# Lattice QCD with chemical potential


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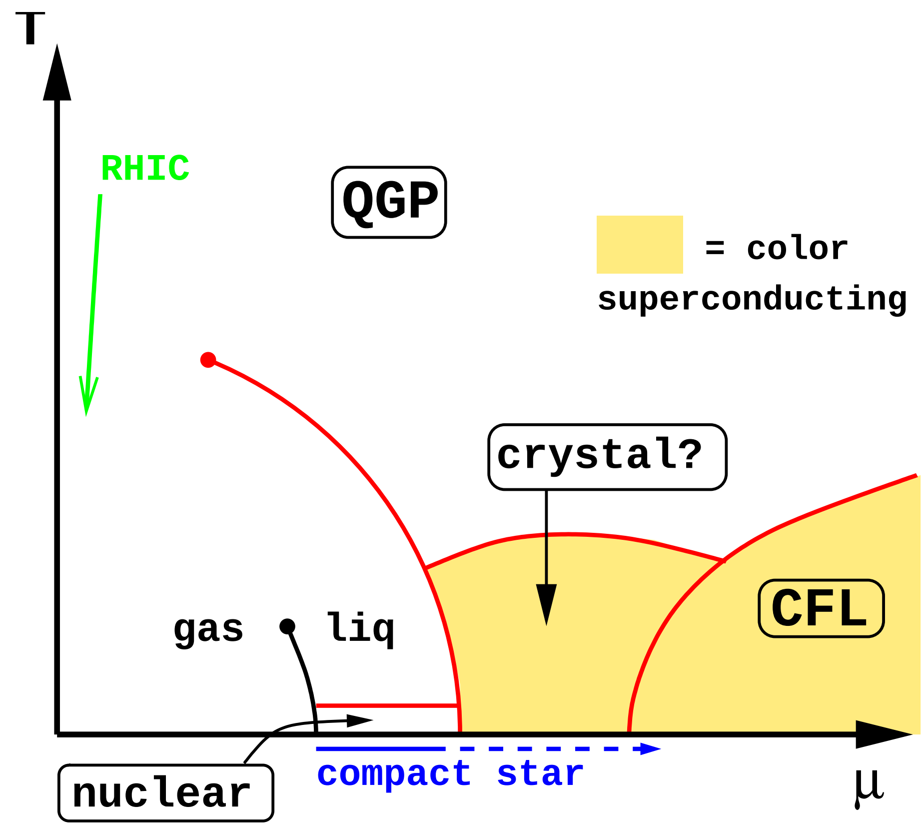
**January 4, 2004**

1. Why chemical potential? What's the problem?
2. A solution: Taylor series expansion.
3. Definitions and results: quark number susceptibilities.
4. Two special bits of phenomenology: fluctuations and strangeness
5. The equation of state: pressure at finite $\mu$
6. Breakdown of the expansion: phase transitions
7. Main results


With: Rajiv Gavai, Pushan Majumdar, Rajarshi Ray.

# Why chemical potential?




Flavour symmetry: one $\mu$ for every independent conserved charge.

M. G. Alford, K. Rajagopal, F. Wilczek, *Phys. Lett.*, B 422 (1998) 247,
R. Rapp, T. Schafer, E. V. Shuryak, M. Velkovsky, *Phys. Rev. Lett.*, 81 (1998) 53.

# What's the problem?

$$Z = \mathrm{e}^{-F/T} = \int DU\ \mathrm{e}^{-S}\ \prod_f \det M(U, m_f, \mu_f) = \int DU\ \mathrm{e}^{-\mathcal{S}(T,\mu)}$$

Dirac operator: $M = m + \partial_\mu \gamma_\mu$

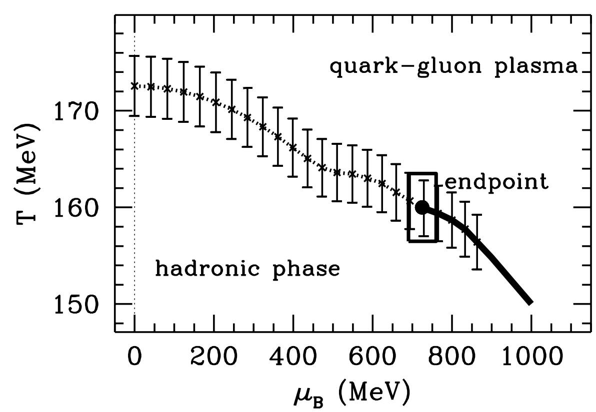
- If there is a $Q$ such that $M^\dagger = Q^\dagger M Q$, then clearly $\det M$ is real.
- $Q = \gamma_5$ for $\mu = 0$. Nothing for $\mu \neq 0$.
- Monte Carlo simulations of $Z$ fail.
- Under CP symmetry $\{U\} \to \{U'\}$ such that $\det M(U) = [\det M(U')]^*$.
- $Z$ remains real and non-negative— thermodynamics is safe.

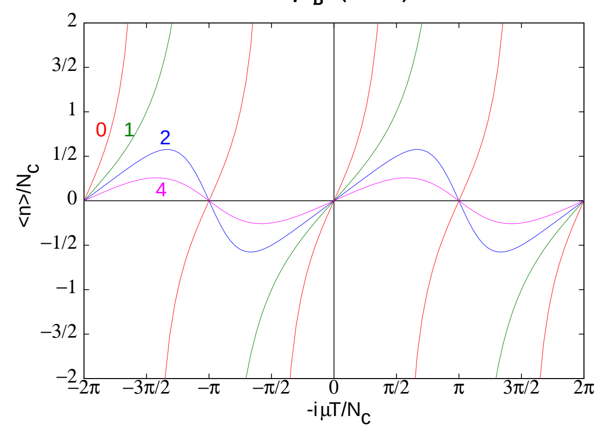
# Recent solutions

- Two parameter reweighting: Z. Fodor and S. D. Katz, *J. H. E. P.*, 03 (2002) 014.
  Express the reweighting in terms of derivatives of Fermion determinant with respect to $\mu$: C. R. Allton *et al.*, *Phys. Rev.*, D 66 (2002) 074507

- Simulate imaginary $\mu$ (positive $\det M$) and do analytic continuation: M. D'Elia and M.-P. Lombardo, hep-lat/0209146, P. De Forcrand and O. Philipsen, *Nucl. Phys.*, B642 (2002) 290
  Special care needed; find Yang-Lee zeroes directly: S, Gupta, hep-lat/0307007.

- Perform a Taylor series expansion of the free energy: R. V. Gavai and S. Gupta, *Phys. Rev.* D 68 (2003) 034506.



# The Taylor Expansion

Since $PV = -F = T \log Z$, the Taylor expansion of $P$ is the same as of $F$!

$$\frac{1}{V} P(T, \mu_u, \mu_d) = \frac{1}{V} P(T, 0, 0) + \sum_f n_f \mu_f + \frac{1}{2!} \sum_{fg} \chi_{fg} \mu_f \mu_g + \cdots$$

where the quark number densities and susceptibilities are—

$$n_f = \frac{T}{V} \frac{\partial \log Z}{\partial \mu_f} \Big|_{\mu_f = 0}$$

$$\chi_{fg} = \frac{T}{V} \frac{\partial^2 \log Z}{\partial \mu_f \partial \mu_g} \Big|_{\mu_f = \mu_g = 0}$$

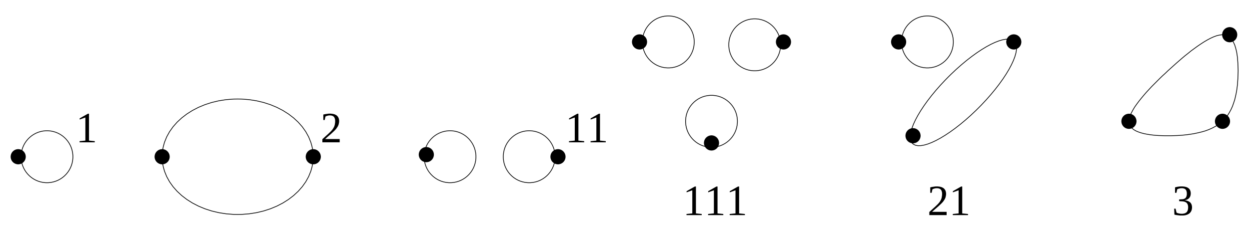
$$\chi_{fgh\cdots} = \frac{T}{V} \frac{\partial^n \log Z}{\partial \mu_f \partial \mu_g \partial \mu_h \cdots} \Big|_{\mu_f = \mu_g = \cdots = 0}$$

# Derivatives

Derivatives of $\log Z$ can be expressed in terms of derivatives of $Z$. The latter can be constructed by the chain rule.

$$Z_f = \frac{\partial Z}{\partial \mu_f} = \int DU \mathrm{e}^{-\mathcal{S}} \,\mathrm{Tr}\, M_f^{-1} M_f'.$$

Note: $M' = \gamma_0$ and $M^{-1} = \psi\overline{\psi}$, so $\mathrm{Tr}\, M^{-1}M' = \psi^\dagger\psi$. Odd derivatives vanish for $\mu_f = 0$ by CP symmetry. S. Gottlieb *et al.*, *Phys. Rev. Lett.*, 59 (1987) 2247




S. Gupta, *Acta Phys. Pol.*, B 33 (2002) 4259

# Why Taylor series expansions?

- Since the reweighting factor, $\exp[\Delta\mathcal{S}]$, is extensive, taking the continuum and/or thermodynamic limits, while keeping the relative error fixed, is an exponentially difficult problem.

- All reweighting results are potentially full of lattice artifacts. The continuum Dirac operator specifies effects of an infinitesimal time translation. On the lattice we deal with finite translations (by lattice spacing $a$). This gives a lattice ambiguity. Reweighting gives no indication of how large the lattice artifacts are.

- Taylor series expansion is prescription dependent beyond 2nd order at every finite lattice spacing $a$, but prescription independent for $a \to 0$. With explicit Taylor expansion one can take the continuum limit.

- With finite statistics, reweighting is dominated by the leading part of the Taylor series expansion. At present no analysis of the statistical errors in the determination of the critical end-point determined by reweighting are available.

# The (linear) quark number susceptibilities

For $N_f = 2$, the linear susceptibilities form the matrix in flavour space which can be diagonalised by rotating to the space of isoscalar ($\mu_0 = \mu_u + \mu_d$) and isovector ($\mu_3 = \mu_u - \mu_d$) chemical potentials—

$$\begin{pmatrix} \chi_{uu} & \chi_{ud} \\ \chi_{ud} & \chi_{uu} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \chi_{uu} + \chi_{ud} & 0 \\ 0 & \chi_{uu} - \chi_{ud} \end{pmatrix}$$

No matter what the representation, there are only two independent QNS (for degenerate masses) which we choose to be

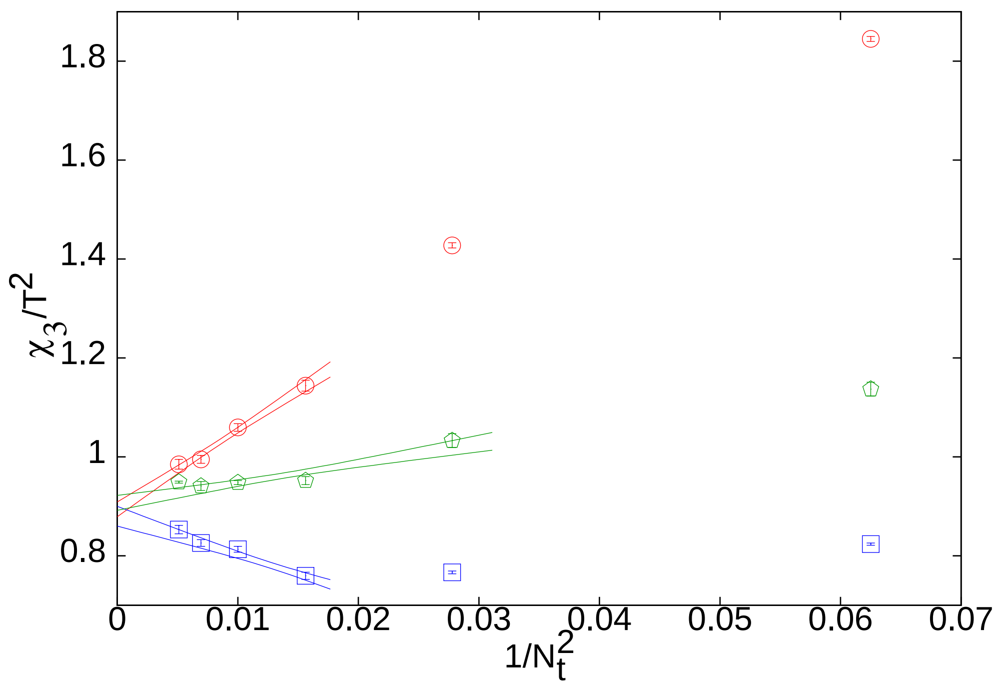
$$\chi_3 = \chi_{uu} - \chi_{ud} = \left\langle \mathrm{Tr}\ M^{-1}M'M^{-1}M' - \mathrm{Tr}\ M^{-1}M'' \right\rangle$$

$$\chi_{ud} = \left\langle \left(\mathrm{Tr}\ M^{-1}M'\right)^2 \right\rangle \quad \text{and} \quad \chi_0 = \chi_3 + 2\chi_{ud}$$

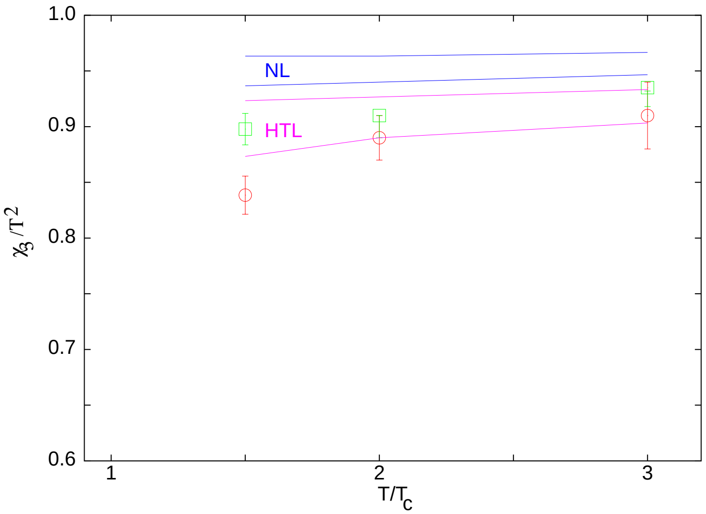
# Finding the continuum limit

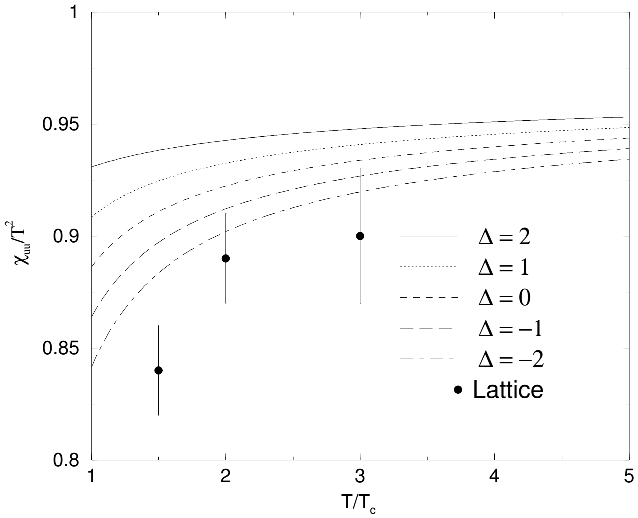
Main technical problem is to control the extrapolation to zero lattice spacing. For this we use two different kinds of Fermions (staggered and Naik) and perform simultaneous extrapolation with both: in the quenched theory.

R. V. Gavai and S. Gupta, *Phys. Rev.* D 67 (2003) 034501 

# Perturbation theory

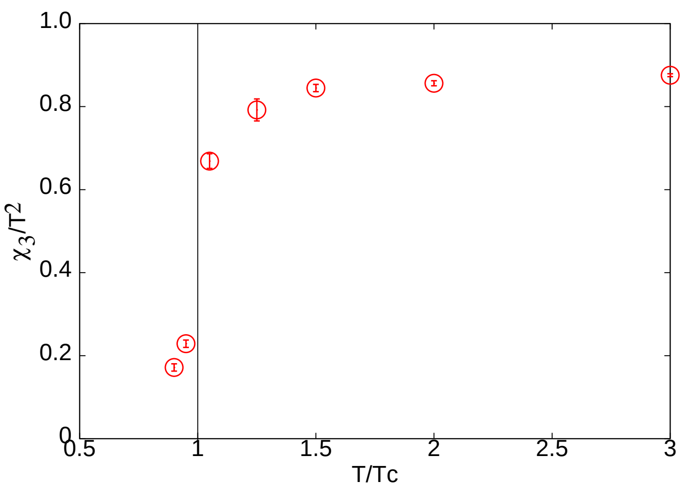


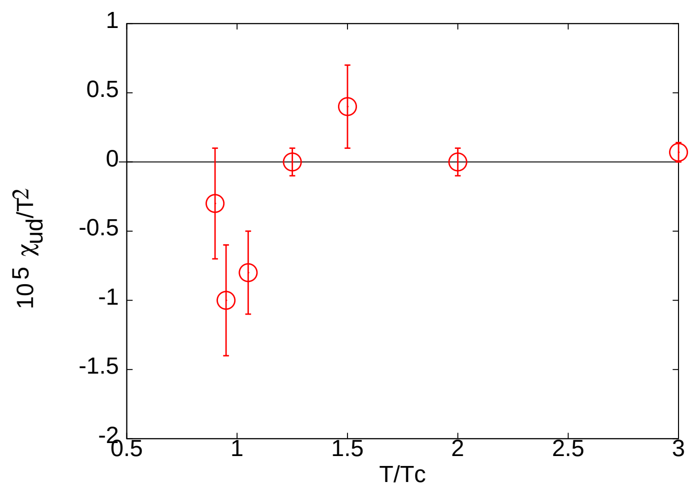




J.P. Blaizot, E. Iancu and A. Rebhan, *Phys. Lett.*, B 523 (2001) 143

A. Vuorinen, *Phys. Rev.*, D 67 (2003) 074032

# $\chi_{ud}$ and $\chi_{uu}$




- Note five orders of magnitude between the two QNS.
- Perturbation theory cannot reach $T < 2T_c$.
- Perturbative result for $\chi_{ud}$ two orders of magnitude too large.

# Quark number susceptibilities: phenomenology

- Fluctuations of conserved quantities in heavy-ion collisions are related to $\chi_{uu}$. Isospin fluctuations are related to $\chi_3 = \chi_{uu} - \chi_{ud}$. Charge fluctuations can also be constructed out of these. M. Asakawa *et al.*, *Phys. Rev. Lett.*, 85 (2000) 2072; S. Jeon and V. Koch, *ibid.*, 85 (2000) 2076

- Under certain conditions strangeness production rates can be related to the strange susceptibility, $\chi_{ss}$. R. V. Gavai *et al.*, *Phys. Rev.*, D 65 (2002) 054506

- The pressure at finite chemical potential is essentially determined by the susceptibility. This is used in hydrodynamic models to obtain single particle inclusive spectra as well as multi-particle correlations. R. V. Gavai and S. Gupta, *Phys. Rev.* D 68 (2003) 034506.

- $\chi_3$ is the zero momentum Euclidean finite temperature longitudinal vector propagator and hence is needed to connect the soft photon production rate to a transport coefficient— the DC electrical conductivity of quark matter. S. Gupta, hep-lat/0301006.

# Event to event fluctuations

Each heavy-ion collision event, followed by the hadronisation, is one realisation of the whole ensemble of possible thermodynamic systems. Within a given rapidity region, the total amount of any conserved charge fluctuates from one event to another. The variance is determined by the response function of QCD matter in equilibrium.

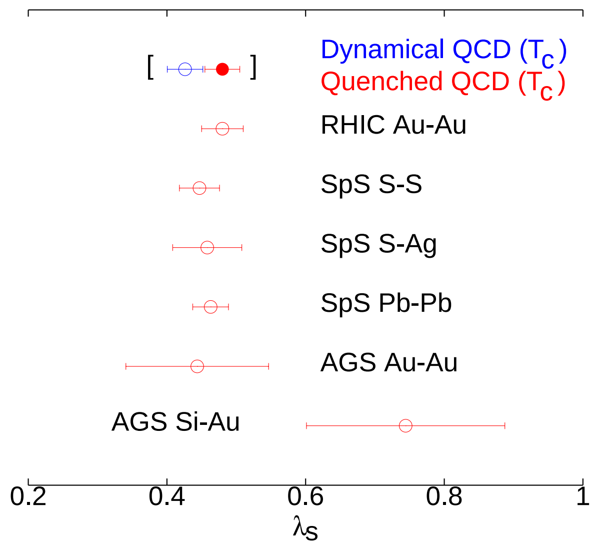
M. Asakawa *et al.*, *Phys. Rev. Lett.*, 85 (2000) 2072
S. Jeon *et al.*, *Phys. Rev. Lett.*, 85 (2000) 2076
D. Bower and S. Gavin, *Phys. Rev.*, C 64 (2001) 051902

From lattice computations it is seen that

$$\chi_B < \chi_Q < \chi_s \qquad (T > T_c)$$
$$\chi_B > \chi_Q > \chi_s \qquad (T < T_c)$$

R. V. Gavai, S. Gupta, P. Majumdar, *Phys. Rev.*, D 65 (2002) 054506

# Strangeness production: quenched continuum

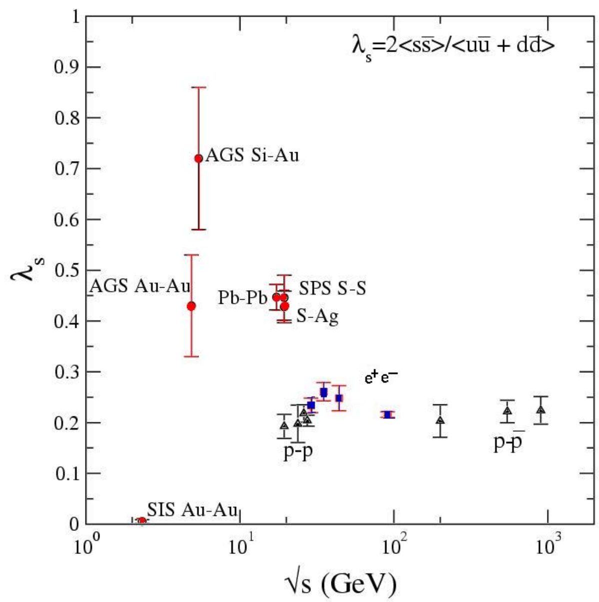


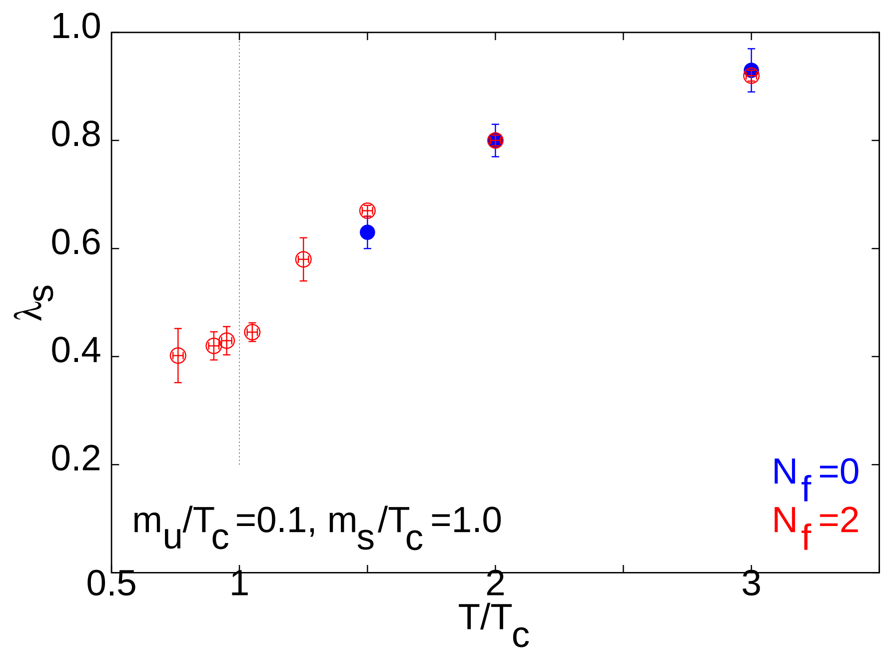

$$\lambda_s = \frac{\langle n_s \rangle}{\langle n_u + n_d \rangle} = \frac{\chi_{ss}}{\chi_{uu}}$$

J. Cleymans, *J. Phys.*, G 28 (2002) 1575,

R. V. Gavai and S. Gupta, *Phys. Rev.*, D 65 (2002) 094515.

# Strangeness production: dynamical quarks




Still need to fine tune bare strange quark mass and take the continuum and thermodynamic limits. R. V. Gavai and S. Gupta, in preparation.

Dependence on bare light quark mass is weak. R. Ray and S. Gupta, in preparation.

# The pressure

$$\Delta P(T,\mu) \equiv P(T,\mu) - P(T,0) = \chi_{uu}(T)\mu^2 + \frac{1}{12}\chi_{uuuu}(T)\mu^4 + \mathcal{O}\left(\mu^6\right)$$

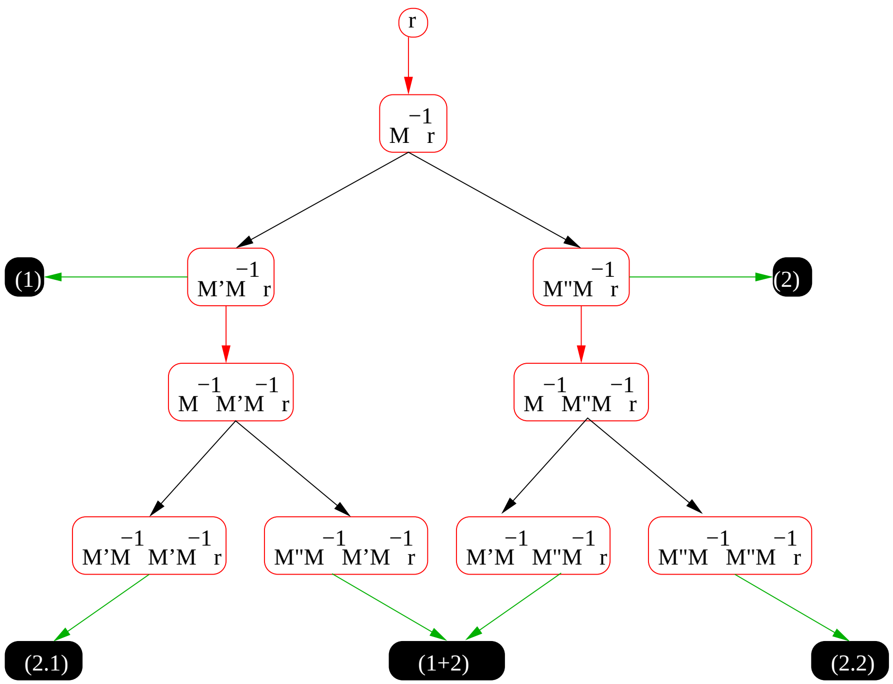
$$= \chi_{uu}\mu^2 \left[1 + \left(\frac{\mu}{\mu_*^{(2)}}\right)^2 \left\{1 + \left(\frac{\mu}{\mu_*^{(4)}}\right)^2 \left(1 + \cdots\right)\right\}\right].$$

where $\mu_*^{(2)} = \sqrt{\frac{12\chi_{uu}}{\chi_{uuuu}}}$ $\mu_*^{(4)} = \sqrt{\frac{30\chi_{uuuu}}{\chi_{uuuuuu}}}$ *etc.*

Well-behaved for $\mu \ll \mu_*$. All results can be obtained in the continuum. Term by term improvement of the series is possible. However, need to compute many terms if high precision is needed.

# Algorithms

Since $\mathbf{I} = \overline{|r\rangle\,\langle r|}$ where $|r\rangle$ is a unit Gaussian complex vector, be easily shown that $\mathrm{Tr}\ A = \overline{\langle r|A|r\rangle}$. Optimisation of the computation of multiple traces reduces to a problem called the Steiner problem. Need 20 matrix inversions to perform a single measurement of upto 8th order susceptibilities.




M. Charikar *et al.*, STAN-CS-TN-97-56.

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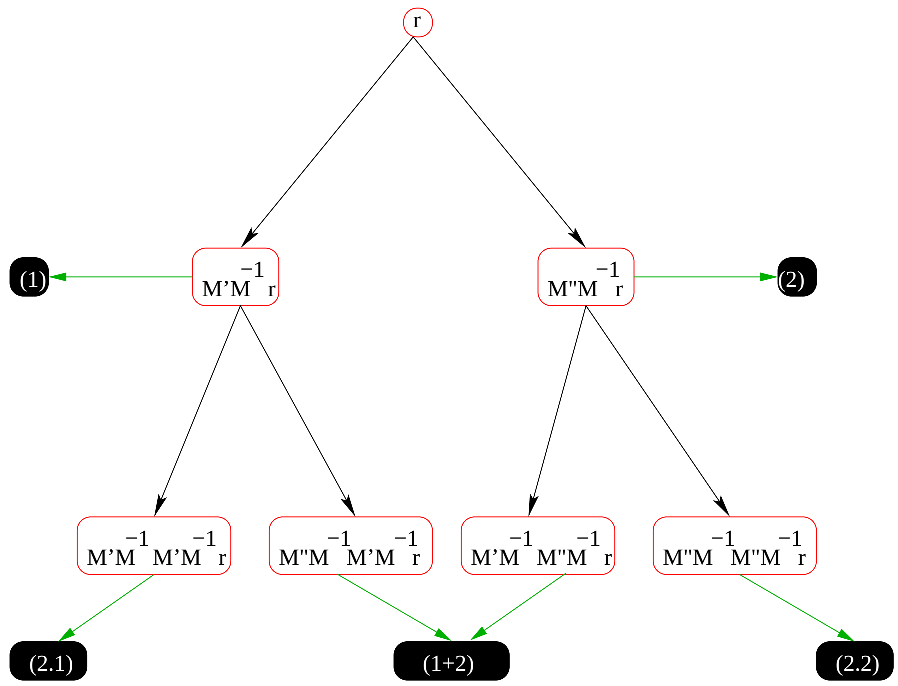


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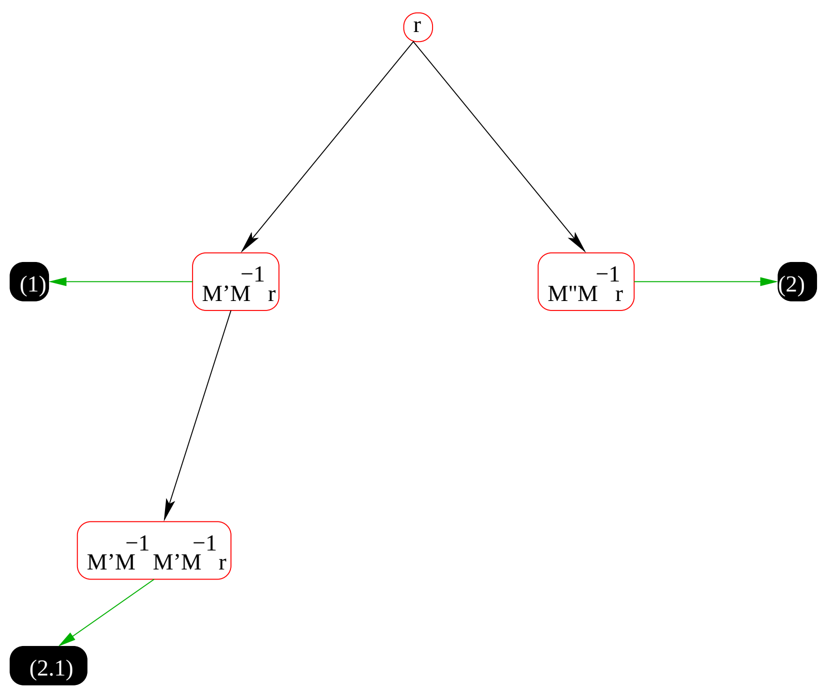


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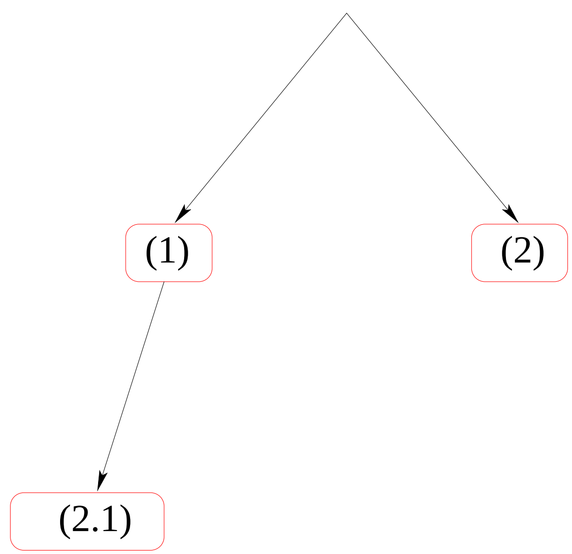
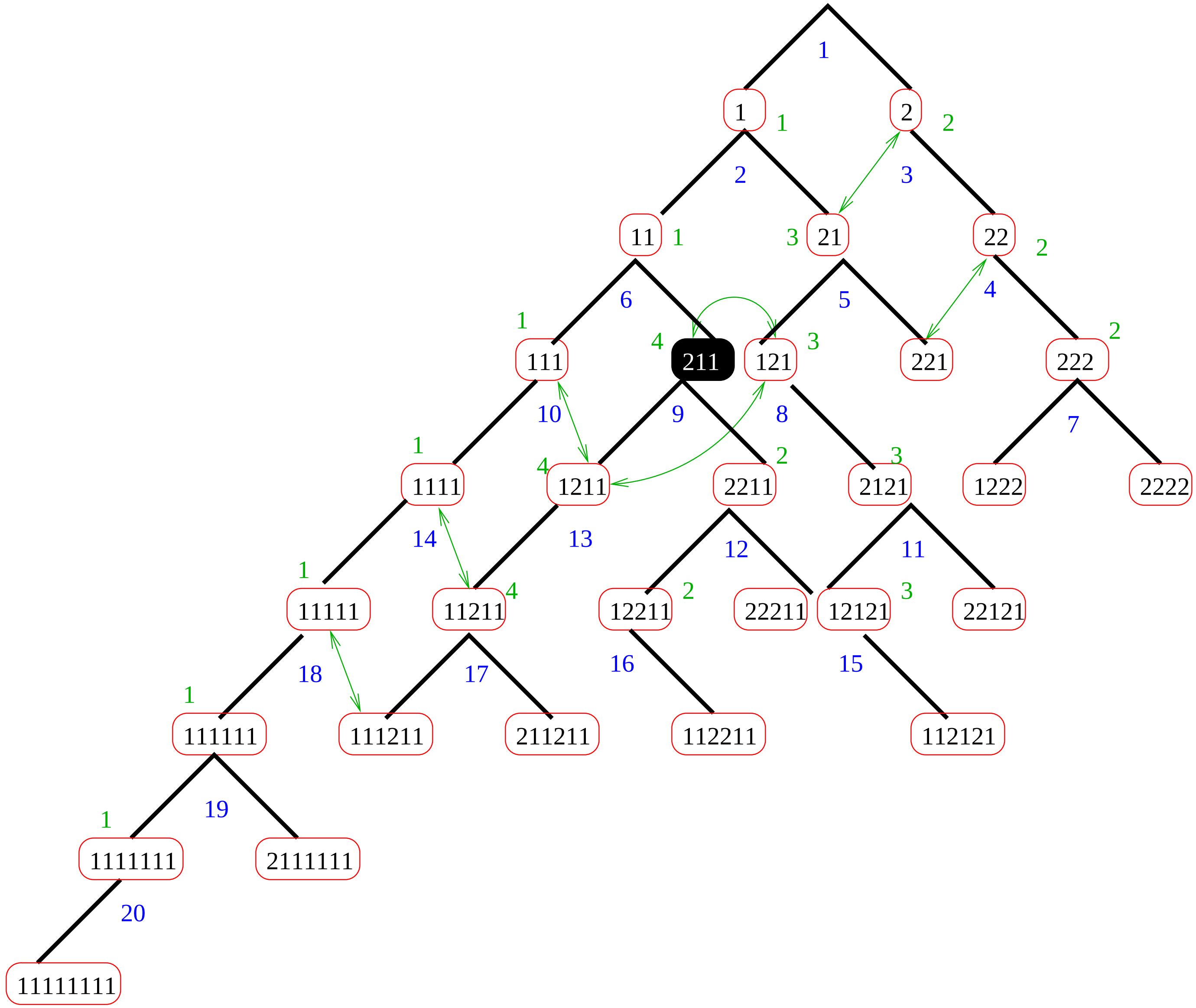
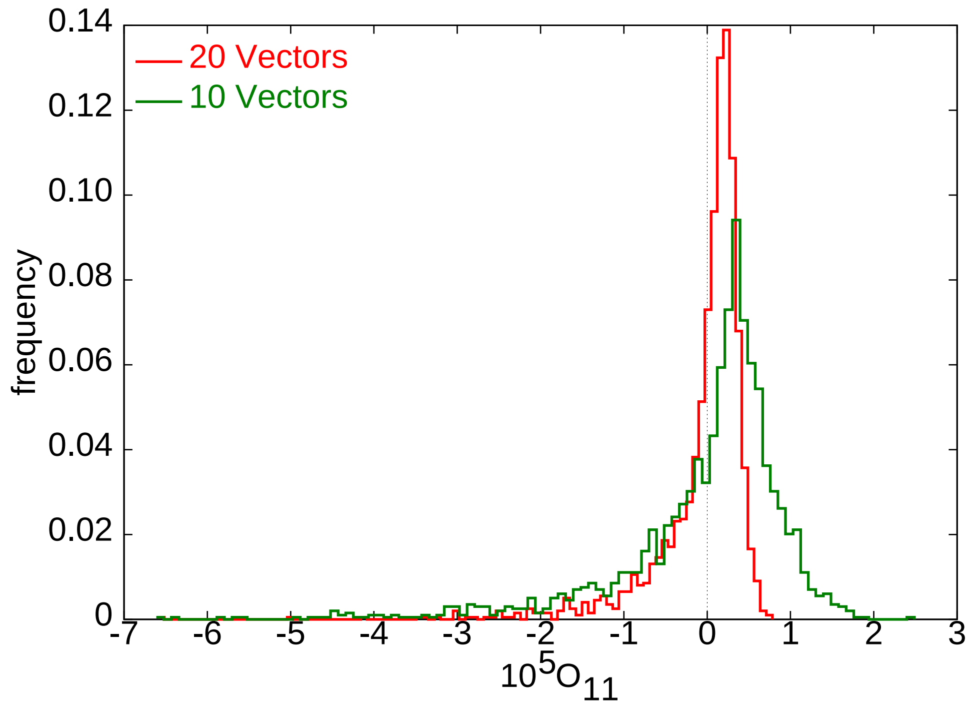



M. Charikar *et al.*, STAN-CS-TN-97-56.

# The actual evaluation tree

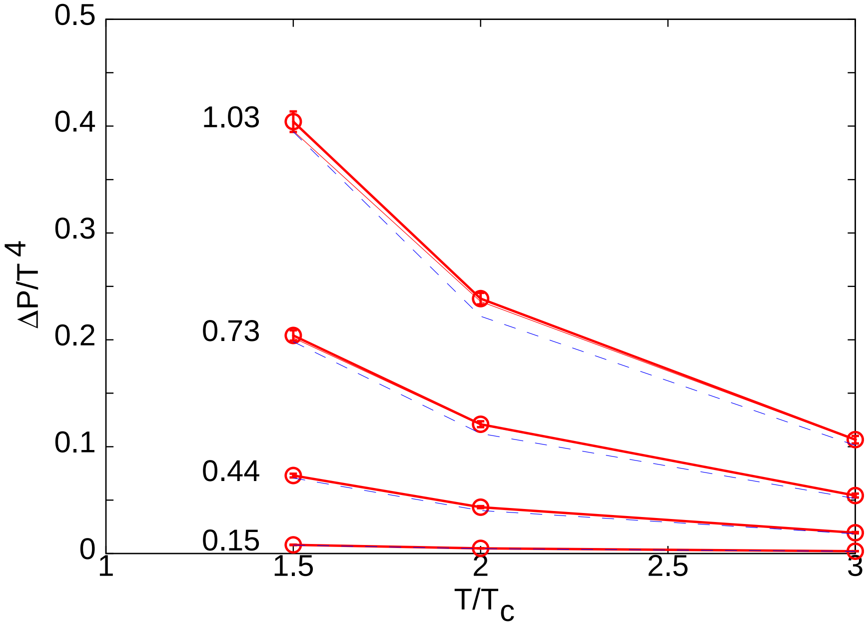


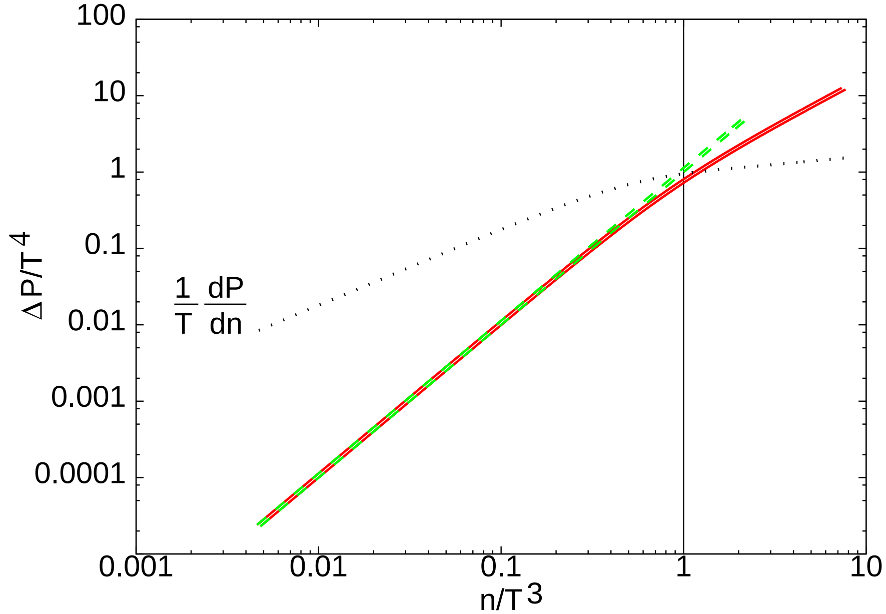
# How many vectors?




The histogram of $O_{11}$ where $\chi_{ud} = \langle O_{11} \rangle$. In the limit of infinite number of vectors the histogram should be skew, a tail to the left and vanishing abruptly at zero. We use 100 vectors to get close to this situation.

# The equation of state




$$\Delta P(T) = P(T, \mu) - P(T, 0)$$

R. V. Gavai and S. Gupta, *Phys. Rev.* D 68 (2003) 034506 and hep-lat/0309014.

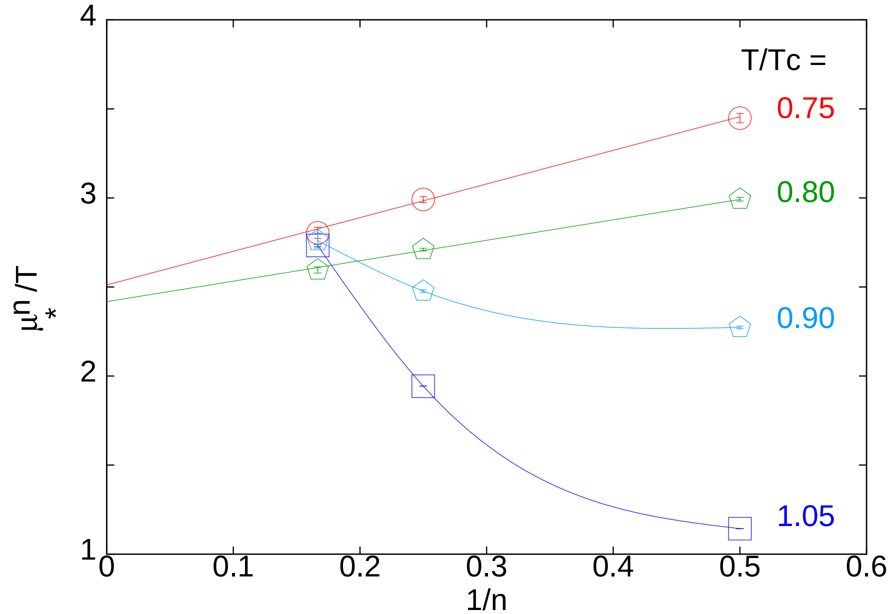
See also

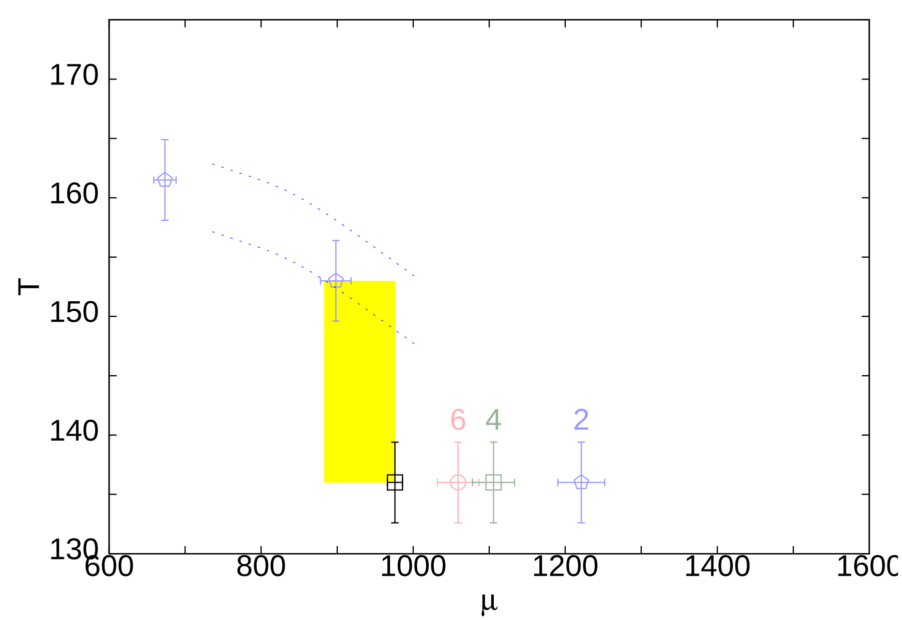
Z. Fodor, S. D. Katz and K. K. Szabo, hep-lat/0208078,

C. R. Allton *et al.*, *Phys. Rev.*, D 68 (2003) 014507.

# Radius of convergence: distance to phase transitions

The series expansion breaks down when a phase transition line is encountered. Use estimates of the radius of convergence, $\mu_*^{(n)}$, to obtain an estimate of the position of the phase transition line.




Qualitative change in the range $0.8 < T/T_c < 0.9$— currently narrowing the range further.

More than one method for computing physics at finite $\mu$. One of these (Taylor series expansion) is a precision technique, allowing contact with experiments.

- Computation of several high order susceptibilities gives estimate of the critical end point by series extrapolation methods.

- Fluctuations and strangeness production rate in heavy-ion collisions are related to susceptibilities. Temperature dependance of $\lambda_s$ is a direct prediction from the lattice.

- Susceptibilities allow extension of the equation of state to finite chemical potential. This enters into any hydrodynamical description of the heavy-ion collision and thus tests particle spectra, HBT radii, elliptic flow *etc*.