Lattice QCD with chemical potential

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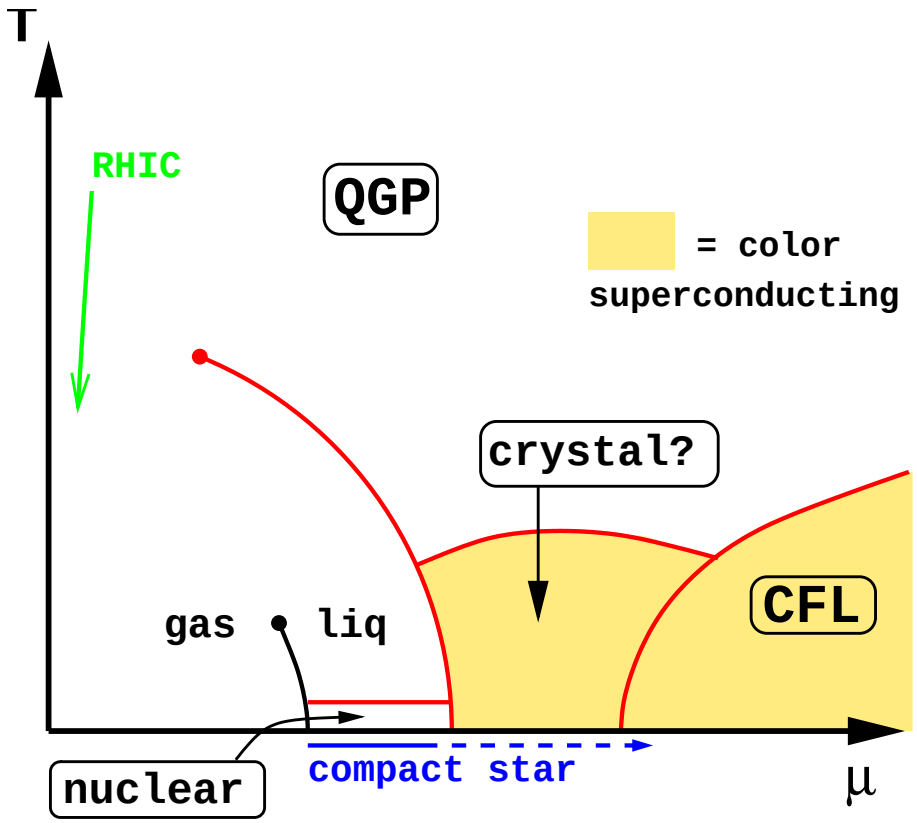
1. Why chemical potential? What's the problem?
2. Reweighting and Taylor series expansion.
3. Quark number susceptibilities: definitions and results.
4. Two bits of physics: fluctuations and strangeness
5. Determining the equation of state: the pressure at finite μ
6. Breakdown of the expansion: phase transitions
7. Condensates and masses
8. Main results

Why chemical potential?



Flavour symmetry: one μ for every independent conserved charge.

M. G. Alford, K. Rajagopal, F. Wilczek, *Phys. Lett.*, B 422 (1998) 247,

R. Rapp, T. Schafer, E. V. Shuryak, M. Velkovsky, *Phys. Rev. Lett.*, 81 (1998) 53.

What's the problem?

$$Z = \mathrm{e}^{-F/T} = \int DU \mathrm{e}^{-S} \prod_f \det M(U, m_f, \mu_f) = \int DU \mathrm{e}^{-\mathcal{S}(T,\mu)}$$

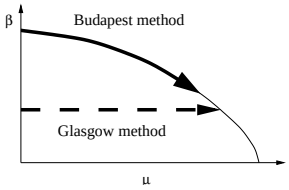
Dirac operator: $M = m + i\partial_\mu \gamma_\mu$

- If there is a Q such that $M^\dagger = Q^\dagger M Q$, then clearly $\det M$ is real.
- $Q = \gamma_5$ for $\mu = 0$. Nothing for $\mu \neq 0$.
- Monte Carlo simulations of Z fail.
- Under CP symmetry $\{U\} \rightarrow \{U'\}$ such that $\det M(U) = [\det M(U')]^*$.
- Z remains real and non-negative— thermodynamics is safe.

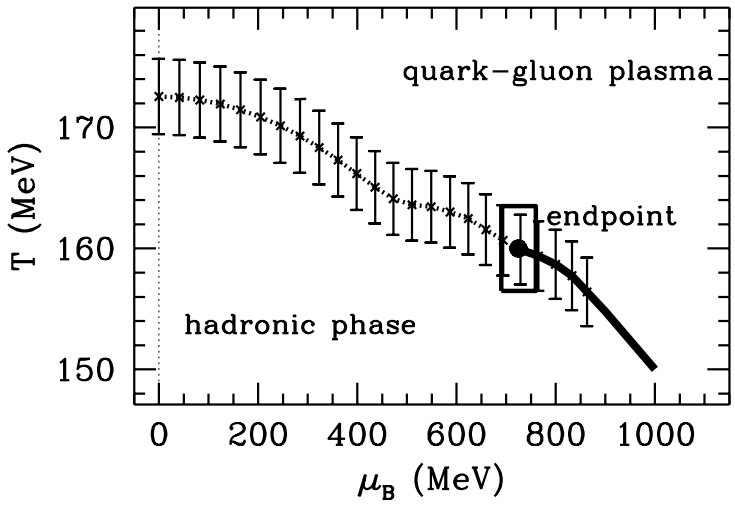
Reweighting: what it is

Do simulations at $\mu_f = 0$, re-express expectation values in terms of these—

$$\langle O \rangle_\mu = \frac{\langle O \exp(-\Delta\mathcal{S}) \rangle}{\langle \exp(-\Delta\mathcal{S}) \rangle} \quad \text{where} \quad \mathcal{S} = S - \sum_f \mathrm{Tr}\ \log M_f,$$



Reweighting done for coarse lattices ($N_t = 4$) and $N_f = 4$, 2 and 2+1.



Reweighting and other direct approaches

- Two parameter reweighting and results.
  Z. Fodor and S. D. Katz, *J. H. E. P.*, 03 (2002) 014.

- Express the reweighting in terms of derivatives of Z with respect to μ.
  C. R. Allton *et al.*, *Phys. Rev.*, D 66 (2002) 074507

- Simulate imaginary μ (positive $\det M$) and do analytic continuation.
  M. D'Elia and M.-P. Lombardo, hep-lat/0209146
  P. De Forcrand and O. Philipsen, *Nucl. Phys.*, B642 (2002) 290

- Special care needed for this: find Yang-Lee zeroes directly.
  S, Gupta, hep-lat/0307007.

- Canonical partition functions and propagator matrix.
  P. Crompton, P. Higgs, MILC, BI, *etc*

Reweighting: lattice artifacts

Chemical potential on the lattice is prescription dependent. Why? The continuum Dirac operator specifies effects of an infinitesimal time translation. On the lattice we deal with finite translations (by lattice spacing a). There are many ways of doing this which lead to the same infinitesimal transformation.

This is the origin of problems with reweighting: it gives no indication of how large the lattice artifacts are.

Taylor series expansion is prescription dependent beyond 2nd order at every finite lattice spacing a, but prescription independent for $a \longrightarrow 0$. With explicit Taylor expansion we can take the continuum limit.

R. V. Gavai and S. Gupta, *Phys. Rev.* D 68 (2003) 034506.

The Taylor Expansion

Since $PV = -F = T \log Z$, the Taylor expansion of P is the same as of F!

$$\frac{1}{V} P(T, \mu_u, \mu_d) = \frac{1}{V} P(T, 0, 0) + \sum_f n_f \mu_f + \frac{1}{2!} \sum_{fg} \chi_{fg} \mu_f \mu_g + \cdots$$

where the quark number densities and susceptibilities are—

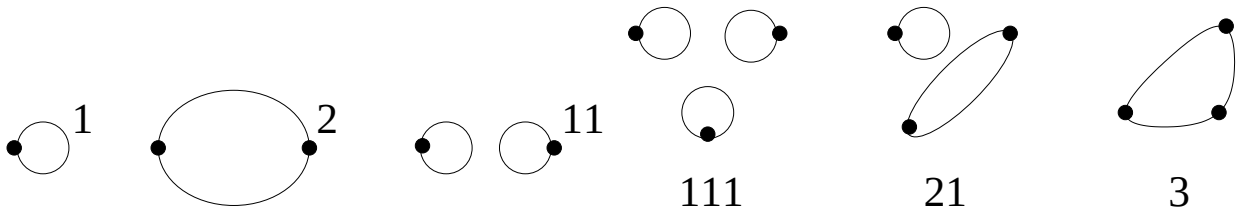
$$\begin{aligned} n_f &= \frac{T}{V} \frac{\partial \log Z}{\partial \mu_f}\bigg|_{\mu_f=0} \\ \chi_{fg} &= \frac{T}{V} \frac{\partial^2 \log Z}{\partial \mu_f \partial \mu_g}\bigg|_{\mu_f=\mu_g=0} \\ \chi_{fgh\cdots} &= \frac{T}{V} \frac{\partial^n \log Z}{\partial \mu_f \partial \mu_g \partial \mu_h \cdots}\bigg|_{\mu_f=\mu_g=\cdots=0} \end{aligned}$$

Derivatives

Derivatives of $\log Z$ can be expressed in terms of derivatives of Z. The latter can be constructed by the chain rule.

$$Z_f = \frac{\partial Z}{\partial \mu_f} = \int DU \mathrm{e}^{-\mathcal{S}} \mathrm{Tr}\ M_f^{-1} M_f'.$$

Note: $M' = \gamma_0$ and $M^{-1} = \psi\overline{\psi}$, so $\mathrm{Tr}\ M^{-1}M' = \psi^\dagger\psi$. Odd derivatives vanish for $\mu_f = 0$ by CP symmetry. S. Gottlieb *et al.*, *Phys. Rev. Lett.*, 59 (1987) 2247



S. Gupta, *Acta Phys. Pol.*, B 33 (2002) 4259

Quark number susceptibilities: phenomena

- Fluctuations of conserved quantities in heavy-ion collisions are related to χ_{uu}. Isospin fluctuations are related to $\chi_3 = \chi_{uu} - \chi_{ud}$, charge fluctuations can also be constructed out of these. M. Asakawa *et al.*, *Phys. Rev. Lett.*, 85 (2000) 2072; S. Jeon and V. Koch, *ibid.*, 85 (2000) 2076

- Under certain conditions strangeness production rate can be related to the strange susceptibility, χ_{ss}. R. V. Gavai *et al.*, *Phys. Rev.*, D 65 (2002) 054506

- The pressure at finite chemical potential is essentially determined by the susceptibility. R. V. Gavai and S. Gupta, *Phys. Rev.* D 68 (2003) 034506.

- χ_3 is the zero momentum Euclidean finite temperature vector propagator and hence closely related to a transport coefficient— the DC electrical conductivity of quark matter. S. Gupta, hep-lat/0301006.

Some notation

With two degenerate flavours of quarks, in flavour space the linear susceptibilities form the matrix

$$\begin{pmatrix} \chi_{uu} & \chi_{ud} \\ \chi_{ud} & \chi_{uu} \end{pmatrix}$$

Transforming to $\mu_0 = \mu_u + \mu_d$ and $\mu_3 = \mu_u - \mu_d$, this matrix becomes

$$\begin{pmatrix} \chi_{uu} + \chi_{ud} & 0 \\ 0 & \chi_{uu} - \chi_{ud} \end{pmatrix}$$

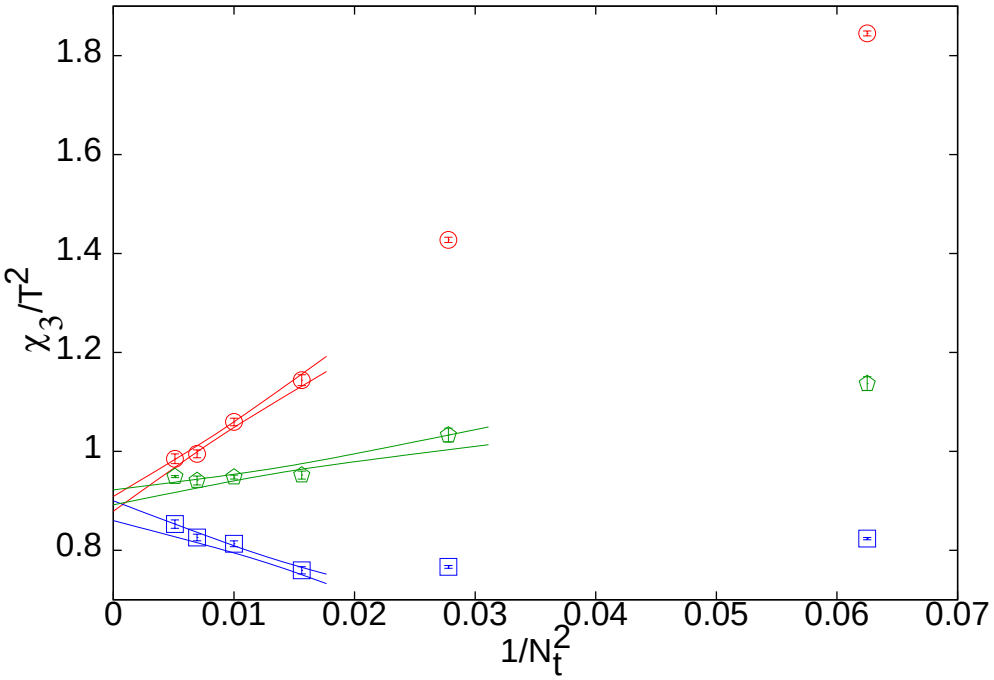
$$\chi_3 = \chi_{uu} - \chi_{ud} = \left\langle \mathrm{Tr}\ M^{-1}M'M^{-1}M' - \mathrm{Tr}\ M^{-1}M'' \right\rangle$$

$$\chi_{ud} = \left\langle \left(\mathrm{Tr}\ M^{-1}M'\right)^2 \right\rangle \quad \text{and} \quad \chi_0 = \chi_3 + 2\chi_{ud}$$

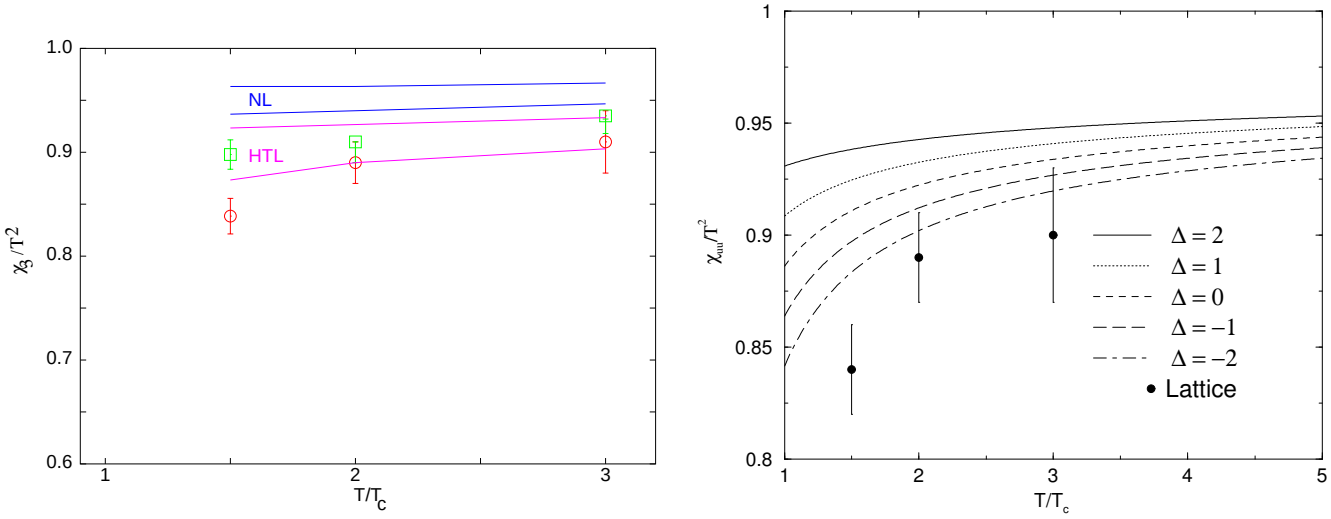
Finding the continuum limit

Main technical problem is to control the extrapolation to zero lattice spacing. For this we use two different kinds of Fermions (staggered and Naik) and perform simultaneous extrapolation with both: in the quenched theory.

R. V. Gavai and S. Gupta, *Phys. Rev.* D 67 (2003) 034501



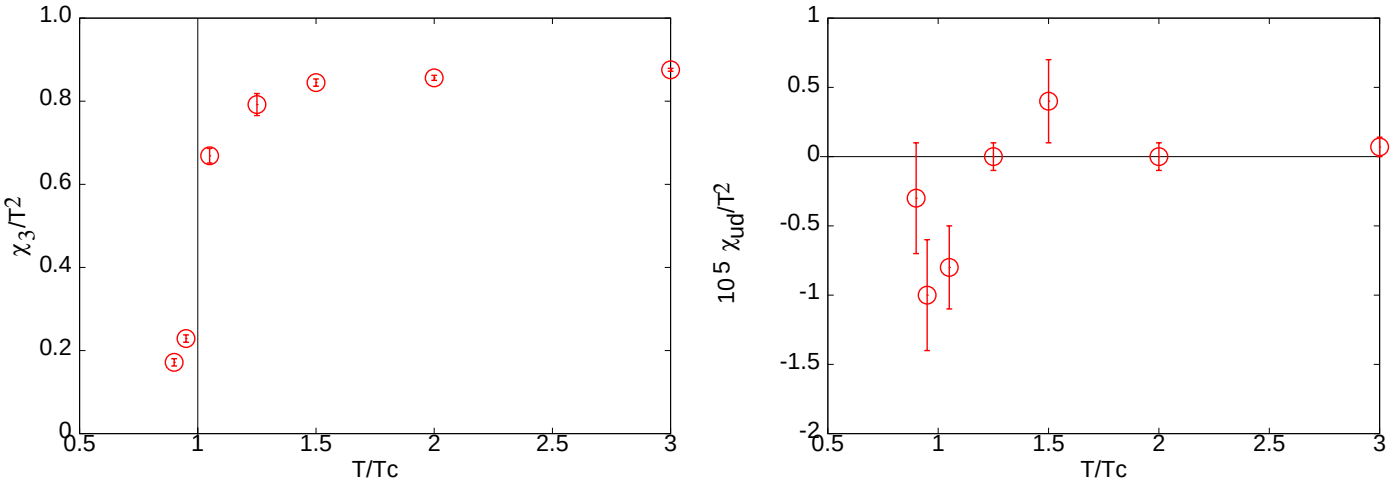
Perturbation theory



J.P. Blaizot, E. Iancu and A. Rebhan, *Phys. Lett.*, B 523 (2001) 143

A. Vuorinen, hep-ph/0212283

χ_{ud} and χ_{uu}



(Note the difference in scales!)

Event to event fluctuations

Each heavy-ion collision event, followed by the hadronisation, is one realisation of the whole ensemble of possible thermodynamic systems. Within a given rapidity region, the total amount of any conserved charge fluctuates from one event to another. The variance is determined by the response function of QCD matter in equilibrium.

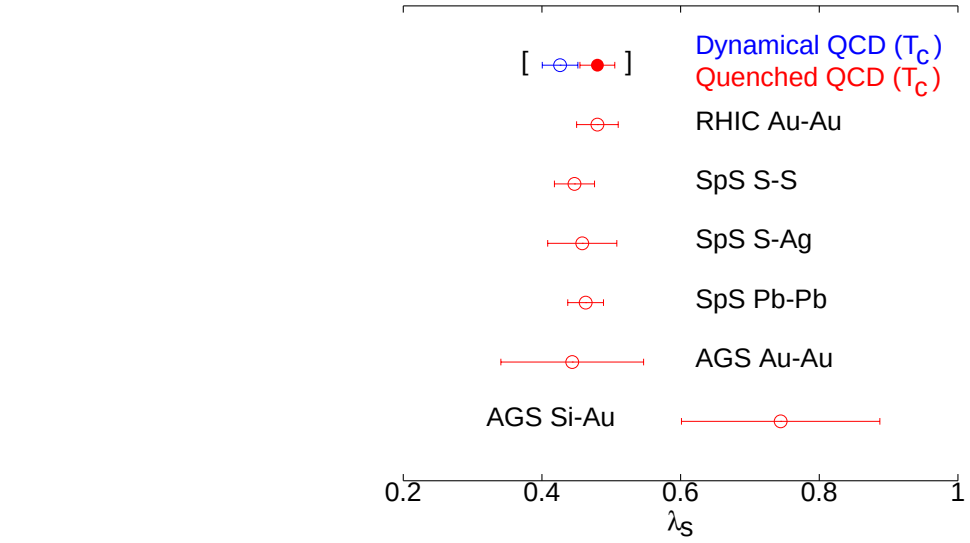
M. Asakawa *et al.*, *Phys. Rev. Lett.*, 85 (2000) 2072
S. Jeon *et al.*, *Phys. Rev. Lett.*, 85 (2000) 2076
D. Bower and S. Gavin, *Phys. Rev.*, C 64 (2001) 051902

From lattice computations it is seen that

$$\chi_B < \chi_Q < \chi_s \qquad (T > T_c)$$
$$\chi_B > \chi_Q > \chi_s \qquad (T < T_c)$$

R. V. Gavai, S. Gupta, P. Majumdar, *Phys. Rev.*, D 65 (2002) 054506

Strangeness production



$$\lambda_s = \frac{\langle n_s \rangle}{\langle n_u + n_d \rangle}$$

J. Cleymans, *J. Phys.*, G 28 (2002) 1575,

R. V. Gavai and S. Gupta, *Phys. Rev.*, D 65 (2002) 094515.

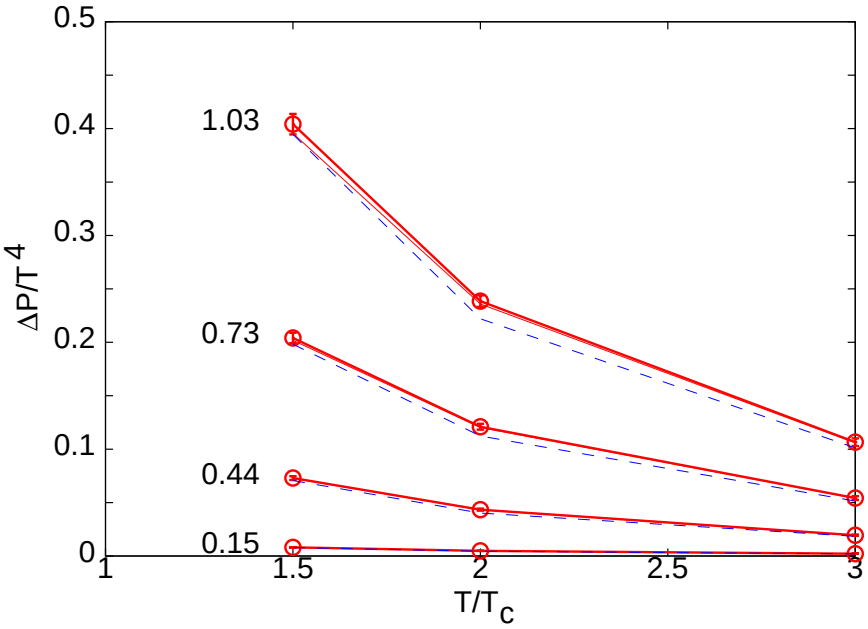
The pressure

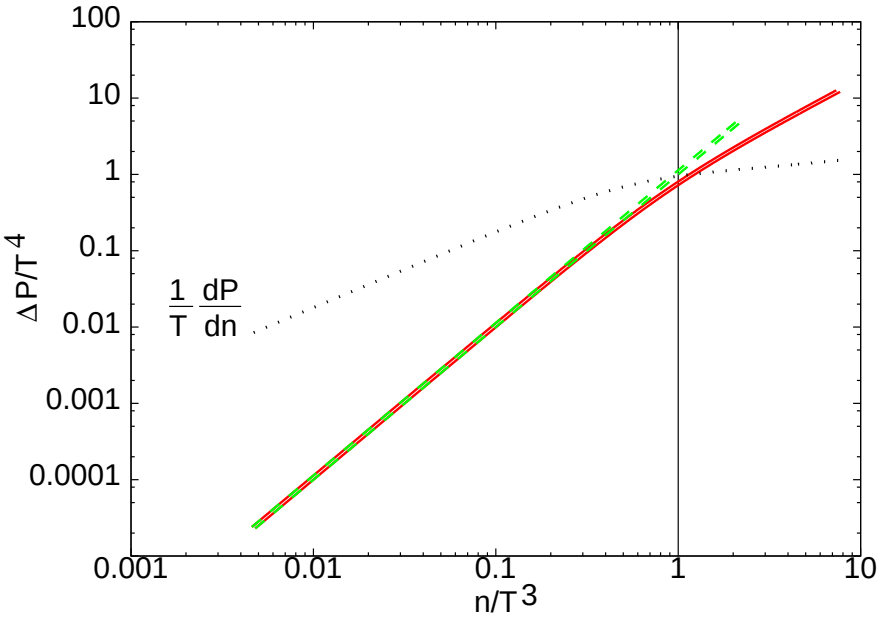
$$
\begin{aligned}
P(T,\mu) &= -F/V = P(T,0) + \chi_3(T)\mu^2 + \frac{1}{12}\chi_{uuuu}(T)\mu^4 + \mathcal{O}\left(\mu^6\right) \\
&= P(T,0) + \left(\frac{\mu}{\mu_*^{(2)}}\right)^2 \left[1 + \left(\frac{\mu}{\mu_*^{(4)}}\right)^2 \left\{1 + \left(\frac{\mu}{\mu_*^{(6)}}\right)^2 + \cdots\right\}\right].
\end{aligned}
$$

where $\mu_*^{(2)} = \sqrt{2/\chi_{uu}}$, $\mu_*^{(4)} = \sqrt{12\chi_{uu}/\chi_{uuuu}}$, $\mu_*^{(6)} = \sqrt{30\chi_{uuuu}/\chi_{uuuuuu}}$, *etc*.

Well-behaved for $\mu \ll \mu_*$. All results can be obtained in the continuum. Term by term improvement of the series is possible.

The equation of state





$$\Delta P(T) = P(T,\mu) - P(T,0)$$

R. V. Gavai and S. Gupta, *Phys. Rev.* D 68 (2003) 034506.

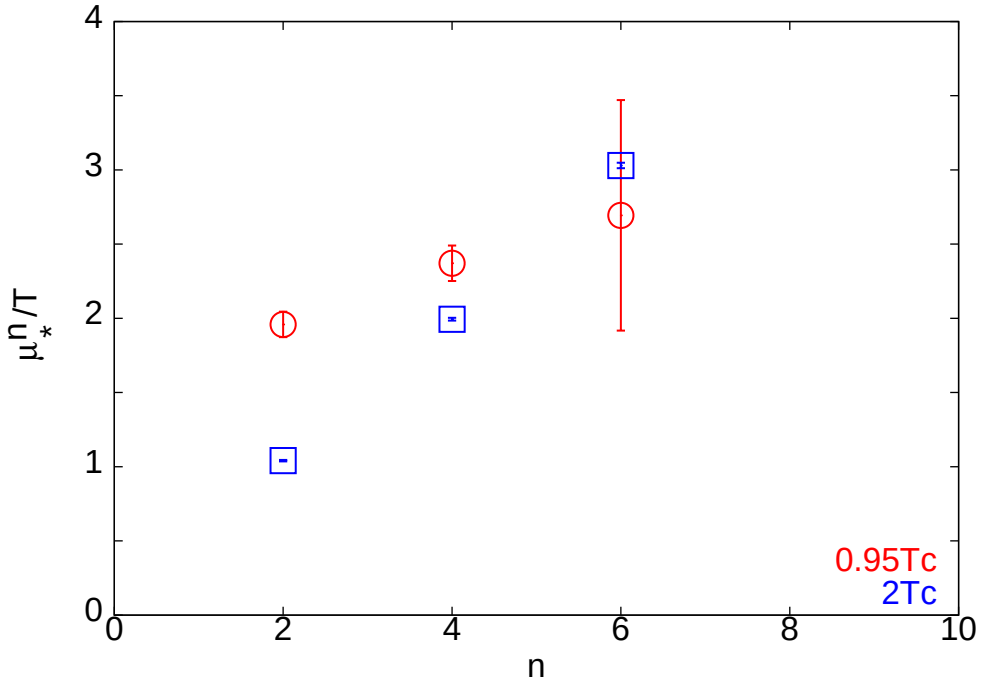
See also

Z. Fodor, S. D. Katz and K. K. Szabo, hep-lat/0208078,

C. R. Allton *et al.*, *Phys. Rev.*, D 68 (2003) 014507.

Radius of convergence: distance to phase transitions

The series expansion breaks down when a phase transition line is encountered. Use any estimate of the radius of convergence to obtain an estimate of the position of the phase transition line.



Qualitative difference between $T < T_c$ and $T > T_c$?

Condensates and masses

Taylor expansions can also be made for expectation values of any operator. We are investigating this for

1. Condensates: $\langle\overline{\psi}\psi\rangle$ changes quadratically with μ, and the quadratic coefficient is the same for isovector and baryon chemical potential. This number is also related to λ_s in strangeness production through a Maxwell relation.

2. Masses: The mass splitting of charged pions at finite isovector chemical potential is linear in μ, but that of the neutral pion is quadratic. This quadratic coefficient is the same as shift in pion mass at finite baryon chemical potential.

O. Miyamura *et al.*, *Phys. Rev.*, D 66 (2002) 077502,

S. Gupta, hep-lat/0202005, S. Gupta and Rajarshi Ray, in progress

Summary of Results

More than one method for computing physics at finite μ. One method (Taylor series expansion) is a precision technique, allowing contact with experiments.

- Computation of several high order susceptibilities may allow estimation of the critical end point by series extrapolation methods.
- Fluctuations and strangeness production rate in heavy-ion collisions are related to susceptibilities.
- Susceptibilities allow extension of the equation of state to finite chemical potential.
- Taylor expansions yield identities between behaviour of various quantities at finite isovector and baryon chemical potential.