Advanced Quantum Mechanics

Rajdeep Sensarma

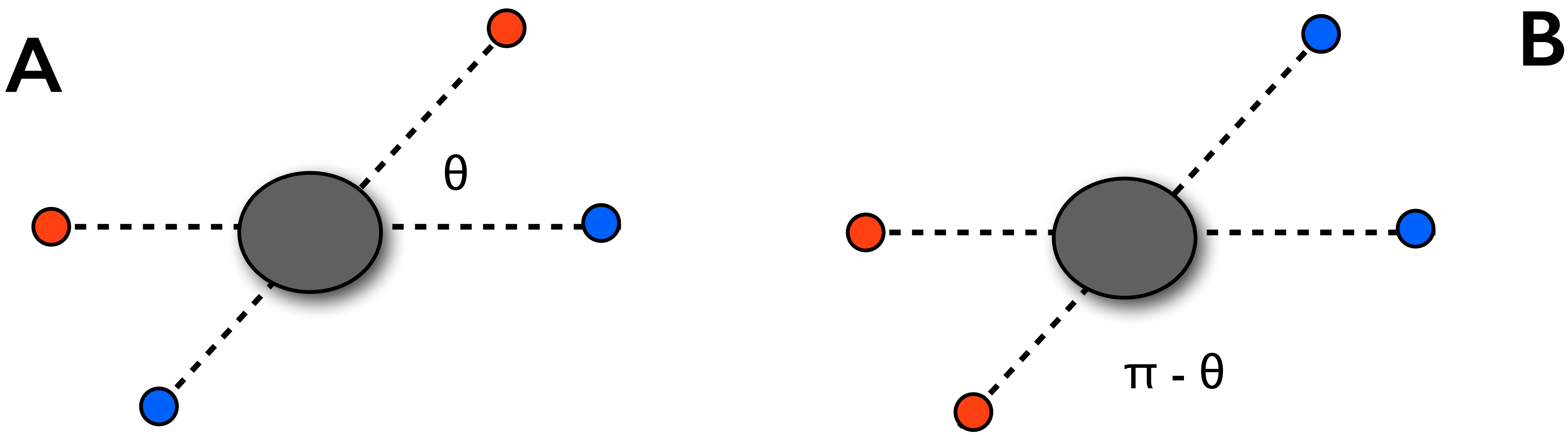
sensarma@theory.tifr.res.in

Scattering Theory

Ref : Sakurai, Modern Quantum Mechanics
Taylor, Quantum Theory of Non-Relativistic Collisions
Landau and Lifshitz, Quantum Mechanics

Scattering of Identical Particles

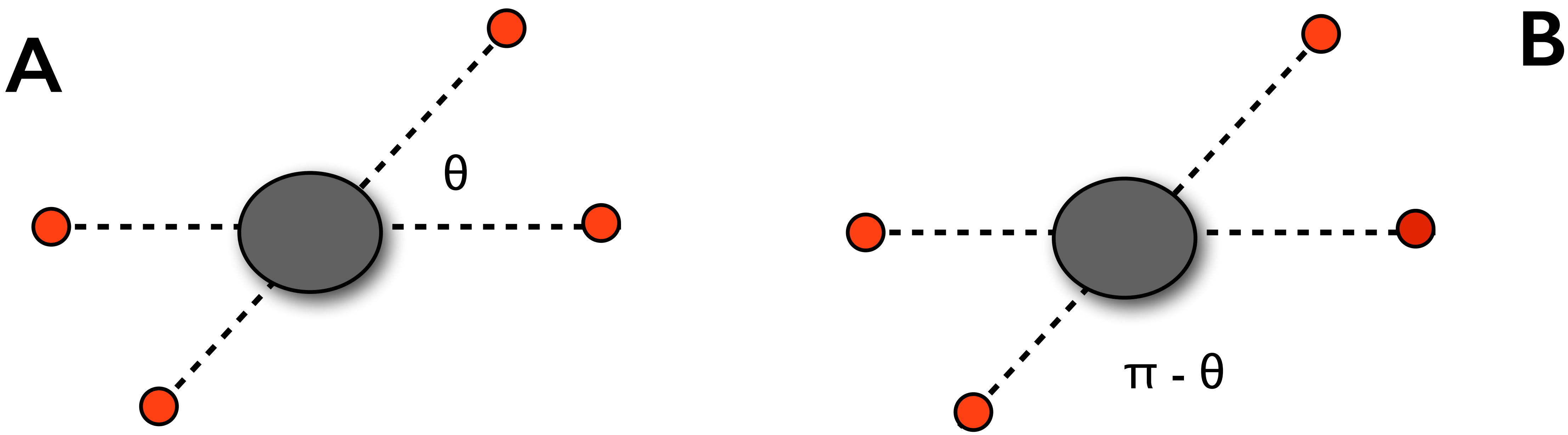
We need to revisit the 2-particle problem of scattering, when the particles are identical.



Scattering of particles in COM frame: particles scattered at angle θ and $\pi - \theta$

Scattering of Identical Particles

We need to revisit the 2-particle problem of scattering, when the particles are identical.



Scattering of particles in COM frame: particles scattered at angle θ and π $-\theta$

- The two outcomes are indistinguishable.
- Previously we only looked at A and not at B
- For identical particles we should add the prob. amplitudes for A and B with appropriate phases

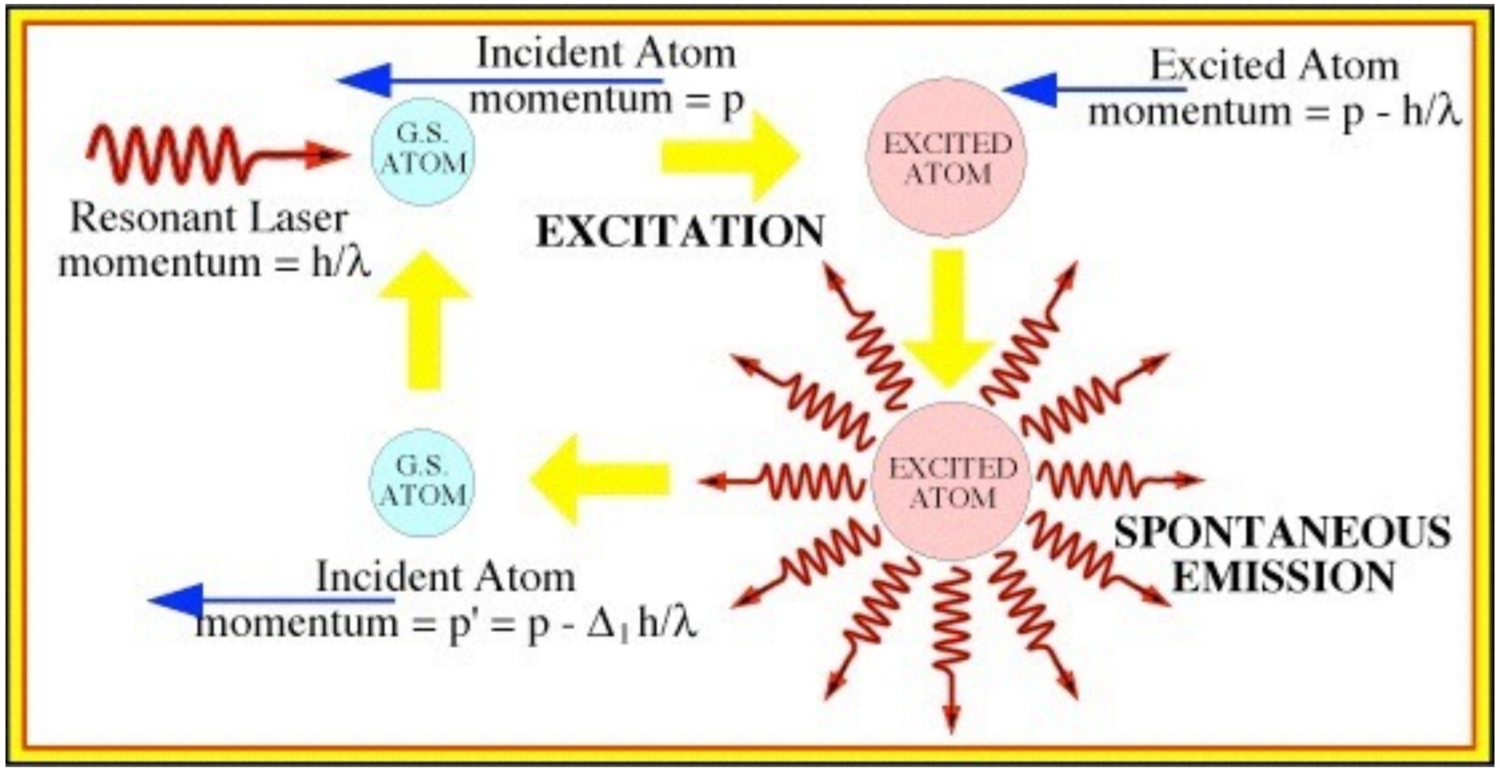
For Bosons: $f(\theta) + f(\pi - \theta)$ For Fermions: $f(\theta) - f(\pi - \theta)$

s-wave scattering differential cross section for identical fermions vanish (at any angle)

Cold Atom Systems

	Bosons	Fermions
Alkali atoms cooled to ~ 10 nK	Rb^{87}, Li^{7}, Na^{23}	K^{40}, Li^{6}

Laser cooling of atoms



$T \sim 10 - 100\ \mu K$

Opposite lasers tuned below atomic transition frequency

Atoms moving toward the light comes into resonance due to Doppler shift

Atoms absorb photon momentum and is slowed down.

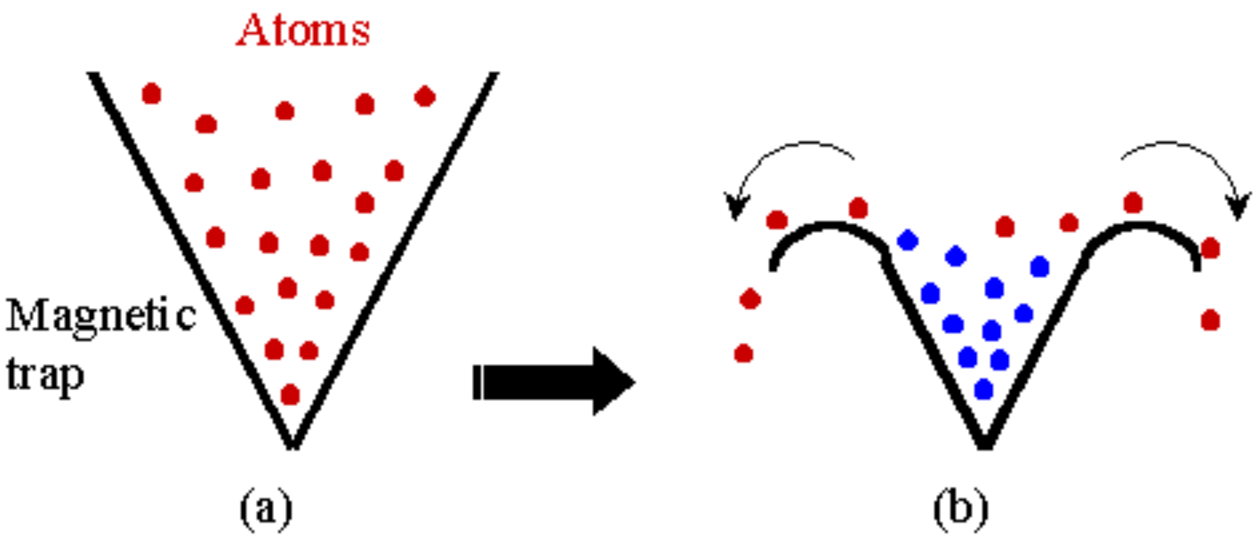
Emission in random directions : avg momentum change is 0

Cold Atom Systems

Alkali atoms cooled to ~ 10 nK

Bosons	Fermions
Rb^{87}, Li^{7}, Na^{23}	K^{40}, Li^{6}

Atoms are trapped using electric/magnetic fields



Further evaporative cooling by opening up the trap

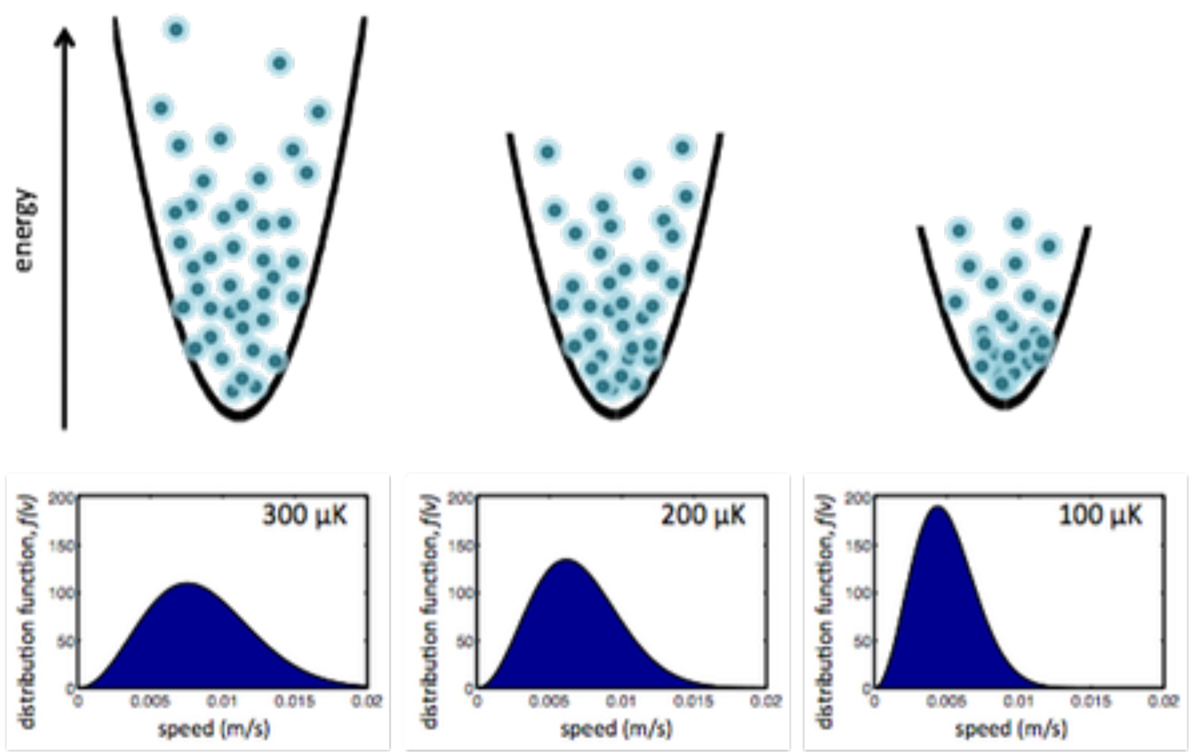


Image courtesy: Le Blanc Group, Univ. of Alberta

Need scattering/ collision between atoms to transfer energy during evap. cooling

Low energy scattering dominated by s-wave scattering

Cannot cool spinless Fermions in this way

Fermions are actually cooled by mixing them with Bosonic species F-B s-wave scattering cross section is not 0

Quantum degenerate gases with 10^5 - 10^7 atoms

Multi-channel scattering

So far, we have not considered internal quantum numbers (like spin) for particles which were involved in scattering.

We will now generalize to the case where the incident particle is in some internal state $|\alpha>$ while the scattered outgoing particle is in an internal state $|\alpha'>$. The scatterer will be taken to be in some initial state $|\beta>$ while the scattering leaves it in the state $|\beta'>$.

E.g.: spin-flip scattering where the spin of the incident particle is flipped in the scattering process.

The quantum states are now specified by relative momentum and the internal qnt. no.s α and β. Note: need dynamics of scatterer to make this work.

Hamiltonian: $$H = \frac{p^2_{\alpha\beta}}{2m} + V(r) + H_{int} \qquad H_{int}|\alpha,\beta\rangle = (\epsilon_\alpha + \epsilon_\beta)|\alpha\beta\rangle$$

Incident State: $e^{i\vec{p}\cdot\vec{r}}|\alpha\beta\rangle$

Scattered State: $e^{i\vec{p}\cdot\vec{r}}|\alpha\beta\rangle + f^{\alpha'\beta'}_{\alpha\beta}(\vec{p},\vec{p'})\frac{e^{ip'r}}{r}|\alpha'\beta'\rangle$

T matrix and scattering amplitude is now a matrix in the space of internal states

Multi-channel scattering: Energetics

Total energy (kinetic+ internal d.o.f.) needs to be conserved in the scattering.

The kinetic energy is not conserved if the energy in the incoming and outgoing internal dof is not the same.

Inelastic Scattering

$$\frac{p'^2}{2m} + \epsilon_{\alpha'} + \epsilon_{\beta'} = \frac{p^2}{2m} + \epsilon_\alpha + \epsilon_\beta \qquad p' \neq p$$

Total energy conservation

$$\frac{p'^2}{2m} = \frac{p^2}{2m} + \epsilon_\alpha + \epsilon_\beta - \epsilon_{\alpha'} - \epsilon_{\beta'}$$

Does not make sense if

$$\frac{p^2}{2m} + \epsilon_\alpha + \epsilon_\beta - \epsilon_{\alpha'} - \epsilon_{\beta'} < 0$$

Consider particles incident with a fixed internal state. For inelastic scattering to a particular internal state, the kinetic energy of the particles should cross a threshold

$$E_{th} = \epsilon_{\alpha'} + \epsilon_{\beta'} - \epsilon_\alpha - \epsilon_\beta$$

Note that if $E_{th} < 0$, the scattering will always occur.

For a given incident internal state and a given incident energy, we can categorize the internal states of the scattered state as either closed, when the energy is below the threshold and open when the energy is above the threshold.

Inelastic Scattering : Energetics and kinematics

Differential Cross Section:

$$\frac{d\sigma}{d\Omega}d\Omega = \frac{\text{No. of particles scattered into the solid angle } d\Omega \text{ around } \hat{k}' = (\theta, \phi) \text{ per unit time}}{\text{Number of incident particles crossing unit area normal to z dirn. per unit time}}$$

Current in incoming channel $v_{\alpha\beta} = \frac{k_{\alpha\beta}}{m}$

Current in outgoing channel $|f_{\alpha\beta}^{\alpha'\beta'}(k_{\alpha\beta}, k'_{\alpha'\beta'})|^2 \frac{k'_{\alpha'\beta'}}{m}$

For inelastic scattering k ≠k', so v≠ v'.

$$\frac{d\sigma_{\alpha\beta}^{\alpha'\beta'}}{d\Omega} = |f_{\alpha\beta}^{\alpha'\beta'}(k_{\alpha\beta}, k'_{\alpha'\beta'})|^2 \frac{k'_{\alpha'\beta'}}{k_{\alpha\beta}}$$

Rate at which 2 particles in volume V in | αβ > are scattered to |α'β'> is K/V

$$\mathcal{K}_{\alpha\beta}^{\alpha'\beta'} = v_{\alpha\beta} \int d\Omega \frac{d\sigma_{\alpha\beta}^{\alpha'\beta'}}{d\Omega} = 2\pi N_{\alpha',\beta'}(E) \int \frac{d\Omega}{4\pi} |\langle \alpha'\beta' | T(k_{\alpha\beta}, k'_{\alpha'\beta'}, E) | \alpha\beta \rangle|^2$$

$$N_{\alpha',\beta'}(E) = \frac{m_r^{3/2}}{\sqrt{2}\pi^2}(E - \epsilon_{\alpha'} - \epsilon_{\beta'})^{1/2}$$

DOS of outgoing states

Born Approx: Replace T by V ⟶ Fermi Golden Rule

Examples of inelastic scattering

electron + atom (ground state) --> electron + atom (excited state)

electron + ion-lattice --> electron + phonons

(Only the target has internal states)

neutron + ion-lattice --> neutron + phonons

electrons scattering off magnetic impurities (both electron and impurity has spin states)

nuclear scattering where hyperfine states are changed in the process.

atom-atom scattering where both atoms have internal (electronic, spin, hyperfine) states

Inelastic Neutron Scattering

Inelastic X-Ray scattering

Stopping of charged particles (like alpha particles) as they pass through a material

Neutrino Oscillations

Inelastic Scattering and Target DOS

Consider Inelastic Neutron Scattering off an insulating magnet

Neutrons interact with electronic spins through $V(r,t) = J\vec{I}_n \cdot \vec{S}(r,t)$

We have assumed that while the system is insulating, i.e. charge motion is frozen

Spins can fluctuate in space and time (about a possible ordered state)

Use full time dependent formalism

$$T^{\alpha}_{\beta}(k_1,\omega_1;k_2,\omega_2) = V^{\alpha}_{\beta}(k_1,\omega_1;k_2,\omega_2) + \int dk_3 \int d\omega_3 V^{\gamma}_{\beta}(k_3,\omega_3;k_2,\omega_2) G^{\gamma}_{0}(k_3,\omega_3) T^{\alpha}_{\gamma}(k_1,\omega_1;k_3,\omega_3)$$

Shortened notation: α, β shorthand for states of both neutron and target

Will think of incoming polarized neutrons with Iz =+,
Look for inelastic scattering in the outgoing channel Iz=-

Relevant potential : $V(r,t) = JI_n^{-1}S^{+}(r,t)$

Will use Born Approximation

Insulating Magnets and Dynamic Structure Factor

Will use Born Approximation

Relevant potential : $V(r,t) = JI_n^{-1}S^+(r,t)$

$$\mathcal{K}_+^-(k',k,) \propto \langle S^+(\vec{q},\omega)S^-(-\vec{q},-\omega)\rangle\delta(\vec{k}'-\vec{k}-\vec{q})\delta(\frac{k'^2}{2m}-\frac{k^2}{2m}-\omega)$$

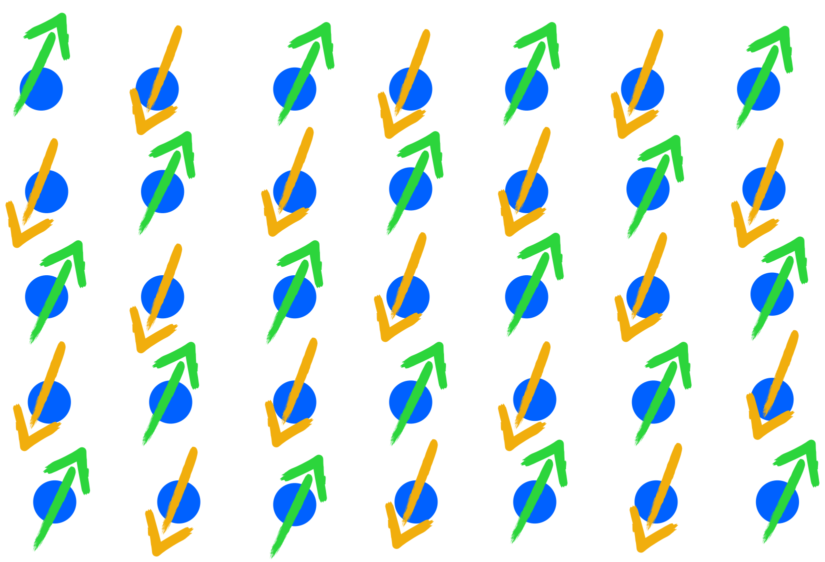
F.T. of $\langle S^+(r,t)S^-(r',t')\rangle$ — mom-conservation — energy-conservation

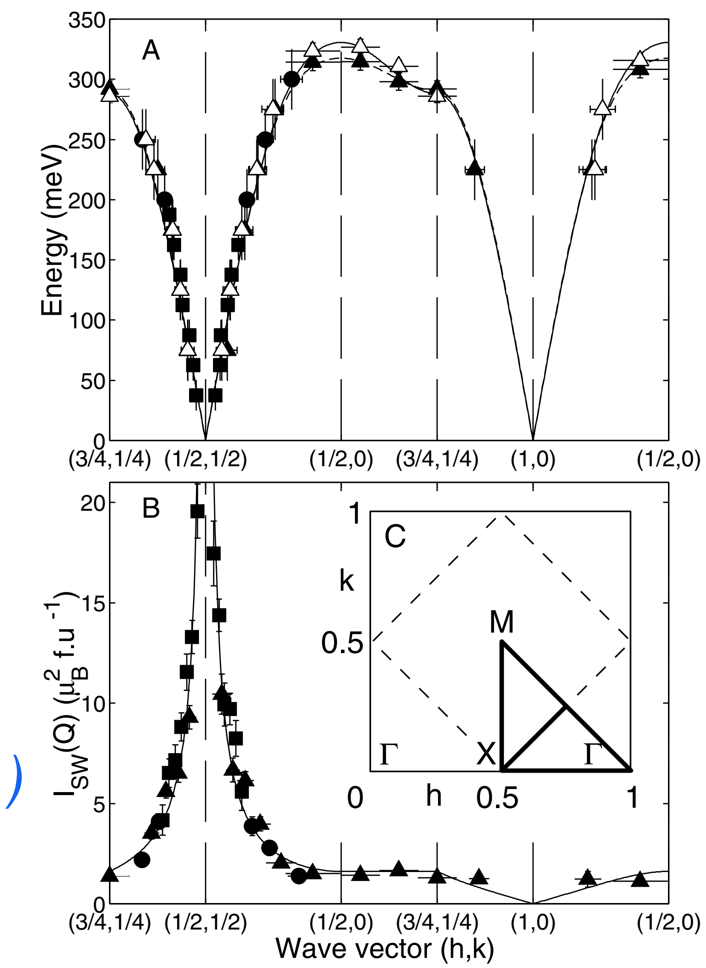
Dynamic Structure Factor

Info about Spin Dynamics

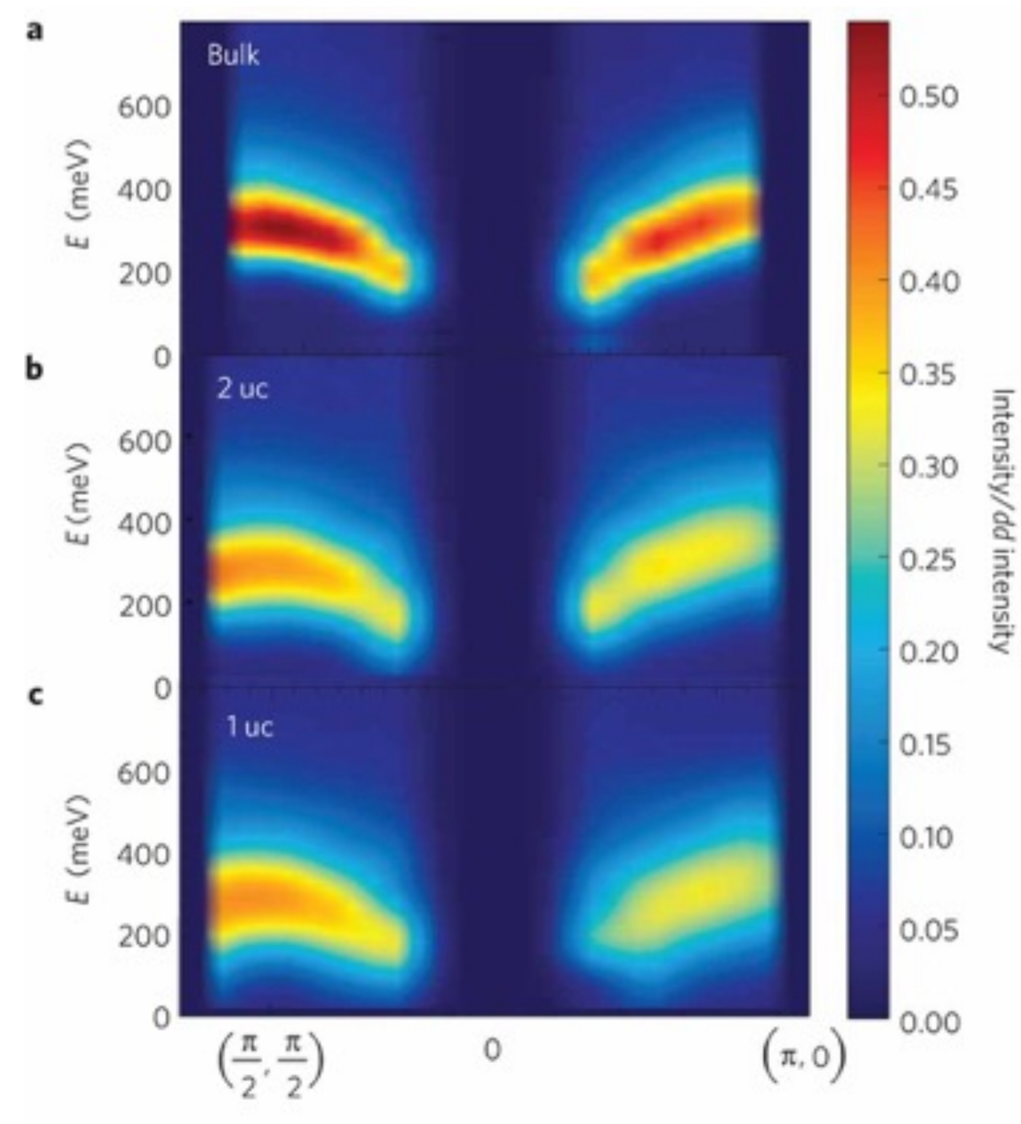
Insulating Magnets and Dynamic Structure Factor

$$\mathcal{K}_{+}^{-}(k', k,) \propto \langle S^{+}(\vec{q}, \omega) S^{-}(-\vec{q}, -\omega)\rangle \delta(\vec{k}' - \vec{k} - \vec{q})\delta(\frac{k'^2}{2m} - \frac{k^2}{2m} - \omega)$$

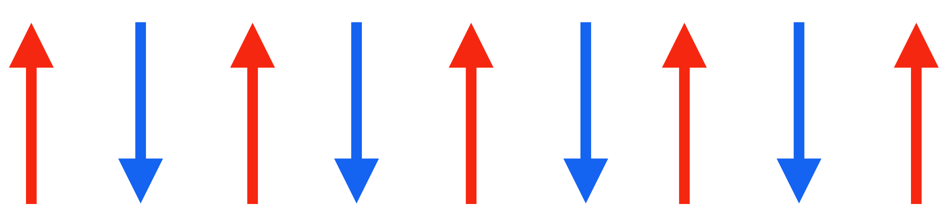




R. Coldea et. al
PRL 86, 5377 (2001)



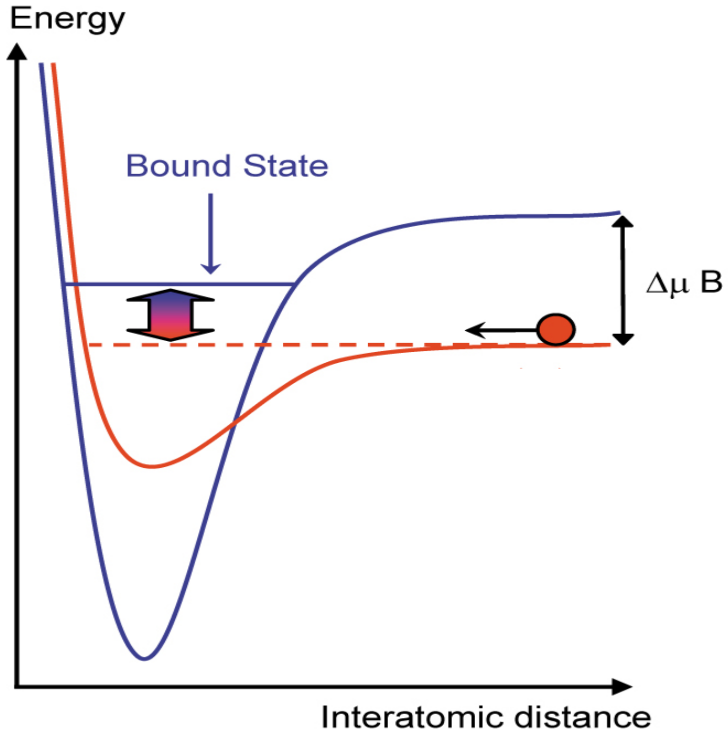
M P M Dean et. al
Nature Mat. 11, 850 (2012)



Spin Wave Excitations captured by
inelastic Neutron Scattering

Elastic Part in Multichannel Scattering

Consider a multichannel scattering problem with 1 open channel



Only Elastic Scattering is possible

However scattering in the open channel can be drastically changed by presence of closed channel

$$T = V + VG_0T$$

matrix eqn. incl. internal states

$$T = (1 - VG_0)^{-1}V = V(1 - VG_0)^{-1}$$

$$G_0 = (E - H_0 + i\eta)^{-1}$$

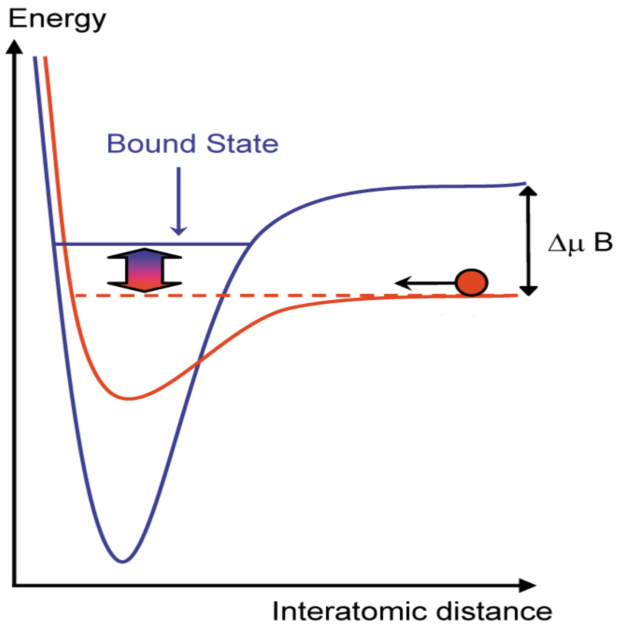
$$T = (E + i\eta - H_0)(E + i\eta - H_0 - V)^{-1}V$$

$$V = V_1 + V_2$$

V_1: diag. in channels

V_2: off diag. coupling

Elastic Part in Multichannel Scattering



$$T = (E + i\eta - H_0)(E + i\eta - H_0 - V)^{-1} V$$

$$V = V_1 + V_2$$ off diag. coupling

diag. in channels

Use $$(A - B)^{-1} = A^{-1}(1 + B(A - B)^{-1})$$

with $A = E + i\eta - H_0 - V_1$ $\quad B = V_2$

$$(E+i\eta-H_0-V_1-V_2)^{-1} = (E+i\eta-H_0-V_1)^{-1}[1+V_2(E+i\eta-H_0-V_1-V_2)^{-1}]$$

$$T = T_1 + (1 - V_1 G_0)^{-1} V_2 (1 - G_0 V)^{-1}$$

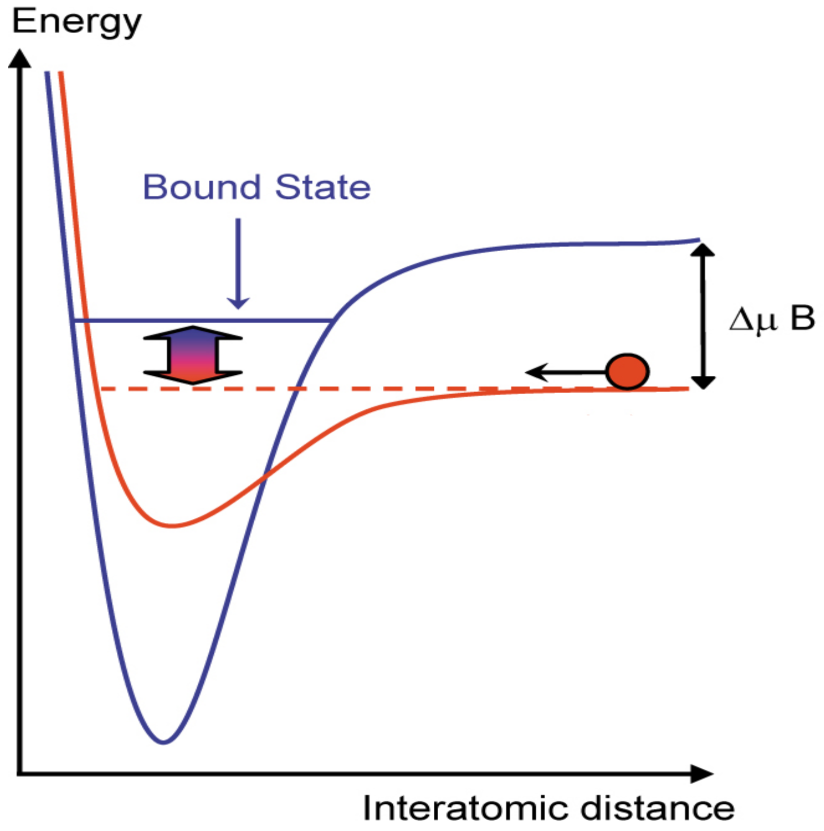
T matrix in absence of off diagonal V_2

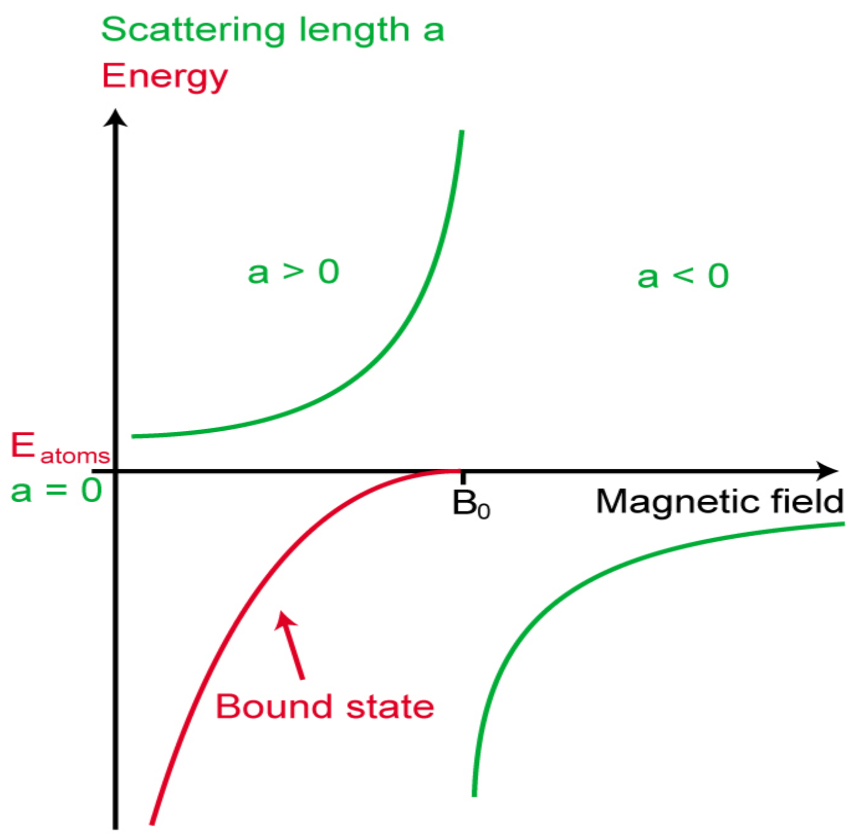
For expectation in the same channel, go to 2nd order in V_2

$$\frac{4\pi a_s}{m} = \frac{4\pi a_P}{m} + \sum_n \frac{|\langle \psi_n | V_2 | \psi_0 \rangle|^2}{E_{th} - E_n}$$

Take low energy limit

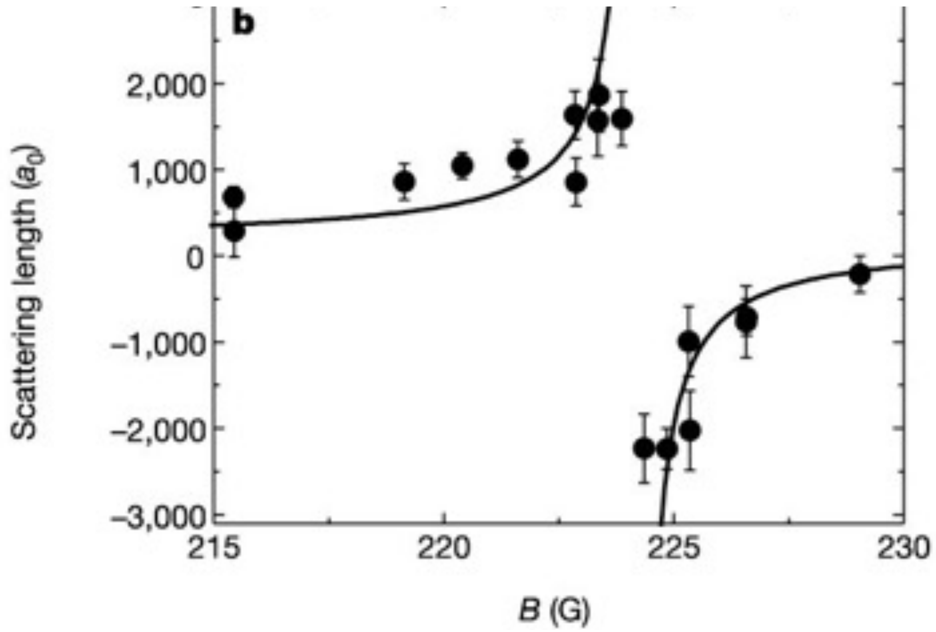
Elastic Part in Multichannel Scattering





C. Regal and D. S. Jin, PRL 90, 230404 (2003)

$$\frac{4\pi a_s}{m} = \frac{4\pi a_P}{m} + \sum_n \frac{|\langle \psi_n | V_2 | \psi_0 \rangle|^2}{E_{th} - E_n}$$



Feshbach Resonance in K^{40}