QCD at finite density

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ILGTI: TIFR

Lattice 2010
Cagliari, Italy
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Attacking the sign problem

Avoiding the sign problem

Connecting to experiments

End

Outline

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The problem

Gauge action positive; not changed by introduction of flavour chemical potentials.
Fermion determinant contains sign problem:

$$\det(D + m + \mu\gamma_0)^* = \det(D + m - \mu^*\gamma_0)$$

Cannot be free of sign problems when μ is real non-zero.

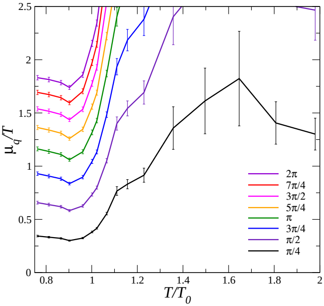
Importance sampling fails: no Monte Carlo procedure.

Problem could be representation dependent; clever reformulation may resolve the problem: for example, by changing to new variables.

How bad is the sign problem?

For $\mu < m_\pi/2$ distribution of signs is Gaussian. At larger μ it becomes Lorentzian. (Analysis in baryonless random matrix theory). Hard in both cases.

Lombardo, Splittorff and Verbaarschot, 0910.5842



Effect of baryons? Effect of finite temperature?

Splittorff *et al.*, Lattice 2010

Contour lines of the variance of the phase of the determinant: problem easier at high temperature.

Bielefeld-Swansea, PR D 71 2005

Reweighting

- Glasgow: generate ensemble at one point in phase diagram, reweight to another point; problem of overlap.
- Finite temperature reweighting; overlap problem smaller. Fodor and Katz, 2001
- Taylor-expand the quark determinant inside the path-integral; amounts to differential reweighting. Bielefeld Swansea, 2002
- Gaussian approximation to the phase of the determinant; used to reweight configurations. Ejiri, 2007

No major methodological developments since 2007. Some applications this year by WHOT-QCD.

3D XY model: world-line formulation

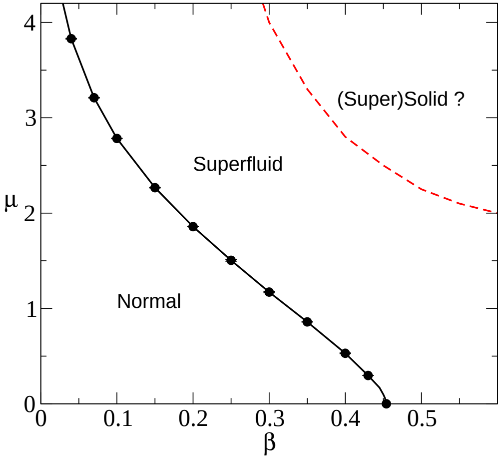
Pure bosonic model: has a sign problem at finite μ:

$$S = -\beta \sum_{x,\hat{\mu}} \cos\left(\theta_x - \theta_{x+\hat{\mu}} - i\mu\delta_{\hat{\mu},\hat{t}}\right).$$

Sign problem completely removed by introducing variables corresponding to current of particles along links. Worm algorithm used to solve this problem.
Banerjee and Chandrasekharan, 1001.3648

Interesting finite-size theory developed: examines the crossing of the ground state levels due to N particles and $N+1$ particles as μ varies. FSS contains small number of parameters to be fitted to observations. Scaling of energy level leads to the conjecture that $\mu_c = M$; determines the phase diagram.

Phase diagram: 3D XY model

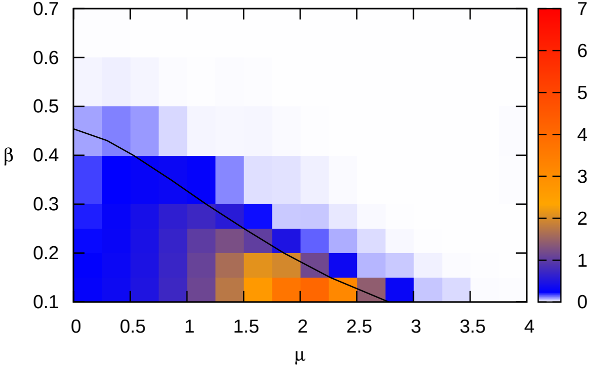


Complex Langevin

Fields complexified, noise remains real. Revived in the last few years. Earlier problems with runaway directions and numerical instability solved by using adaptive step-size integrators. Many test systems now amenable to analysis using this technique. Proof of convergence seemed within reach.
Aarts, Seiler, Stamatescu, 0912.3360

But new problem unearthed: convergence to wrong result. Conjectured not to be due to the sign problem; but resemble the results of using a complex noise. Something yet to be understood.
Aarts and James, 1005.3468

Errors in complex Langevin simulation of 3D XY model



Contour plot of $\Delta S = (S_{CL} - S_{WL})/S_{WL}$.

Aarts and James, 1005.3468

Outline

Maclaurin (Taylor) series expansion

The pressure in a grand canonical ensemble allows a Maclaurin series expansion:

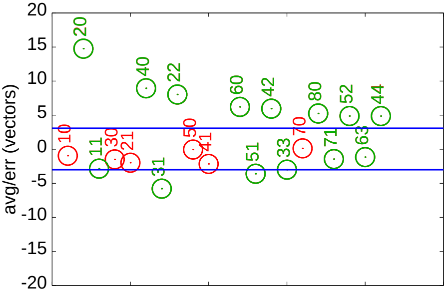
$$P(T,\mu) = P(T) + \frac{\mu^2}{2!}\chi^{(2)}(T) + \frac{\mu^4}{4!}\chi^{(4)}(T) + \cdots$$

The coefficients are evaluated at $\mu = 0$ where there is no sign problem.

Evaluate the susceptibilities $\chi^{(n)}$ directly as expectation values of operators.
Gavai, SG, 2003

Evaluate the susceptibilities by constructing the pressure (or its derivatives) at series of imaginary chemical potentials and then fitting extrapolating functions to the data.
Cosmai *et al.*, Falcone *et al.*: Lattice 2010

Statistical significance of measurements



Covariance over configurations: $\sigma_{O_4,O_6} \simeq \sigma_{O_6,O_8} \simeq 0.7$

1. Staggered: 4.24^3 lattice, $m_\pi = 230$ MeV, $T = 0.75\,T_c$, 400 vectors. (Red symbols: supposed to vanish) SG, 2004

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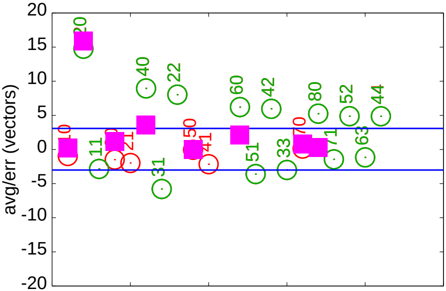

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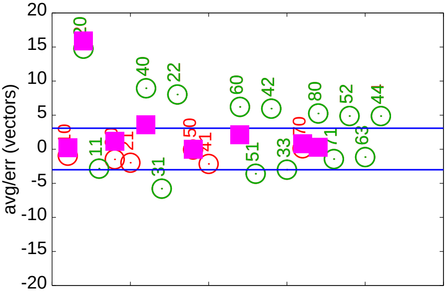

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3. Asqtad quarks: up to 50% of the noise due to stochastic estimators with 400–800 vectors. MILC, 1003.5682

CPU effort

$LT = 4$ lattices

At T_c autocorrelations: 200–250 trajectories
Number of CG inversions per trajectory: 200
One measurement every decorrelated configuration: 500×18 CG inversions
Measurement/configuration: $500 \times 18/(200 \times 200) = 0.24$

At $2T_c$ autocorrelations: 4 trajectories
Number of CG inversions per trajectory: 100
One measurement every decorrelated configuration: 100×18 CG inversions
Measurement/configuration: $100 \times 18/(100 \times 4) = 4.5$

Series Analysis

Series analysis for spin models

Analysis of series for critical behaviour since 1960s. Well-developed when series coefficients are exactly known. First step: evaluate radius of convergence. Then check whether singularity is due to physical parameter values.

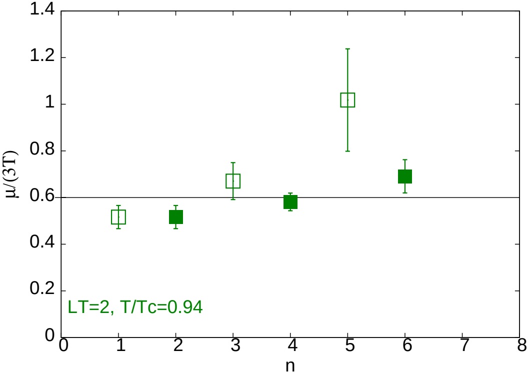
Domb and Green, vol 2

Series analysis for $\mu \neq 0$ QCD

Similar idea, but needs to be adapted to specific problem. Series coefficients have statistical errors; coefficients are volume dependent. Some subtleties.

Gavai, SG, 2004, 2008

Finite volume effects



Gavai, SG, 0806.2233

Filled symbols: $r_n = \sqrt{(n+3)!\chi^{(n+1)}/(n+1)!\chi^{(n+3)}}$
Unfilled symbols: $r_n = ((n+2)!\chi^{(2)}/2!\chi^{(n+2)})^{1/n}$
$LT \geq 4$ and $Lm_\pi \geq 5$; plateau develops.

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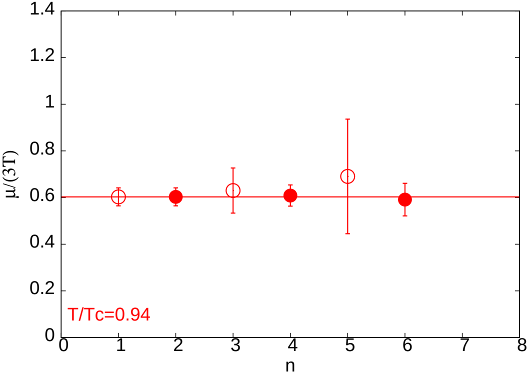

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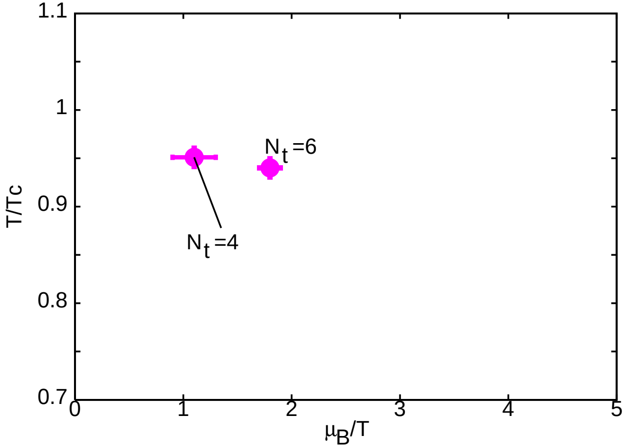
Finite volume effects and order of expansion

1. Increasing order of series expansion and finite volume scaling closely tied together.
2. Susceptibility never diverges on finite volume, but grows higher and sharper with increasing volume. Major effect: growth of peak; minor effect: shift of peak.
3. Series expansion of such a sequence of functions should show lack of divergence for each volume if pushed to large enough order.
4. At finite order, signal of eventual divergence should build up.
5. With increasing volume, there should be a plateau of stability for radius of convergence before radius diverges.

Cutoff dependence and the effect of strange quarks



Staggered: $N_f = 2$, $m_\pi = 230$ MeV, $LT \geq 4$ Gavai, SG, 0806.2233

P4: $N_f = 2 + 1$, $m_\pi = 220$ MeV, $LT = 4$ Schmidt, 2010

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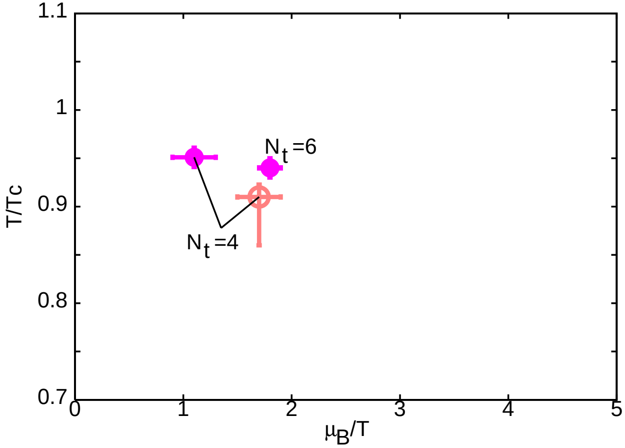

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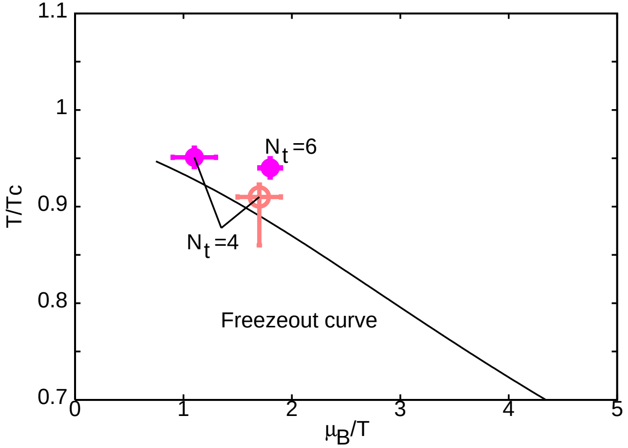

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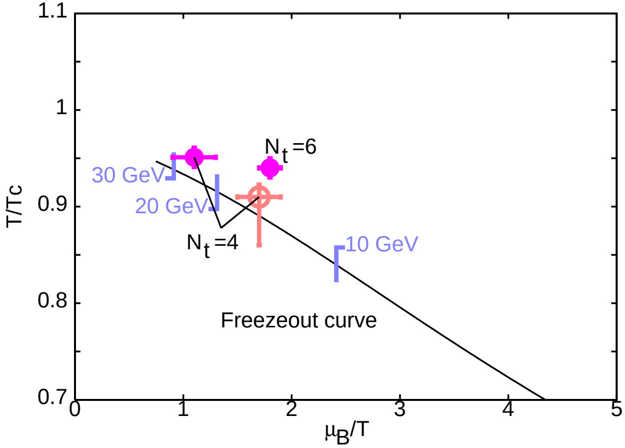

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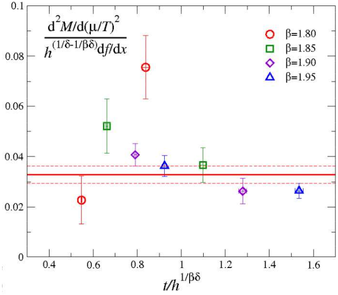
The critical line

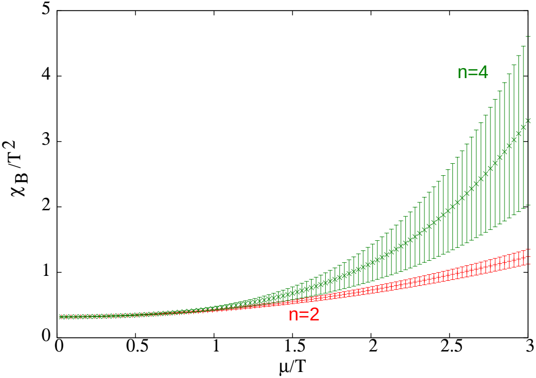


Critical line in the chiral limit? Curvature at finite μ:

$$T_c(\mu) = T_c \left[1 + \kappa \left(\frac{\mu}{T_c} \right)^2 + \cdots \right]$$

Ejiri *et al.*, Mukherjee *et al.*, Klein *et al.*, Falcone *et al.*, Lattice 2010

Extrapolating measurements



Gavai, SG, 0806.2233

Infinite series diverges, but truncated series finite and smooth: sum is bad. Resummations needed to reproduce critical divergence. Padé resummation useful.

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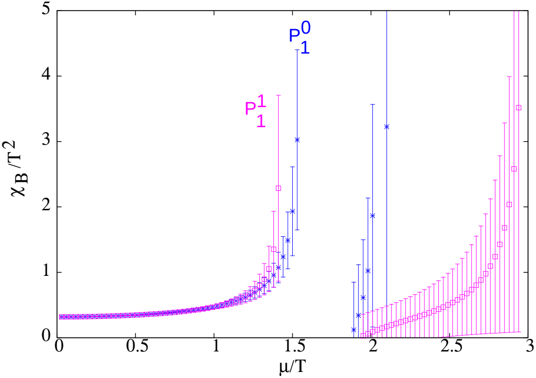

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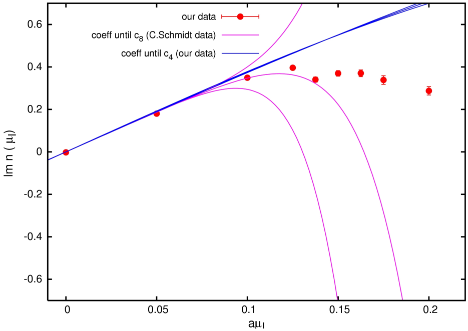
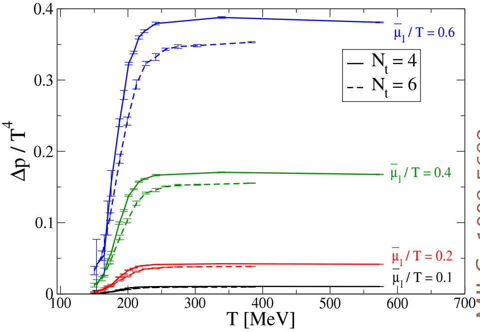
Series at imaginary μ



Falcone *et al.*, Lattice 2010

More terms in the series needed. Does a resummation help?

The pressure



MILC, 1003.5682

$\Delta p = p(T, \mu) - p(T, 0)$. May be interesting to try a resummation.

Outline

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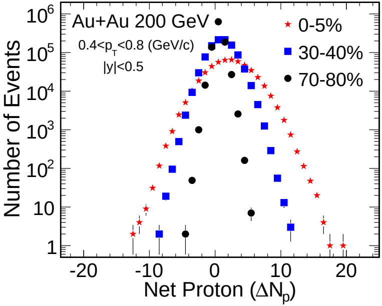
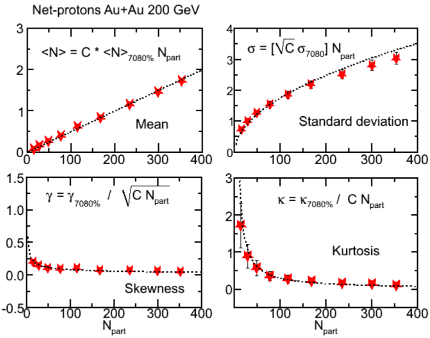
Event distributions of conserved charges



STAR, 1004.4959

- Fluctuations of conserved quantities are Gaussian: provided large volume and equilibrium
- Proton number a substitute for baryon number: how good?
- Is this Gaussian due (entirely or largely) to thermal fluctuations?

Look beyond Gaussian



STAR: QM 2009, Knoxville

- Higher cumulants scale down with larger powers of V.
- N_{part} is a proxy for V.
- Cumulants observed to scale correctly as N_{part}.
- Can one connect to QCD?

How to compare experiment with lattice QCD

The cumulants of the distribution are related to Taylor coefficients—

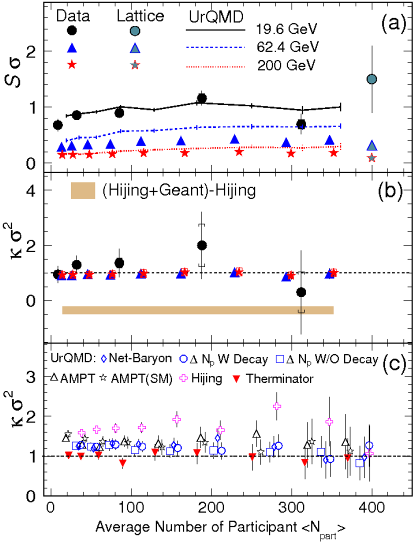
$$[B^2] = T^3 V \left(\frac{\chi^{(2)}}{T^2}\right), \quad [B^3] = T^3 V \left(\frac{\chi^{(3)}}{T}\right), \quad [B^4] = T^3 V \chi^{(4)}.$$

V is unknown, so direct measurement of QNS not possible. Define variance $\sigma^2 = [B^2]$, skew $\mathcal{S} = [B^3]/\sigma^3$ and Kurtosis, $\mathcal{K} = [B^4]/\sigma^4$. Construct the ratios

$$\mathcal{S}\sigma = \frac{[B^3]}{[B^2]}, \qquad \mathcal{K}\sigma^2 = \frac{[B^4]}{[B^2]}, \qquad \frac{\mathcal{K}\sigma}{\mathcal{S}} = \frac{[B^4]}{[B^3]}.$$

These are comparable with experiment provided lattice data **extrapolated to relevant T and μ**: use Padé approximants. SG, 0909.4630

Extrapolate lattice data to finite μ



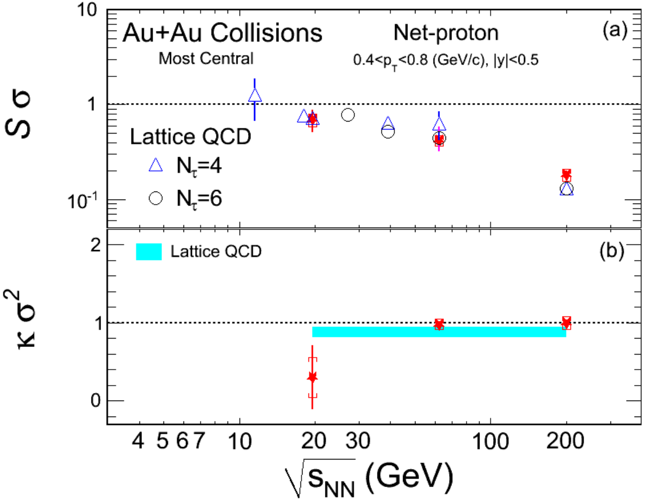
STAR Collaboration, 1004.4959 (2010)

Surprising agreement with lattice QCD:

- implies non-thermal sources of fluctuations are very small
- T does not vary across the freezeout surface.
- tests QCD in non-perturbative thermal region

Gavai, SG, 1001.3796

Experiment vs lattice QCD



Outline

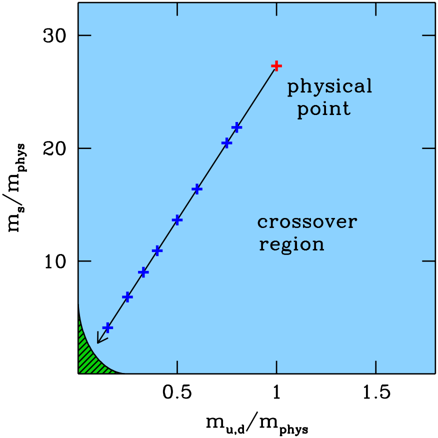
Other topics in this conference

1. Phase diagram at imaginary μ, Phillipsen *et al.*, Cosmai *et al.*
2. Correlators for $\mu \neq 0$, Iida *et al.*
3. Phase diagram in strong coupling, Miura *et al.*, Nakano *et al.*, Ohnishi *et al.*
4. Canonical ensemble simulations, Liu *et al.*
5. Unitary Fermi gas Goulko, Endres, Nicholson, Lee
6. Other topics Myers, Cristoforetti, Wettig, Palumbo

Summary

1. Some algorithmic progress in direct attack on the sign problem. Several related problems now solvable. Many interesting results now available: 3D XY model most recent.
2. Analytic continuation methods yielded many results. Application to phase diagram and EOS since 2003. Applied to correlators, number density, *etc.*.
3. Imaginary μ is an alternative method for analytic continuation. Many studies of systematics reported in parallel sessions. Consistency with Taylor expansion now being established.
4. Methods exist to compare lattice results with experiments now being done at the RHIC. First results very encouraging.

Backup: Lattice results for the Columbia Plot



In $N_f = 2+1$:

$$m_\pi^{crit} \begin{cases} = 0.07 m_\pi & (N_t = 4) \\ < 0.12 m_\pi & (N_t = 6) \end{cases}$$

Endrodi etal, 0710.0988

Similarly for $N_f = 3$.

Karsch etal, hep-lat/0309121

Backup 2: Analysis of BiBrooG data a la ILGTI