,1/2)

Check that we have a Lorentz invariant action → Check that $\bar{\psi}\gamma^{\mu}\psi$ Transforms like 4 vector

To do this, we need to find the form of Lorentz generators J_i and K_i

L and R spinors transform as spin 1/2 under rotn.

$$J_i = \begin{pmatrix} \frac{\sigma^i}{2} & 0 \\ 0 & \frac{\sigma^i}{2} \end{pmatrix} \quad K_i = \begin{pmatrix} \frac{-i\sigma^i}{2} & 0 \\ 0 & \frac{i\sigma^i}{2} \end{pmatrix}$$

L and R spinors transform differently under boosts

$$\gamma_0 J_i = J_i \gamma_0 \qquad \gamma_0 K_i = -K_i \gamma_0$$

J is pseudovector under rotn.
K involves one space and 1 time co-ord —> changes sign under parity.

$$\psi \to U(\Lambda)\psi \quad \to \quad \psi^\dagger \to \psi^\dagger U^\dagger(\Lambda) \quad \text{and} \quad \psi^\dagger\gamma_0 \to \psi^\dagger U^\dagger(\Lambda)\tilde{\gamma}_0$$

with $U(\Lambda) = e^{-i(\vec{J}\cdot\vec{\omega}+\vec{K}\cdot\vec{\eta})}$ $\quad U^\dagger(\Lambda) = e^{i(\vec{J}\cdot\vec{\omega}-\vec{K}\cdot\vec{\eta})}$ $\quad U^{-1}(\Lambda) = e^{i(\vec{J}\cdot\vec{\omega}+\vec{K}\cdot\vec{\eta})}$

Note: γ_0 changes form under LT, but maintains its anticommutation with K in new basis

Since γ_0 changes sign of K while commuting across, but keeps J unchanged, commuting it across $U^\dagger$ will convert it to U^{-1}

$$\psi^\dagger\gamma_0 \to \psi^\dagger U^\dagger(\Lambda)\tilde{\gamma}_0 = \psi^\dagger\tilde{\gamma}_0 U^{-1}(\Lambda)$$

Thus $\bar{\psi}\psi$ is a Lorentz scalar

Massless Dirac Fermions [(1/2,0) + (0,1/2)

Consider the matrix $\Gamma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \frac{i}{4}(2\gamma^\mu\gamma^\nu - \{\gamma^\mu, \gamma^\nu\}) = \frac{i}{2}(\gamma^\mu\gamma^\nu - g^{\mu\nu}1_4)$

$$[\Gamma^{\mu\nu}, \gamma^\alpha] = \frac{i}{2}[\gamma^\mu\gamma^\nu, \gamma^\alpha] = \frac{i}{2}(\gamma^\mu[\gamma^\nu, \gamma^\alpha] + [\gamma^\mu, \gamma^\alpha]\gamma^\nu) = \frac{i}{2}(\gamma^\mu[\gamma^\nu, \gamma^\alpha] - [\gamma^\alpha, \gamma^\mu]\gamma^\nu)$$

$$= \frac{i}{2}(\gamma^\mu 2\gamma^\nu\gamma^\alpha - \gamma^\mu 2g^{\nu\alpha} - 2\gamma^\alpha\gamma^\mu\gamma^\nu + 2g^{\alpha\mu}\gamma^\nu) = i([\gamma^\mu\gamma^\nu, \gamma^\alpha] - \gamma^\mu g^{\nu\alpha} + g^{\alpha\mu}\gamma^\nu)$$

So
$$[\Gamma^{\mu\nu}, \gamma^\alpha] = \frac{i}{2}[\gamma^\mu\gamma^\nu, \gamma^\alpha] = i(g^{\alpha\nu}\gamma^\mu - g^{\alpha\mu}\gamma^\nu)$$

$$[\Gamma^{\mu\nu}, \Gamma^{\alpha\beta}] = \left(\frac{i}{2}\right)^2 [\gamma^\mu\gamma^\nu, \gamma^\alpha\gamma^\beta] = \left(\frac{i}{2}\right)^2 ([\gamma^\mu\gamma^\nu, \gamma^\alpha]\gamma^\beta + \gamma^\alpha[\gamma^\mu\gamma^\nu, \gamma^\beta])$$

$$= \frac{i^2}{2}(g^{\nu\alpha}\gamma^\mu\gamma^\beta - g^{\mu\alpha}\gamma^\nu\gamma^\beta + g^{\nu\beta}\gamma^\alpha\gamma^\mu - g^{\mu\beta}\gamma^\alpha\gamma^\nu)$$

$$= i(g^{\nu\alpha}\Gamma^{\mu\beta} - g^{\mu\alpha}\Gamma^{\nu\beta} + g^{\nu\beta}\Gamma^{\alpha\mu} - g^{\mu\beta}\Gamma^{\alpha\nu})$$

$\mathbf{\Gamma}^{\mu\nu}$ satisfies the Lie Algebra for Lorentz generators

So $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$

These relations only use Dirac Algebra and not specific forms for γ matrices

Massless Dirac Fermions [(1/2,0) + (0,1/2)

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

and

$$[S^{\mu\nu}, \gamma^\alpha] = \frac{i}{2}[\gamma^\mu\gamma^\nu, \gamma^\alpha] = i(g^{\alpha\nu}\gamma^\mu - g^{\alpha\mu}\gamma^\nu)$$

γ^μ transforms as a Lorentz 4-vector

$\bar{\psi}\gamma^\mu\psi$ transforms as a Lorentz 4-vector

$$S = \int d^4x \frac{1}{2}[\bar{\psi}\gamma^\mu \vec{\partial}_\mu \psi]$$ is a Lorentz scalar

These relations only use Clifford Algebra and not specific forms for γ matrices

4 x 4 matrices satisfying Clifford Algebra is not unique

In fact any unitary transformation will keep $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ invariant.

Thus there are many equivalent basis to write Dirac spinors. The form of the spinors as well as the γ matrices depend on the basis, but the form of the action is invariant.

Chiral basis

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

Charge conjugated spinor

$$\psi^c = \begin{pmatrix} \sigma^2\psi_R^* \\ -\sigma^2\psi_L^* \end{pmatrix}$$

Majorana spinor

$$\psi^M = \begin{pmatrix} \psi_L \\ -\sigma^2\psi_L^* \end{pmatrix}$$

Real spinor
(Real G No.s)

γ matrices are purely imaginary

Massive Dirac Fields

Let us now come to the question of a mass term

Since $\bar{\psi}\psi$ is a Lorentz scalar $\quad im\bar{\psi}\psi = im(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L)$

is a possible real Lorentz scalar mass term

This is not the only possible quadratic Lorentz scalar

We had earlier shown that $\psi_L^\dagger\psi_R$ and $\psi_R^\dagger\psi_L$ are individually Lorentz scalars

An independent scalar can be formed out of the difference of this two terms $-m(\psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L)$

Define projection operators to obtain Weyl Fermions from Dirac Fermions

In the chiral basis, following projection operators project the Dirac spinor into Left(Right) handed Weyl Fermions

$$L(R) = \frac{1}{2}(1 \pm \gamma_5) \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{24}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$$

γ_5 transforms as a Lorentz scalar, but changes sign under parity i.e. transforms as a pseudo-scalar.

Note: Explicit for of γ_5 will be different in diff. basis, but its relation with γ^μ holds in all basis

Massive Dirac Fermions

$$-m(\psi_L^\dagger \psi_R - \psi_R^\dagger \psi_L) \rightarrow -\frac{m}{2}\bar{\psi}\gamma_5\psi$$

So the most general Dirac action for a massive field has the form

$$S = \frac{1}{2}\int d^4x[\bar{\psi}\gamma^\mu \vec{\partial}_\mu + im - m^{'}\gamma_5\psi]$$

If parity is a good symmetry of the system m′ = 0

$$S = \frac{1}{2}\int d^4x\ \bar{\psi}[\gamma^\mu \vec{\partial}_\mu + im]\psi \ \ = \frac{1}{2}\int d^4x\ \bar{\psi}[\gamma^\mu \partial_\mu + im]\psi$$

upto surface terms

The Saddle point equation for this action is $[i\gamma^\mu \partial_\mu - m]\psi = 0$

Multiply by γ^0, and use $(\gamma^0)^2=1$

$$[i(\gamma^0)^2\partial_t + i\gamma^0\vec{\gamma}\cdot\nabla - m\gamma^0]\psi = 0 \qquad \gamma^0\vec{\gamma} = \vec{\alpha}, \quad \gamma^0 = \beta$$

$$i\partial_t\psi = [-i\vec{\alpha}\cdot\nabla + m\beta]\psi$$

Dirac Equation of 1 paticle Rel. QM

Global Symmetries of Dirac Action

$$S = \frac{1}{2}\int d^4x\ \bar{\psi}[\gamma^\mu \vec{\partial}_\mu + im]\psi \ = \frac{1}{2}\int d^4x\ \bar{\psi}[\gamma^\mu \partial_\mu + im]\psi$$

Global phase rotation $\psi \to e^{i\phi}\psi$

Chiral Transformation $\psi \to e^{i\chi\gamma_5}\psi$

The action is invariant under these transformations

The conserved Noether current is given by

$$j^\mu = i\bar{\psi}\gamma^\mu\psi \qquad\qquad j_5^\mu = i\bar{\psi}\gamma^\mu\gamma_5\psi$$

and the corresponding conserved charges are

$$Q = i\int d^3x\bar{\psi}\gamma^0\psi = i\int d^3x\ \ [\psi_L^\dagger\psi_L + \psi_R^\dagger\psi_R]$$

$$Q_5 = i\int d^3x\bar{\psi}\gamma^0\gamma_5\psi = i\int d^3x\ \ [\psi_L^\dagger\psi_L - \psi_R^\dagger\psi_R]$$

Things we have not touched

- Constructing creation/annihilation operators and Fock space from fields
- Calculating Experimentally measureable quantities

    Scattering amplitude as transition amplitude —> Calculation through fn integrals.

- Finite Chemical potential ——> finite density of particles
- Imaginary time and finite temperature calculations ——> Correlation fn. and response fn.
- Interacting theories

    Symmetry Considerations and possible interaction terms.

    Perturbation Theory calculation of 2n point functions —> scattering, correlation fn. etc.

    Saddle Points, symmetry breaking and effective theories

    Renormalization —— the other guiding principle

Wait for QFT-I next semester.