The critical point of QCD: lattice and experiment

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1 The context
2 On lattice
3 In experiment
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5. Supersymmetry in nuclear energy levels. Why?
Computing in QCD

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3. What are phases of compressed baryonic matter? Are there phase transitions? Can some of these phases be found in compact stellar objects? What is the equation of state of such matter? What are the transport coefficients in such matter: do they transport momentum and energy efficiently?

Some of these questions can be answered in lattice computations. For others the field theory technology is not yet well-developed.
3 conserved charges: $B$, $Q$ and $S$. Hence 3 chemical potentials; and $T$: 4 knobs to turn differently in different corners of the universe. In heavy-ion collisions change 2 quantities $T$ and $\mu_B$. But only one knob to tune: $\sqrt{S}$. 
Heavy-ion collisions: kinematics

The context

(a) (b) (c)
Heavy-ion collisions: evolution

- AGS (Brookhaven): Au, 10 GeV, ??– 1990
- RHIC (Brookhaven), d, Au, 5–200 GeV, 2000 – 2020
- LHC (CERN), Pb, 5.4 TeV, 2012 – ??
- FAIR (GSI Darmstadt), U, 10–40 GeV, 2015 –

Particle detectors

- QGP
- pre-thermal
- normal
- perturbative

Z

t

10 m

10 fm
All this is qualitative: need to make one quantitative computation: now done, estimate of the critical point Gavai, SG.
The context

The phase diagram: topology through symmetry

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Outline

1. The context

2. On lattice

3. In experiment
The sign problem

QCD is solved on (super) computers by evaluating the path integral

\[ Z = \int \prod_{x\nu} dU_{x\nu} \det D \exp[-S_E(U_{x\nu})] \quad U_{x\nu} = \exp[iaA_\nu(x)] \]

\( \mathbf{x} \): spacetime point, \( \nu \): directions. Continuum limit: number of points goes to infinity, therefore infinite dimensional integral. Lattice cutoff, \( a \to 0 \) using QCD beta function. But even at small \( a \), many points; need to use Monte Carlo techniques.
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Works when weight is real. Gauge part always real. Quark part? At zero \( \mu_B \) one finds \( \det D \) real. Therefore Monte Carlo works. At finite \( \mu_B \) additional term \( \mu_B \gamma_0 \) like complex gauge field. Hence Monte Carlo fails: fermion sign problem.
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Sign problems everywhere: QCD at finite \(\mu_B\), Chern-Simmons theory, high temperature superconductors, many nano-systems · · · . First solution in QCD.
The method

Taylor expansion of the pressure in $\mu_B$

$$P(T, \mu_B) = \sum_n \frac{1}{n!} \chi^{(n)}(T) \mu_B^n$$

has Taylor coefficients that need to be evaluated only at $\mu_B = 0$ where there is no sign problem. The baryon number susceptibility (second derivative of $P$) has a related Taylor expansion

$$\chi_B(T, \mu_B) = \sum_n \frac{1}{n!} \chi^{(n+2)}(T) \mu_B^n.$$  

$\chi_B$ diverges at the critical point. Series expansion can show signs of divergence. If all the coefficients are positive, then the divergence is at real $\mu_B$.

The method is perfectly general and can be applied to any theory.
The implementation

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- Simulation algorithm is R-algorithm. MD time step has been changed by factor of 10 without any change in results.
Remaining issues

- Series expansion carried out to 8th order. What happens when order is increased? Intimately related to finite volume effects: next.
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- What if a different estimator of the critical point is used: must agree, at least in the large volume limit.
- Can the phase diagram be more complicated? Yes, we only find the nearest critical point to $\mu_B = 0$. 
$N_t = 6$: Radius of convergence

Filled symbols: $(c_0/c_n)^{1/n}$. Open symbols: $\sqrt{c_{n-1}/c_{n+1}}$. 

LT=4; $T/T_c=0.94$
Critical end point

- Multiple criteria agree:
  - Small window in $T$ where all the coefficients are positive.
  - Stability of radius of convergence with order and estimator
  - Finite size effects follow correct trend; more planned for the future.
  - Pinching of the radius of convergence with $T$.
- This gives $T^E / T_c = 0.94 \pm 0.01$ and $\mu^E_B / T^E$ as below

<table>
<thead>
<tr>
<th>$N_t$</th>
<th>$V = (4/T)^3$</th>
<th>$V \rightarrow \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$1.3 \pm 0.3$</td>
<td>$1.1 \pm 0.1$</td>
</tr>
<tr>
<td>6</td>
<td>$1.8 \pm 0.1$</td>
<td>?</td>
</tr>
</tbody>
</table>

- Very naively: extrapolate to $V \rightarrow \infty$ by same factor, extrapolate to $a \rightarrow 0$ as $a^2$ (staggered quarks), then $\mu^E_B \approx 325$ MeV. Somewhat lower at $m_\pi = 140$ MeV. Many assumptions, many caveats. May be in the range $\mu^E_B = 250–400$ MeV with $T^E = 165–175$ MeV.
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Gaussian Fluctuations

Fluctuations are Gaussian

Suggestion by Stephanov, Rajagopal, Shuryak: measure the width of momentum distributions. Better idea, use conserved charges, because at any normal (non-critical) point in the phase diagram:

\[ P(\Delta B) = \exp \left( -\frac{(\Delta B)^2}{2VT\chi_B} \right) \cdot (\Delta B = B - \langle B \rangle). \]

Bias-free measurement possible: Asakawa, Heinz, Muller; Jeon, Koch.

Why Gaussian?

At any non-critical point the appropriate correlation length \((\xi)\) is finite. If the number of independently fluctuating volumes \((N = V/\xi^3)\) is large enough, then net \(B\) has Gaussian distribution: central limit theorem (CLT).
Is the current RHIC point non-critical?

**Answer**

Check whether CLT holds.
Recall the scalings of extensive quantity such as $B$ and its variance $\sigma^2$, skewness, $S$, and Kurtosis, $\mathcal{K}$, given by

$$B(V) \propto V, \quad \sigma^2(V) \propto V, \quad S(V) \propto \frac{1}{\sqrt{V}}, \quad \mathcal{K}(V) \propto \frac{1}{V}.$$ 

**Caveat**

Make sure that the nature of the physical system does not change while changing the volume. Perhaps best accomplished by changing rapidity acceptance while keeping centrality fixed. Alternative tried by STAR is to change the number of participants.
In experiment

STAR measurements

Net-protons Au+Au 200 GeV

\[ \langle N \rangle = C \times \langle N \rangle_{7080\%} N_{\text{part}} \]

Mean

\[ \sigma = \sqrt{C \times \sigma_{7080\%} N_{\text{part}}} \]

Standard deviation

\[ \gamma = \gamma_{7080\%} / \sqrt{C N_{\text{part}}} \]

Skewness

\[ \kappa = \kappa_{7080\%} / C N_{\text{part}} \]

Kurtosis

Can we compare STAR’s measurements of $\sigma_{70-80\%}$ and $K_{70-80\%}$ with lattice QCD?

Two questions to be answered before this is feasible:

1. $N_{part}$ is a proxy for the volume. In changing this is the physics unchanged? Do the fluctuations give initial information or near-freezeout information? Need to develop a complete theory of diffusion+hydro (Son and Stephanov, hep-ph/0401052, Bower and Gavin, hep-ph/0106010, Bhalerao and SG, 0901.4677).

2. Have all other sources of non-Gaussianity have been subtracted out? What about jetty fluctuations, for example? Need studies of systematics.
In experiment

What to compare with QCD

The cumulants of the distribution are related to Taylor coefficients—

\[
[B^2] = T^3 V \left( \frac{\chi^{(2)}}{T^2} \right), \quad [B^3] = T^3 V \left( \frac{\chi^{(3)}}{T} \right), \quad [B^4] = T^3 V \chi^{(4)}.
\]

\(T\) and \(V\) are unknown, so direct measurement of QNS not possible (yet). Define variance \(\sigma^2 = [B^2]\), skew \(S = [B^3]/\sigma^3\) and Kurtosis, \(\mathcal{K} = [B^4]/\sigma^4\). Control all backgrounds in the measurements of \([B^n]\). Then construct the ratios

\[
m_1 = S\sigma = \frac{[B^3]}{[B^2]}, \quad m_2 = \mathcal{K}\sigma^2 = \frac{[B^4]}{[B^2]}, \quad m_3 = \frac{\mathcal{K}\sigma}{S} = \frac{[B^4]}{[B^3]}.
\]

These are comparable with QCD (Table III of Gavai, SG, 2008).

Is there an internally consistent check that all backgrounds and systematic effects are removed and comparison with lattice QCD possible?
How to compare with QCD

As $T$ and $\mu_B$ are varied, the QCD predictions will lie on a surface in the space of measurements $(m_1, m_2, m_3)$. If the data lies on this surface then all non-thermal backgrounds are removed. Then a comparison with QCD and a measurement of $T$ and $\mu_B$ is immediate. Similarly for Q and S.
Lattice results along the freezeout curve

Open symbols: $T_c = 192$ GeV, filled symbols: $T_c = 175$ GeV. Boxes: $N_t = 4$, circles: $N_t = 6$. 
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$Sqrt(S_{NN})$ (GeV) vs $m^2$
Lattice results along the freezeout curve

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In experiment

One way to find the critical point

- Construct E-to-E distributions of $B$, $Q$ and $S$. Since there are non-trivial linkages between them, comparison of the three distributions is important. Construct distributions in limited acceptance in order to simulate a grand canonical ensemble.

- Issues related to missed particles, in particular uncharged baryons and strange particles ($\text{nucleons and } K^0$). Require studies to see the effects of these.

- Observe the scaling of $B$, $Q$ and $S$ as a function of volume: if central limit theorem, then normal point. Otherwise close to critical point.

- Close to critical point the kurtosis does not scale with volume and may become very large due to critical exponent effects.

- Effect of hydrodynamic evolution needs to be included.