# Fluctuations in QCD

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Fluctuations, seeing quarks, strangeness, higher order fluctuations; critical end point and the wing critical line.

# Plan

1. Fluctuations of conserved quantities: QCD results for susceptibilities and the specific heat.

2. Linkage: using multiple quantum numbers together to figure out the nature of excitations.

3. Some results on strange quarks

4. Higher order fluctuations: non-poissonian behaviour is not fishy.

5. The critical end point and the wing critical line in the QCD phase diagram

# Theoretical overview of fluctuations

In thermal equilibrium (grand canonical ensemble) a microstate of energy $E$ and conserved quantum numbers $Q_i$ are found with probability $\exp(-E/T - \sum_i \mu_i Q_i)$. Fluctuations of $E$ and $Q_i$ are Gaussian except at a critical point.

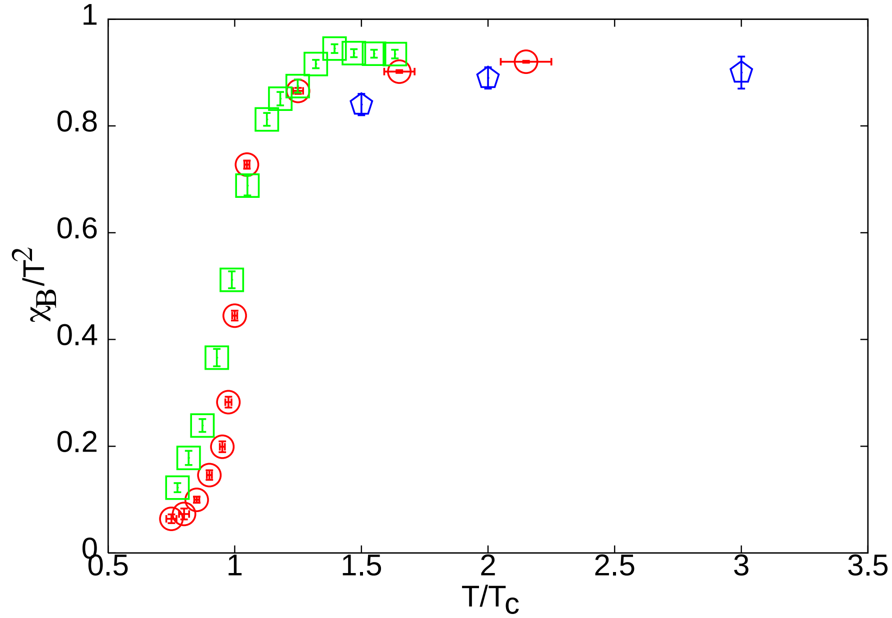
The fluctuations can be computed by using the second derivative of the partition function

$$Z(T, \mu_i) = \sum_{states} \exp(-E/T - \sum_i \mu_i Q_i).$$

In QCD we compute the specific heat or susceptibilities of baryon number, electric charge or hypercharge (alternatively, strangeness).

$$\chi_{Q_i} = \frac{\partial^2 \log Z}{\partial \mu_i^2}$$

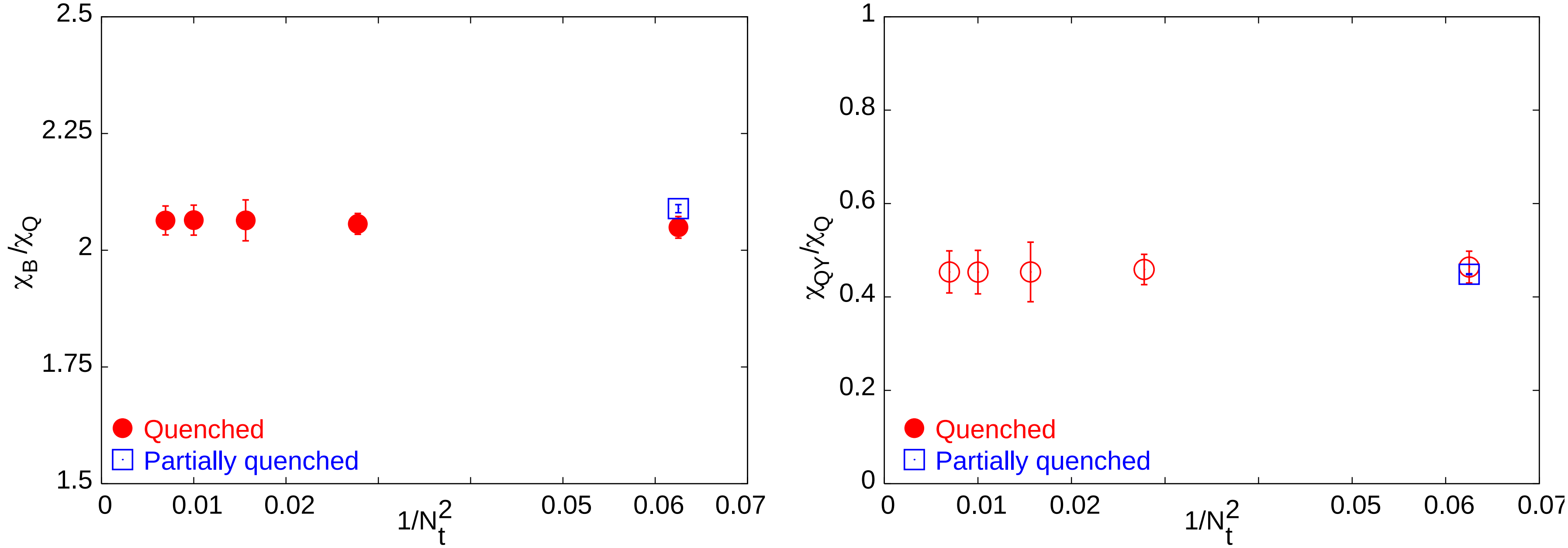
# Fluctuations




Quenched continuum (Mumbai), MILC $a = 1/8T$, $N_f = 2$, Mumbai $N_f = 2$ continuum extrapolation through quenched

Weak coupling expansion: Blaizot, Iancu and Rebhan, Vuorinen

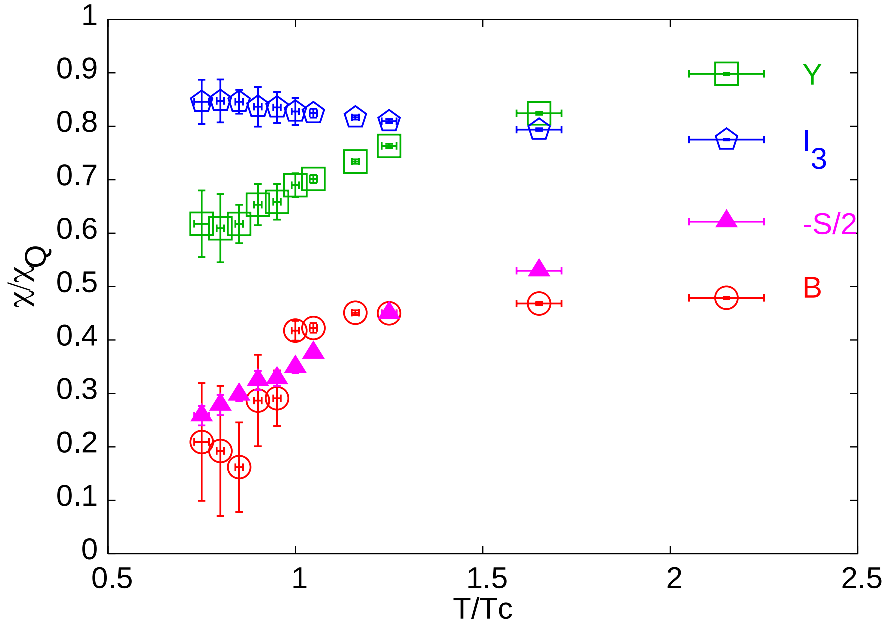
# Ratios are robust




Above $T_c$ ratios of QNS are almost independent of lattice spacing, and insensitive to quark masses (as long as $m < T$).

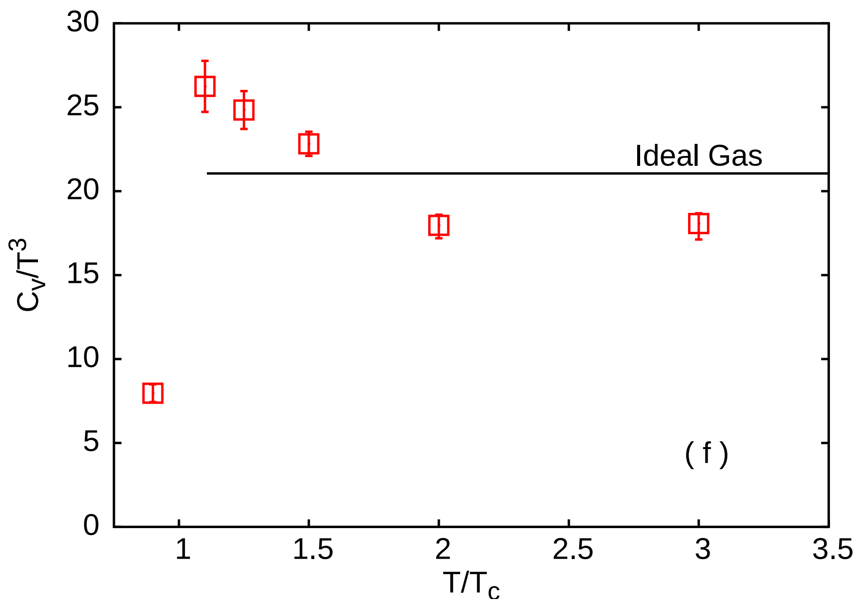
Gavai and SG, PRD 73 (2006) 014004

# Other fluctuations




Hierarchy of fluctuations: $S$, $Q$, $Y$, $B$

# Energy fluctuations




Departure from ideal-gas behaviour even at the highest temperature.
Gavai, SG, Mukherjee, Phys.Rev.D71:074013,2005

# Quasiparticles: linkage of quantum numbers

Identify a particle by a complete set of quantum numbers. When there are many conserved quantum numbers the problem is simple. Look at two quantum numbers simultaneously— say $U$ and $D$.

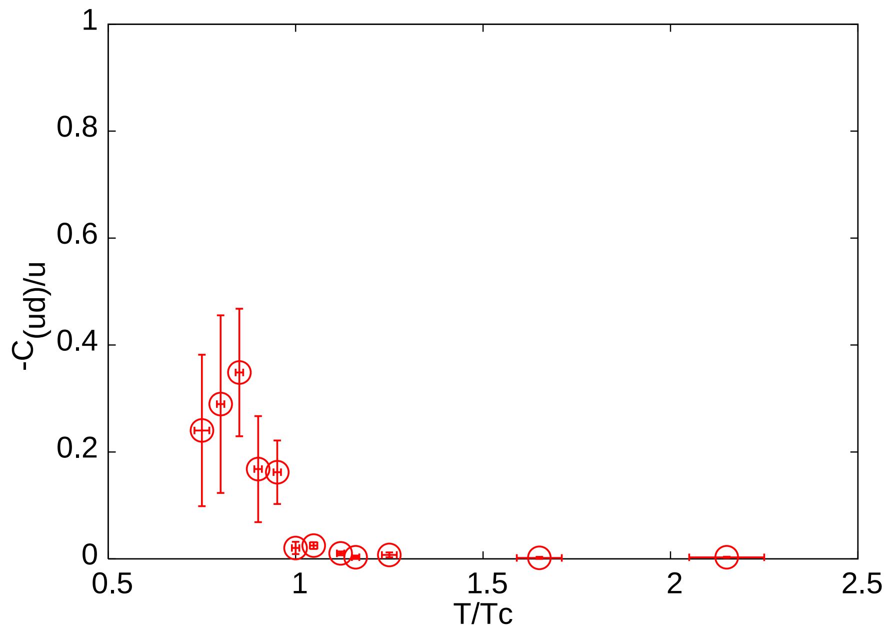
$T = 0$: whenever $U = 1$ is excited $D = -1$ is excited along with it.

$T > T_c$: when $U = 1$ is excited $D = \pm 1$ should be excited along with it if the medium contains quarks. Otherwise, by observing what value of $D$ is preferentially excited, you find something about the quantum numbers of the excitations.

Similarly one could study the linkages $U|B$ or $U|Q$, or $D|B$ etc.

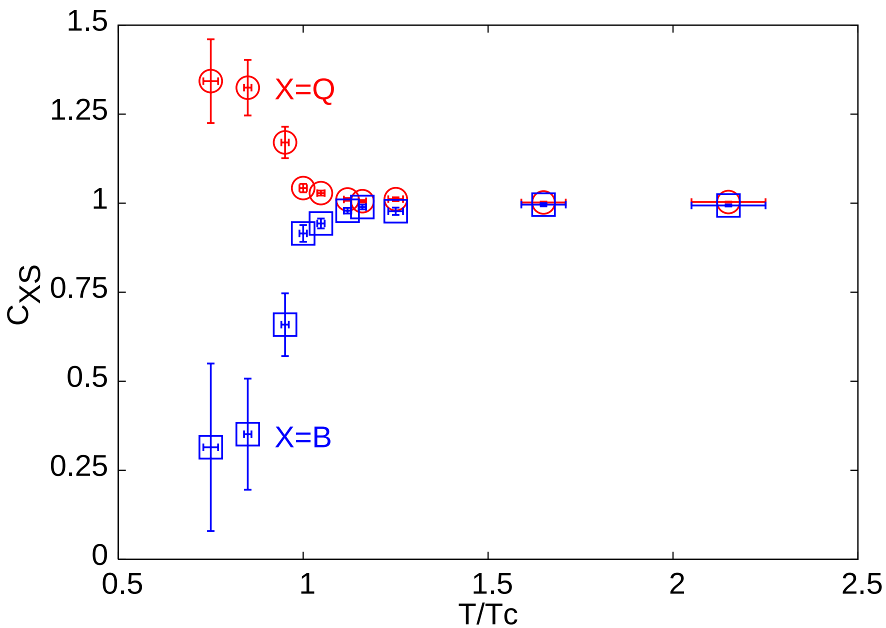
$$C_{(XY)/Y} \equiv \frac{\langle XY \rangle - \langle X \rangle \langle Y \rangle}{\langle Y^2 \rangle - \langle Y \rangle^2} = \frac{\chi_{XY}}{\chi_Y}$$

# U and D are not linked




$u$ and $\bar{d}$ can be carried by the same particle below $T_c$ but not above $T_c$.

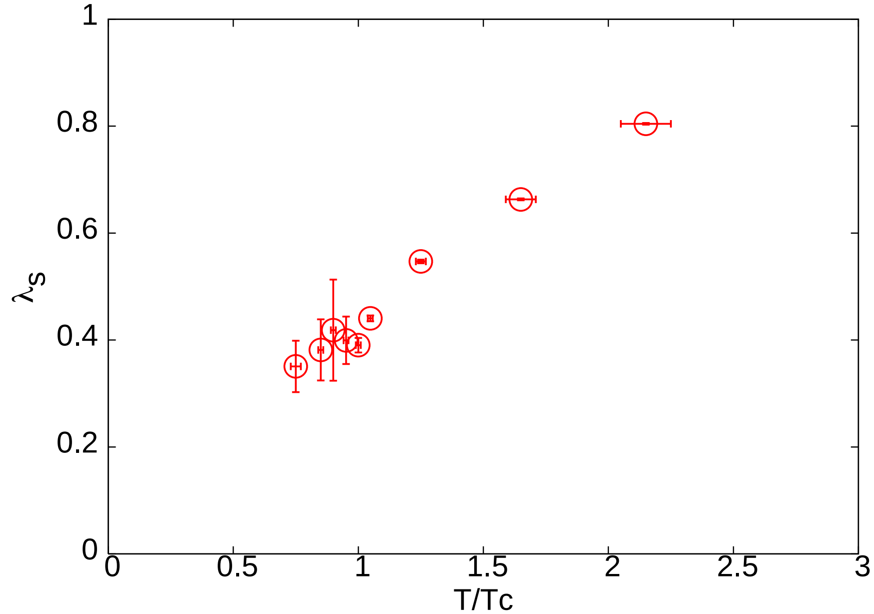
# Strangeness is carried by quarks




$C_{BS} = -3C_{(BS)|S}$ and $C_{QS} = 3C_{(QS)|S}$. Below $T_c$ strange baryons are relatively heavy and therefore sparse in the plasma, but kaons are not so heavy. Above $T_c$: strange quarks.

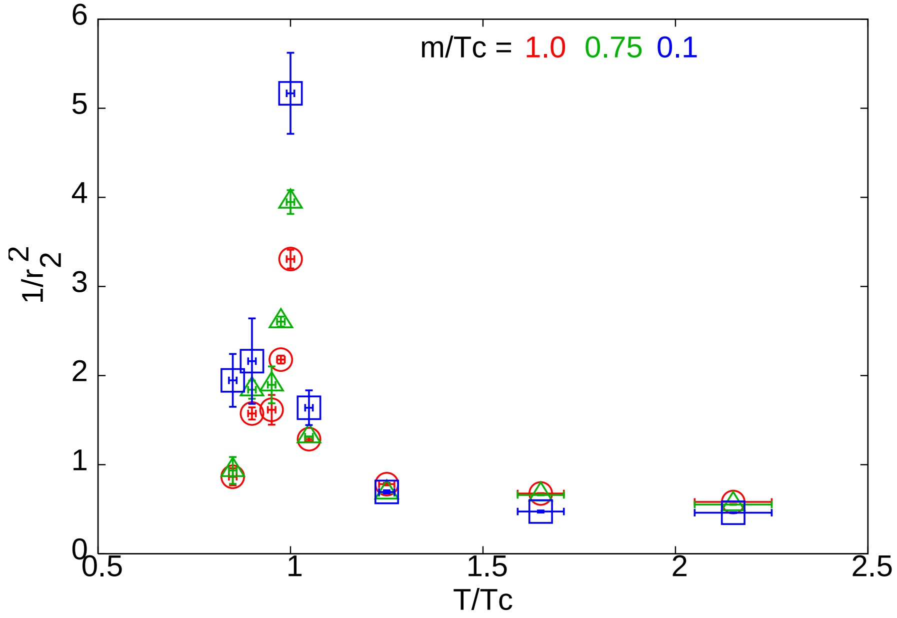
Koch, Majumder, Randrup, PRL 95 (2005) 182301, Gavai and SG, PRD 73 (2006) 014004

# Strange quark abundance




Strange quark abundance $\lambda_s = \chi_s/\chi_u$ determined for realistic light and strange quark masses. The ratio is robust.
Gavai and SG, PRD 73 (2006) 014004

# Higher order fluctuations




Strong dependence on quark masses: peak near $T_c$ not accessible to weak coupling theory (Vuorinen).

Gavai and SG, Phys.Rev.D72:054006,2005; D68:034506,2003
Allton etal, Phys.Rev.D71:054508,2005

## Global symmetries determine the phase diagram

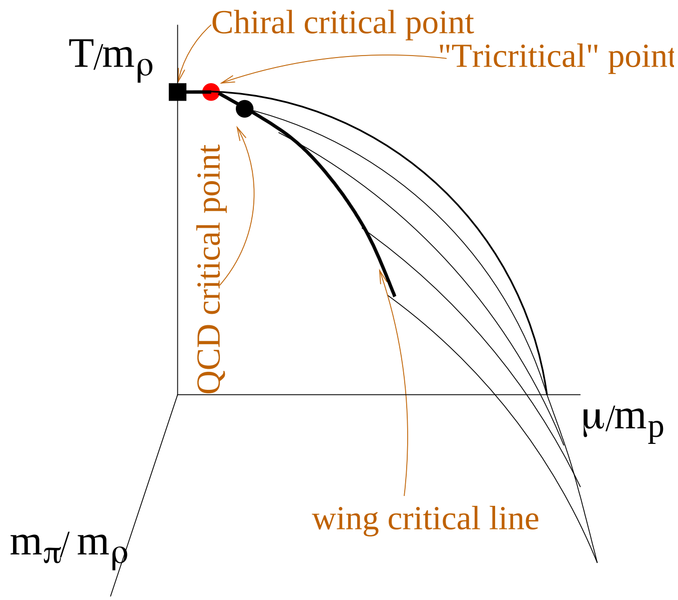
Two flavours of light quarks: approximate $SU(2) \times SU(2)$ chiral symmetry, in the limit broken spontaneously to diagonal $SU(2)$, (pseudo) Goldstone bosons are the (light pseudo-scalar) pions.

Five tunable parameters: $T$ (temperature), $\mu_u$ and $\mu_d$ (two chemical potentials), $m_u$ and $m_d$ (two masses). Gibbs phase rule allows large order multi-critical points.

Order parameter for chiral symmetry restoration: $\langle\overline{\psi}\psi\rangle$, tuned by changing $T$ and $\mu = (\mu_u + \mu_d)/2$, excitations in this "radial" direction are heavy scalar mesons.

Order parameter for pion condensation: $\langle\overline{\psi}\gamma_5\tau_2\psi\rangle$, non-zero value may be induced by tuning isospin chemical potential $\mu_3 = (\mu_u - \mu_d)/2$, excitations in this direction give a massless charged pion.

# The phase diagram




Berges and Rajagopal
Halasz, Jackson, Schrock, Stephanov and Verbaarschot
1998

Section of the 5-d phase diagram along a surface of $\mu_3 = 0$ and $m_u = m_d$: phases distinguished by $\langle\overline{\psi}\psi\rangle$. Other interestingly ordered phases at larger $\mu$.

# The sign problem

$$Z = \mathrm{e}^{-F(T,\mu)/T} = \int DU \; \mathrm{e}^{-S} \prod_f \det M(U, m_f, \mu_f)$$

where the Dirac operator is a lattice discretisation of $M = m + \partial_\mu \gamma_\mu$.

- If there is a $Q$ such that $M^\dagger = Q^\dagger M Q$, then clearly $\det M$ is real.
- $Q = \gamma_5$ for $\mu = 0$. Nothing for $\mu \neq 0$. Monte Carlo simulations of $Z$ fail.
- Thermodynamics remains valid, free energy is fine.

## Developing methods to work with the sign problem

- Reweighting: Simulate at some parameter set, reweight the Fermion determinant ($M$, inside the path integral) to another parameter set. Fodor and Katz (hep-lat/0104001) used density of states method, Bielefeld-Swansea (hep-lat/0204010) used Taylor expansion of determinant.
- Analytic continuation: Find $F$ and derivatives at some parameter set, make analytical continuation of $F$ (outside the path integral) to another parameter set.
  - Imaginary chemical potential: $\exp(i\mu)$ like a $U(1)$ gauge field, no sign problem. de Forcrand and Philipsen (hep-lat/0205016), d'Elia and Lombardo (hep-lat/0209146) used regular imaginary $\mu$, Azcoiti et al (hep-lat/0409157) extended to two couplings.
  - Taylor Expansion of free energy: Gavai and SG (hep-lat/030301), Bielefeld-Swansea (hep-lat/0305007)

## The Taylor expansion of the pressure for 2 flavours

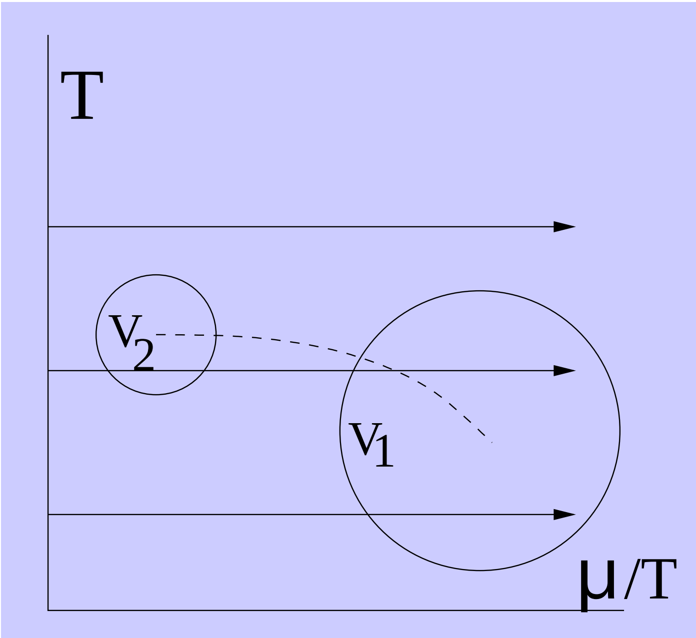
$$P(T, \mu_u, \mu_d) = \left(\frac{T}{V}\right) \log Z(T, \mu_u, \mu_d)$$

$$P(T, \mu_u, \mu_d) = P(T, 0, 0) + \sum_{n_u, n_d} \chi_{n_u, n_d} \frac{\mu_u^{n_u}}{n_u!} \frac{\mu_d^{n_d}}{n_d!}$$

$m_u = m_d$ implies that $\chi_{n_u,n_d} = \chi_{n_d,n_u}$, for any $\mu_u = \mu_d$. One QNS is

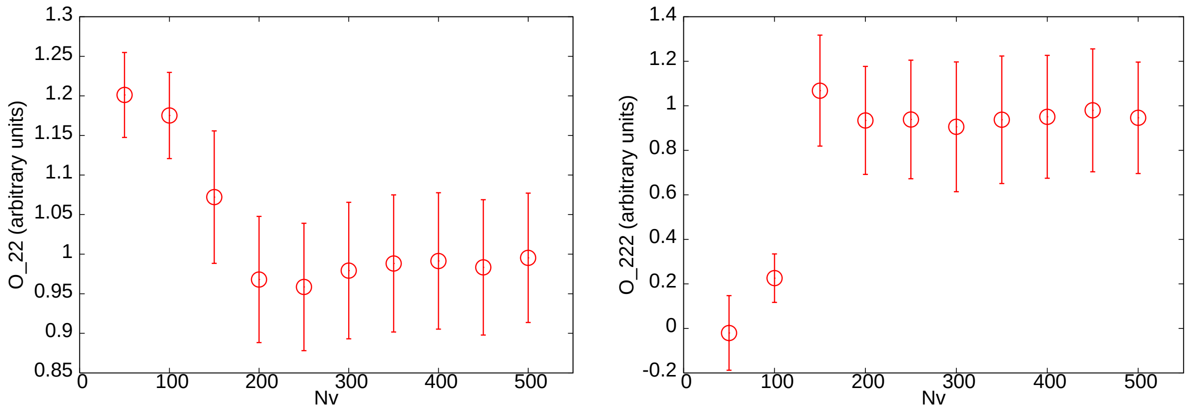
$$\chi_B(T, \mu_B) = \left.\frac{\partial^2 P(T, \mu_u, \mu_d)}{\partial \mu_B^2}\right|_{\mu_u = \mu_d = \mu_B/3}$$

$\chi_B(T^E, \mu_B^E)$ diverges in the infinite volume limit: pseudo critical behaviour at finite volumes. van Hove's theorem



# Evaluating fermion traces

Numerical estimates of traces are made by the usual noisy method, which involves the identity $I = \overline{|r\rangle\langle r|}$, where $r$ is a vector of complex Gaussian random numbers. We need upto 500 vectors in the averaging.




Central value: measurement with exactly $N_v$ vectors; bars: config-to-config variation. Statistics of vectors $(N_v)$ is the big issue. Statistics of configs secondary. Bielefeld: $N_v = 50$–$100$, Mumbai: $N_v = 400$–$500$.

## Evaluating fermion traces

Distribution of each trace is Gaussian. Product of traces such as $\chi_{2222}$ strongly non-Gaussian. Proof: For a Gaussian random number of unit variance

$$\langle x_i^2 \rangle = 1, \qquad \langle x_i^4 \rangle = 3 \qquad \text{implies } \left[x_i^4\right] \equiv \langle x_i^4 \rangle - 3\langle x_i^2 \rangle^2 = 0,$$

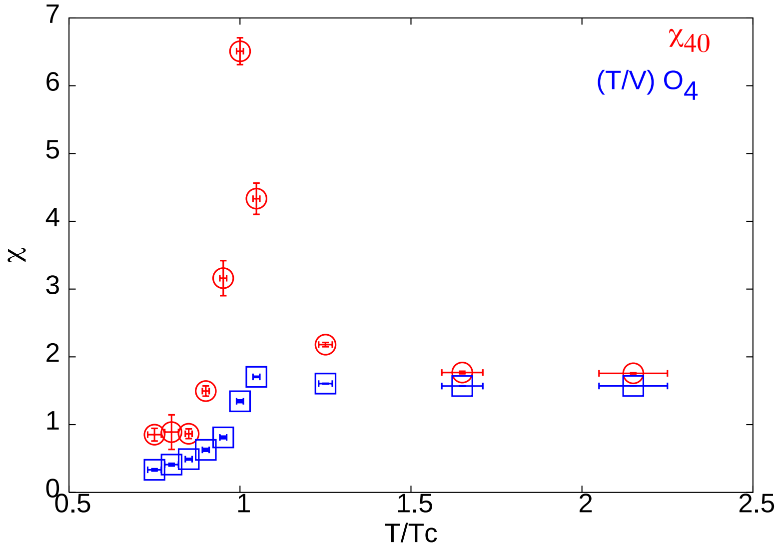
but for a product of independent Gaussian numbers $v = x_1 x_2 \cdots x_n$,

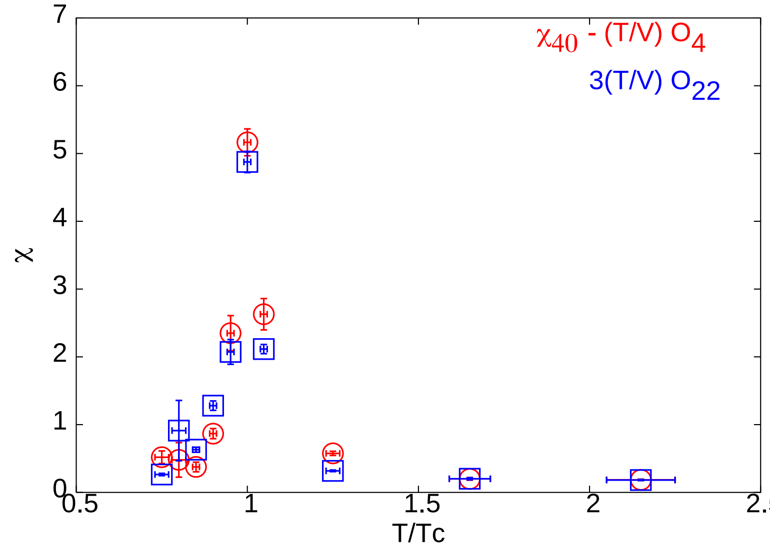
$$\langle v^2 \rangle = 1, \qquad \langle v^4 \rangle = 3^n \qquad \text{implies } \left[v^4\right] = 3^n - 3.$$

The distribution of $v$ can be written down in closed form in special cases.

Central limit theorem applies: distribution of $\overline{v}$ is Gaussian (proof straightforward). But to reduce the 4th cumulant to substantially below the 2nd, one needs statistics $\gg 3^n$. So, the number of vectors $N_v \gg \mathcal{O}(n3^n)$.

# 4th order NLS peaks at $T_c$




Peak due to multiloop operators: correspond to products of traces.

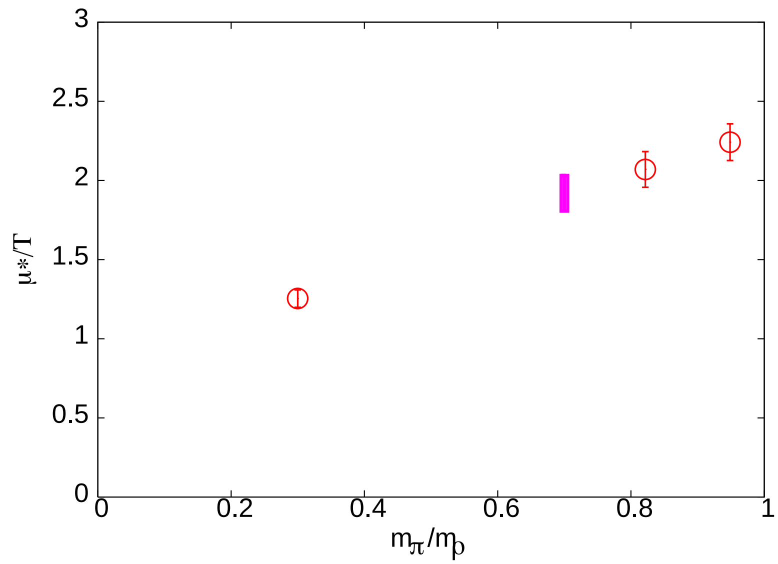
Proper control of $(T/V)\langle O_{22}\rangle$ is needed to obtain the correct value of $\chi_{40}$ (and $\chi_{22}$) near $T_c$. Similiar control of products of traces needed for all higher susceptibilities in this region.

Proper choice of $N_v$ crucial.

# Quark mass dependence

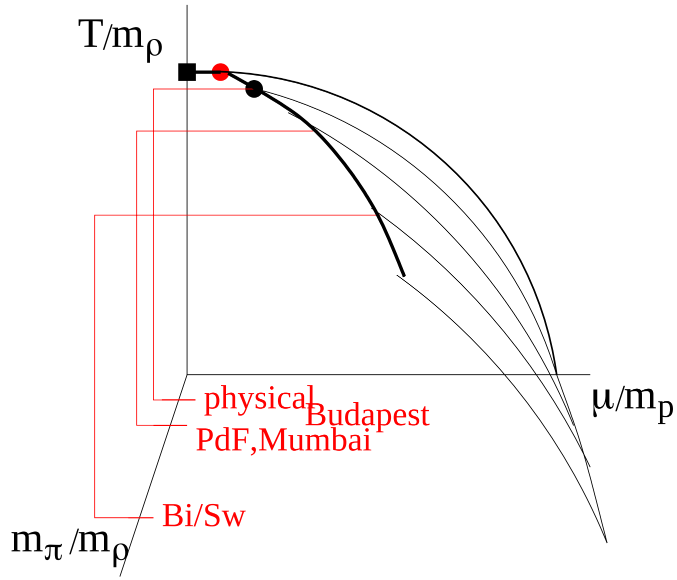
Radius of convergence: $\mu^*/T = \sqrt{|12\chi_B^{(2)}/\chi_B^{(4)}|}$.

Peak in $\chi_{40}$ implies decreasing radius of convergence. The radius of convergence seems to be very sensitive to quark mass in a region near $T_c$. Interpolation to $m_\pi/m_\rho = 0.7$ is consistent with the break point in Bielefeld-Swansea result.




Partially quenched computation. Sea quarks correspond to the lowest mass shown here.

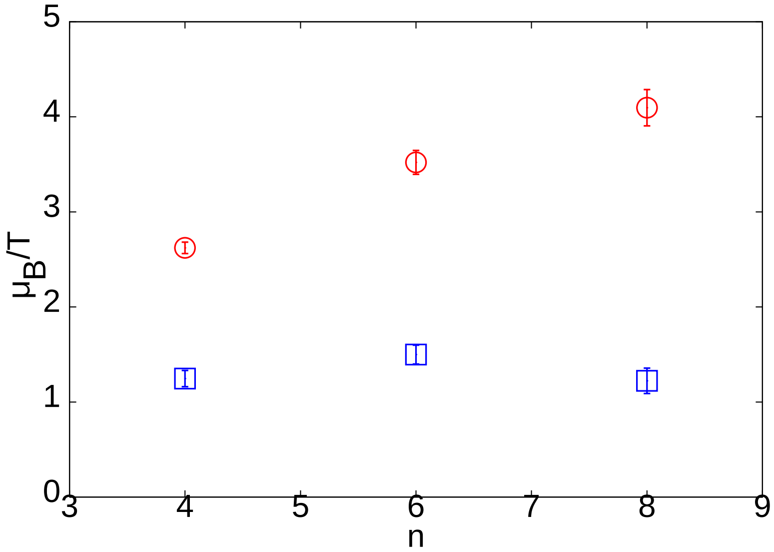
# The wing critical line

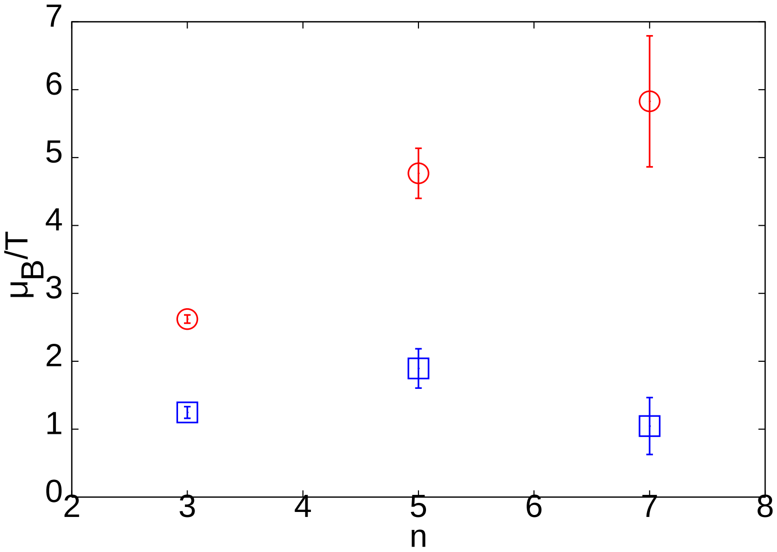



How curved is the wing critical line? Data from different simulations can be combined to answer this question since they are at similiar values of $m_p/m_\rho$ but different $m_\pi/m_\rho$. Criticality may be harder to observe at larger $m_\pi/m_\rho$, but simulations are easier.

# Radius of convergence

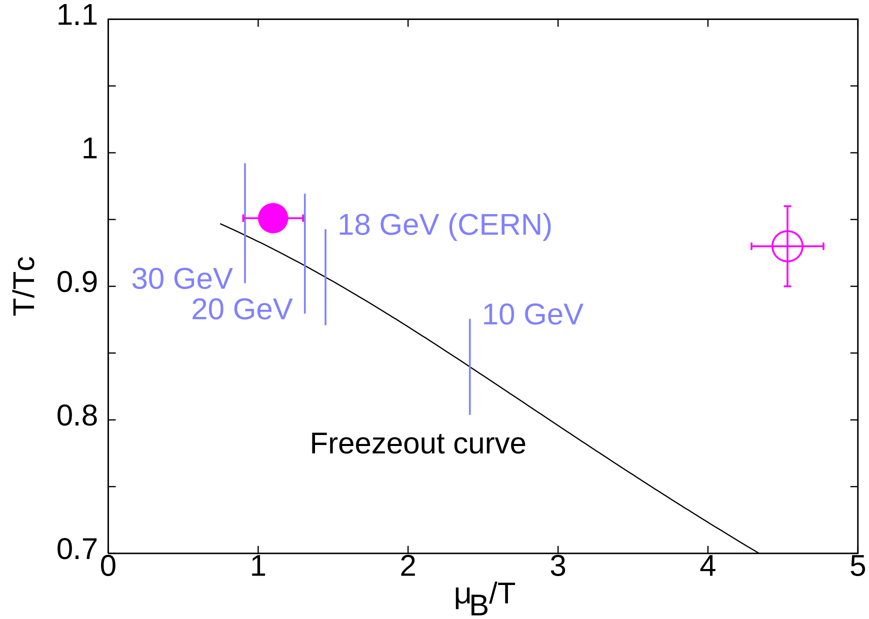
Notation: If $f(x) = \sum_n f_{2n} x^{2n}$ then $\rho_{2n} = \left|\frac{f_0}{f_{2n}}\right|^{1/2n}$ and $r_{2n+1} = \sqrt{\left|\frac{f_{2n}}{f_{2n+2}}\right|}$




Old Budapest results roughly consistent with our small volume analysis. A threshold $Lm_\pi \approx 5$ is needed to study the thermodynamic limit.

# The QCD critical end point




R. V. Gavai and SG, PRD 71 (2005) 114014

Strong finite volume effect; strong quark mass effect. When $Lm_\pi \to \infty$, $a = 1/4T$ and $m_\pi/m_\rho = 0.3$ then $T^E/m_\rho \approx 0.17$ and $\mu^E/m_\rho \approx 0.19$.

## What to control in a reliable computation

1. Statistics of random vectors: $N_v \simeq 400$–700 required. One test: off-diagonal higher order susceptibilities must be independent of lattice volume.
2. Statistics of configurations: secondary problem. All configurations should be statistically independent, otherwise systematic effects. Measure autocorrelation times ($\tau$). Statistical errors: $\sigma^2_{actual} = (1 + 2\tau)\sigma^2_{apparent}$.
3. Spatial volume: must be large enough to contain more than 5 Compton wavelengths of the pion. Even larger if one wants to study critical indices.
4. Quark mass is crucial. State of the art is a quark mass such that $m_\pi/m_\rho$ is 50% larger than the physical value.
5. Lattice spacing errors currently undetermined. Require two computations at the same $m_\pi/m_\rho$ with two actions having different lattice spacing effects.

# Summary

1. Continuum limit of baryon number fluctuations predicted within approximately 5% accuracy. Similiar results for fluctuations of other conserved quantities.

2. Linkage studies can directly reveal the nature of the quasiparticles in the fireball. Variables similiar to fluctuations.

3. Strangeness abundance predicted within 5% accuracy in equilibrium.

4. Non-Poisson component of fluctuations important near $T_c$ and expected to grow as one approaches the critical end point.

5. The critical end point may be accessible to energy scans at colliders. Agreement between all lattice computations if one takes volume effects and quark mass dependence into account.