An introduction to lattice field theory

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Outline

The quantum problem

A quantum problem is completely specified either through the Hermitean **Hamiltonian**, $H(t)$, or the unitary **time-evolution operator**,

$$U(0,T) = \mathrm{e}^{i\int_0^T dt H(t)}$$

There are path integral representations for this operator. Modern QFT starts from these.

The finite temperature quantum problem is completely understood if one computes the partition function

$$Z(\beta) = \mathrm{Tr}\,\mathrm{e}^{-\int_0^\beta dt H(t)},$$

which is formally the same problem in Euclidean time. The same path integral suffices to solve this problem: only boundary conditions change.

Path integral: generalizing the 2-slit experiment

$$\langle x_i, t_i |\, U(t_i, t_f)\, | x_i, t_i \rangle = \langle x_i, t_i | x_f, t_f \rangle$$

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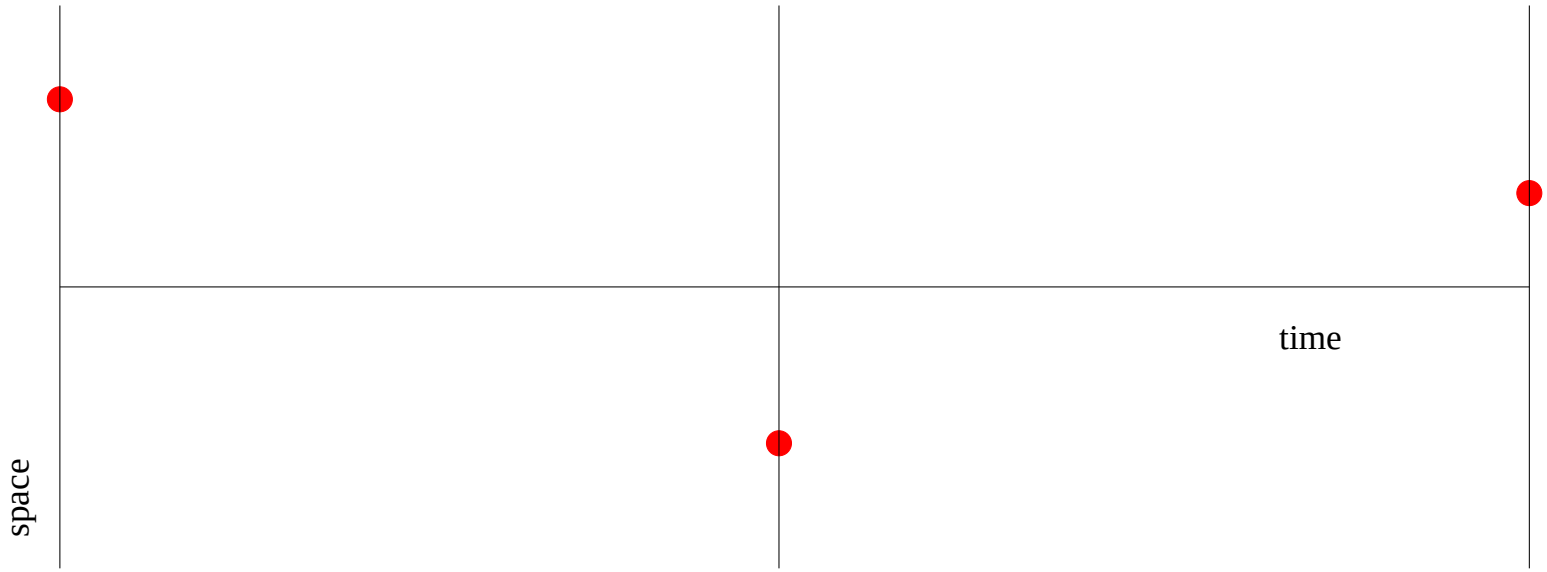

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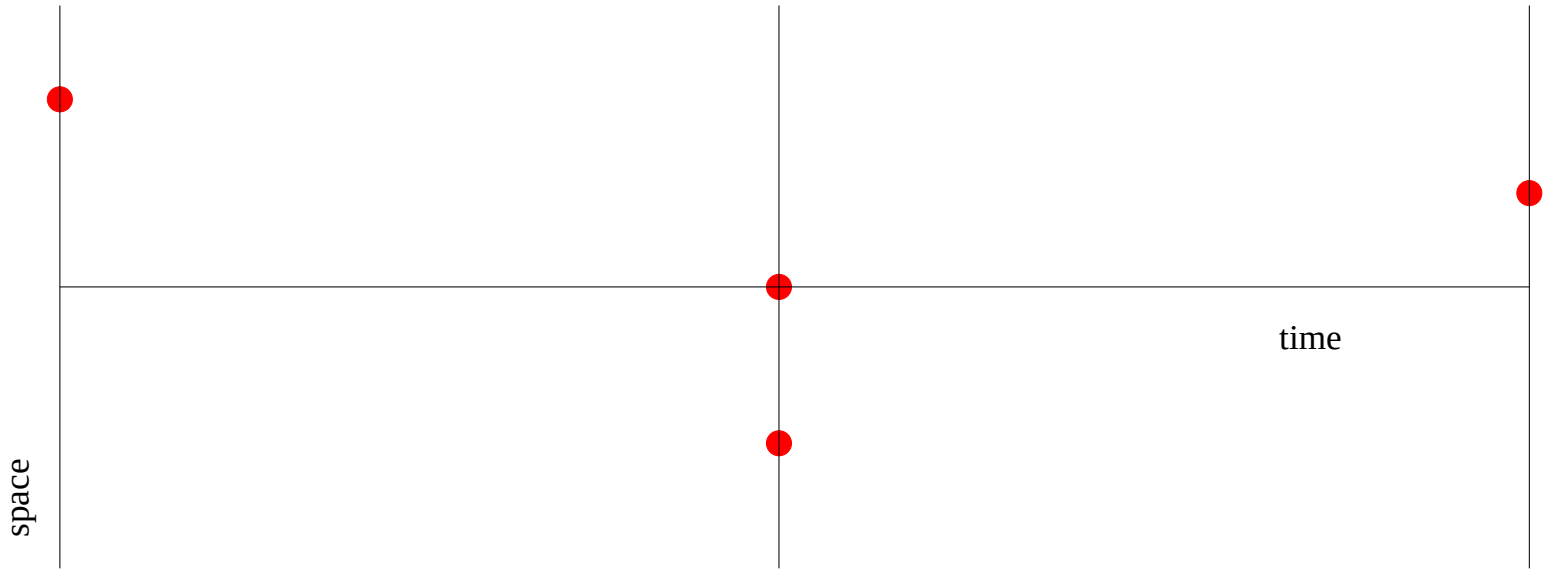

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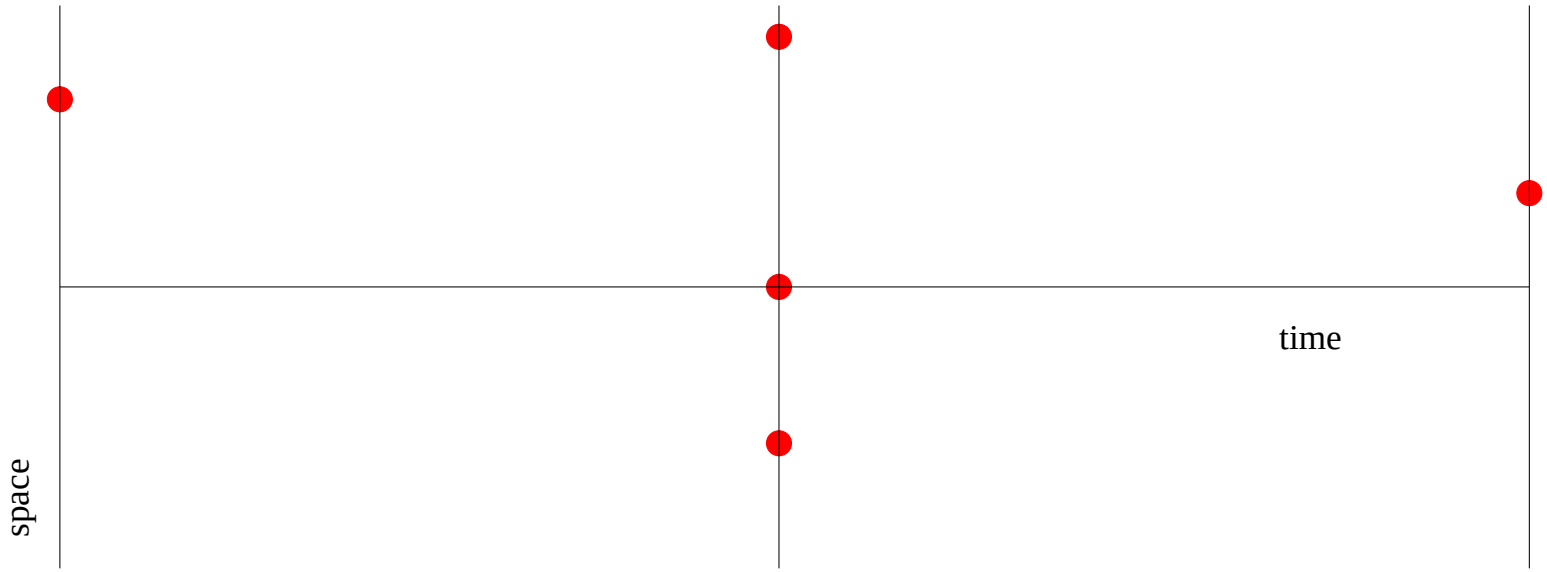

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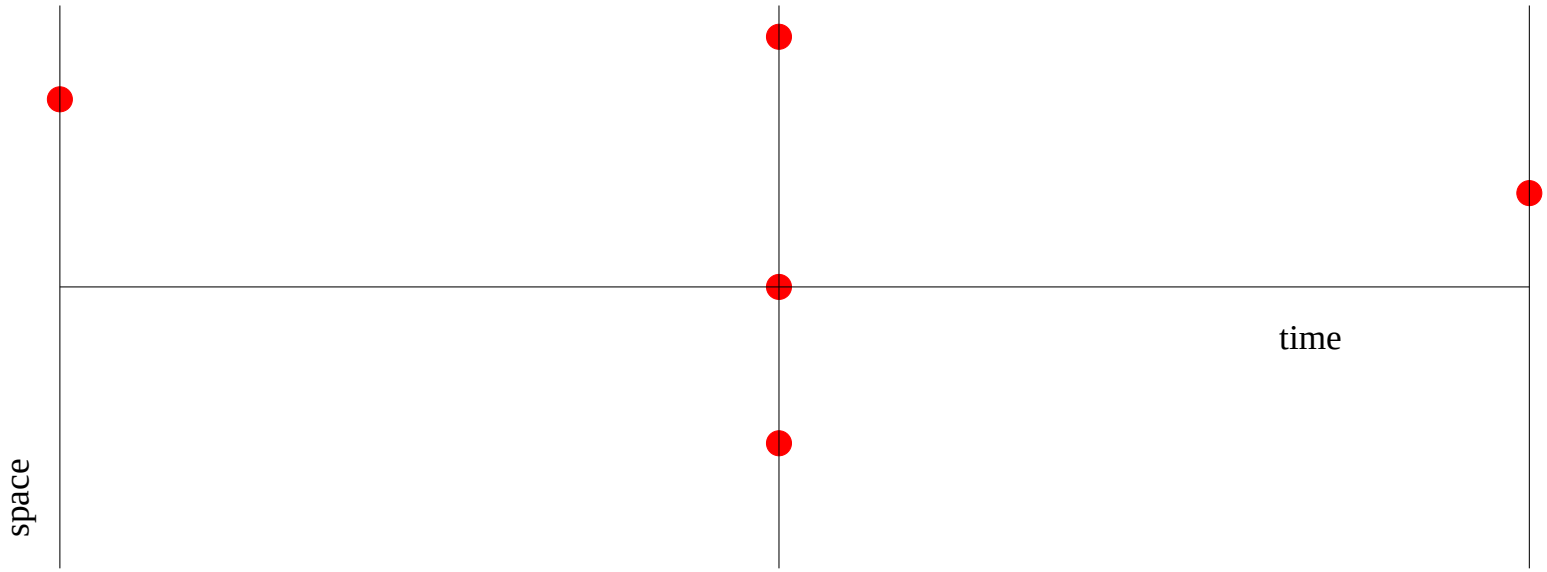

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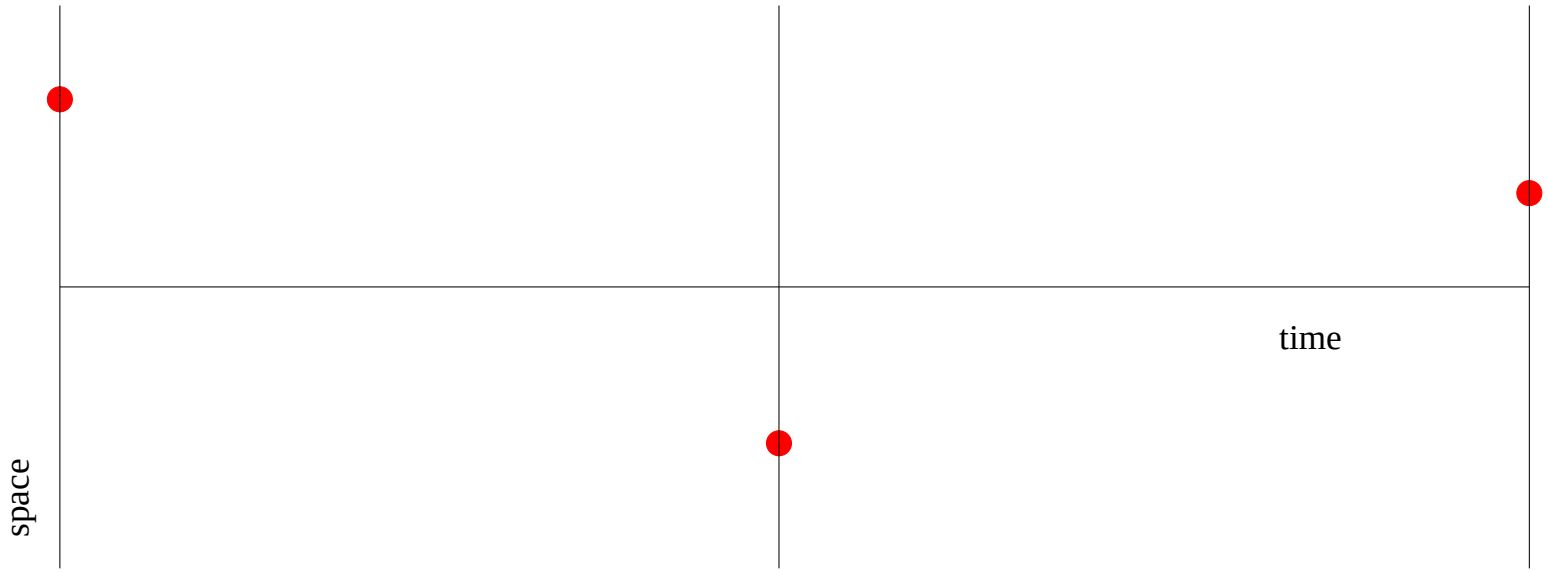

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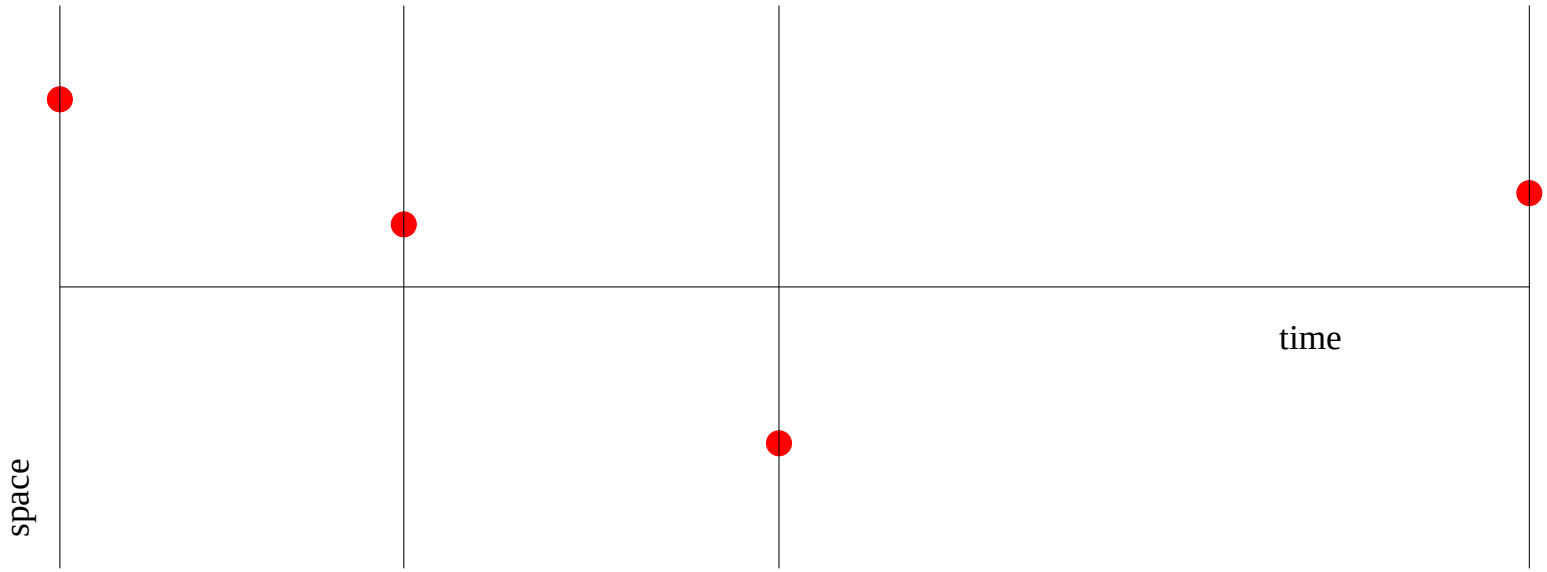

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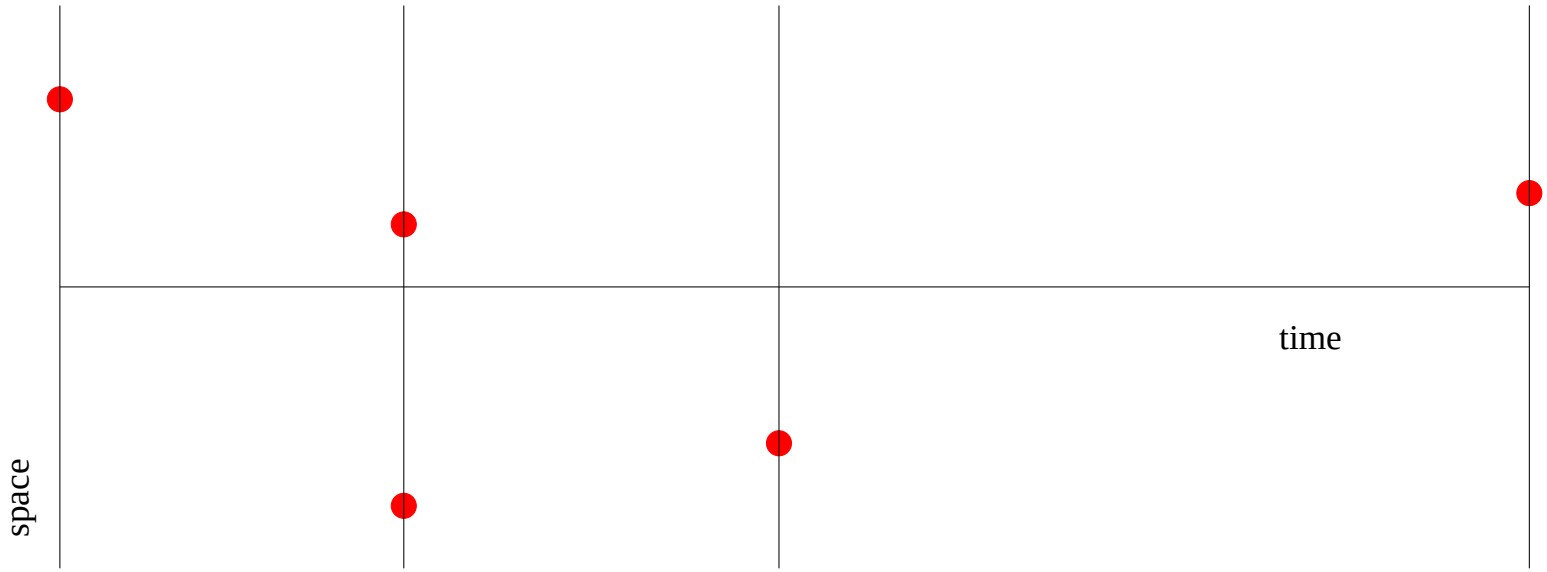

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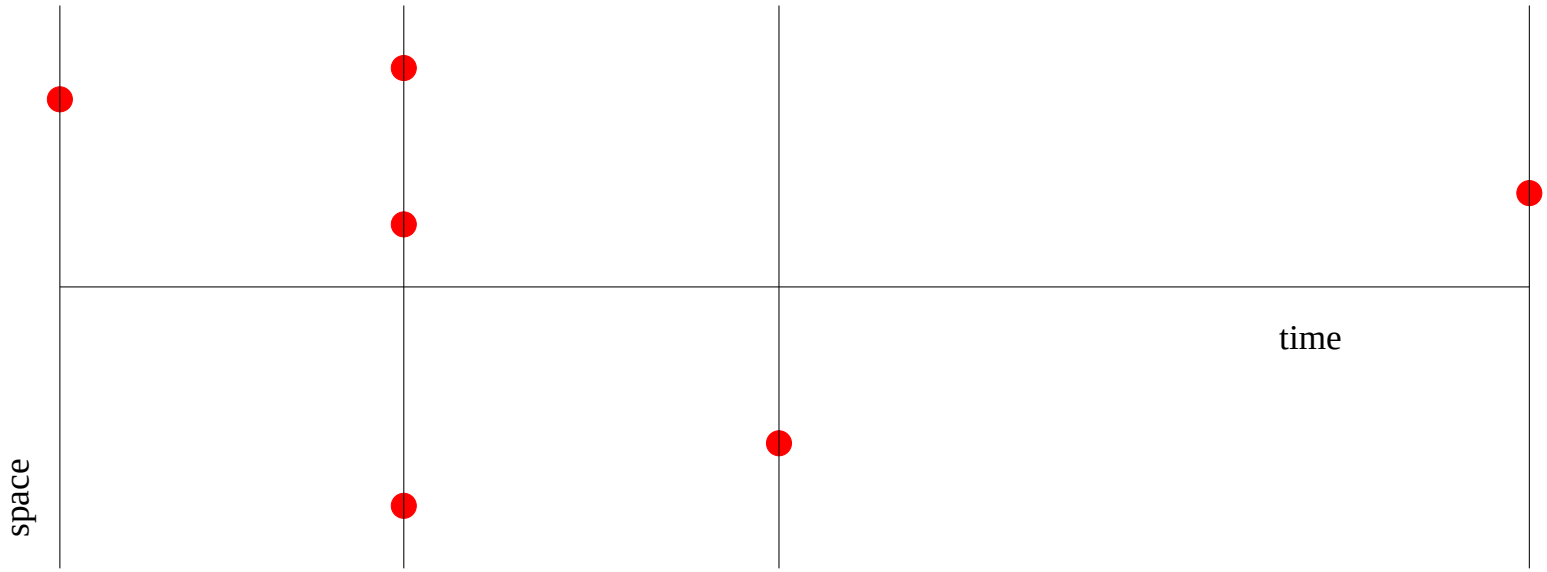

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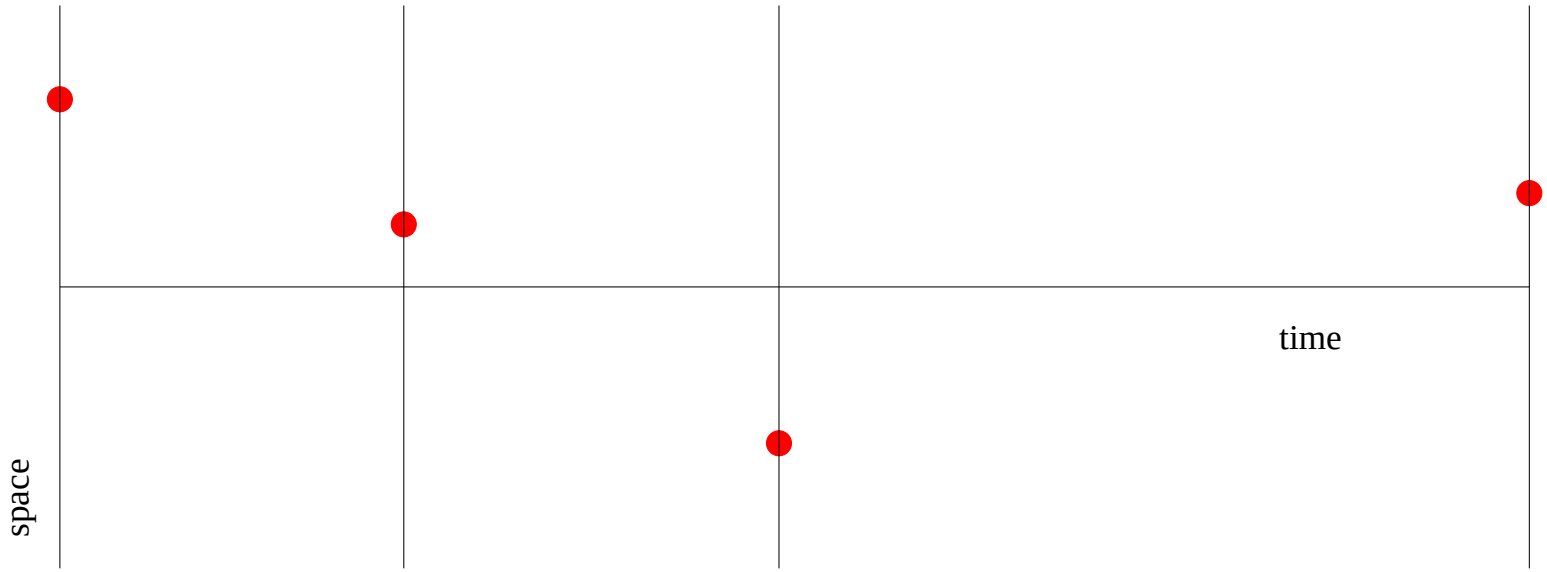

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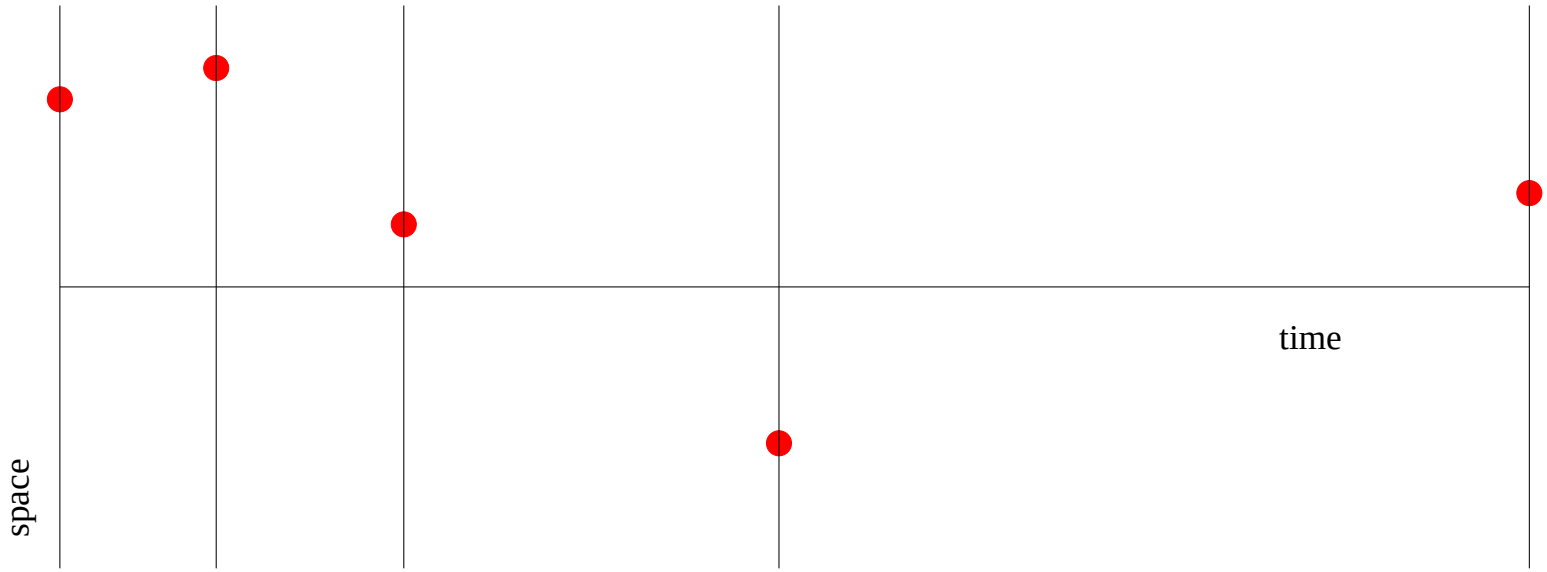

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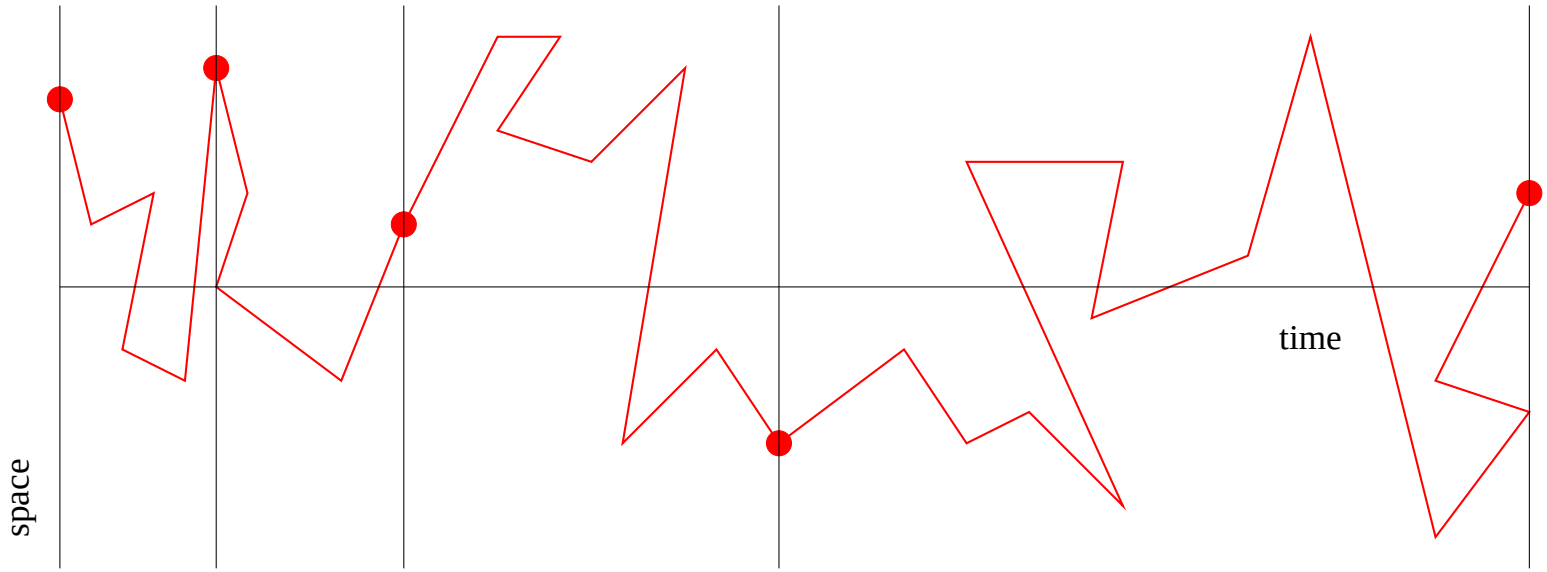

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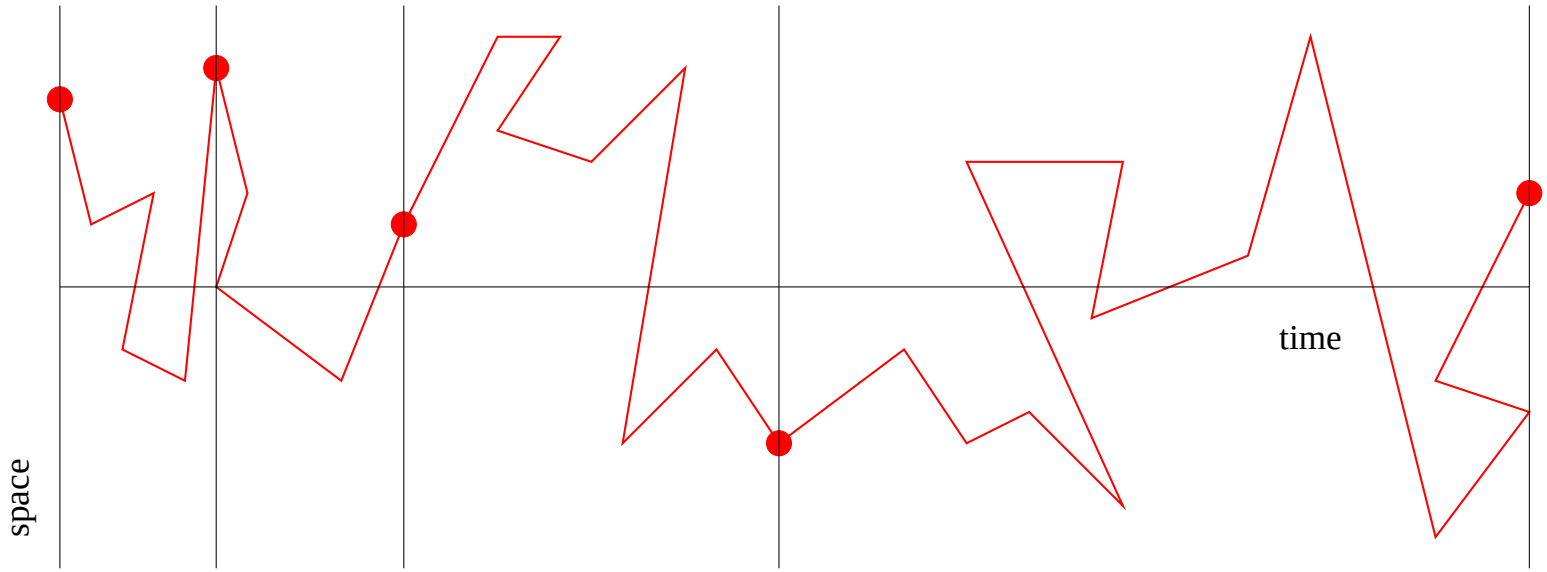


Sum over all paths: path integral

Dirac (1933), Feynman (1948)

A path integral is matrix multiplication

Define $\delta t = T/N_t$ The amplitude for a quantum state $|x_0\rangle$ at initial time 0 to evolve to the state $|x_N\rangle$ at the final time T can be written as

$$\begin{aligned}\langle\alpha_0|\, U(0,T)\,|\alpha_0\rangle \;=\; \sum_{\psi_1,\psi_2,\cdots,\psi_{N-1}} & \langle\alpha_0|\, U((N-1)\delta t, T)\,|\psi_{N-1}\rangle \\ & \langle\psi_{N-1}|\, U(\delta t,(N-1)\delta t)\,|\psi_{N-2}\rangle\cdots \\ & \langle\psi_1|\, U(\delta t,0)\,|\alpha_0\rangle\,,\end{aligned}$$

where we have inserted complete sets of states at the end of each interval. The notation also distinguishes between the states at the end points and the basis states $|\psi_i\rangle$ at the intermediate points. This sum over all intermediate states is called the **path integral**. The choice of the basis states $|\psi_i\rangle$ is up to us, and we can choose them at our convenience.

Diagonalising the evolution operator

If V is unitary and $V^\dagger H V$ is diagonal, then

$$U(0,T) = V \begin{pmatrix} e^{-iE_0 T} & 0 & \cdots \\ 0 & e^{-iE_1 T} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} V^\dagger.$$

We will follow the convetion $E_0 < E_1 < \cdots$. The sum over intermediate states is diagonal, and the Vs act only on the initial and final states to give

$$\langle \alpha_0 | \, U(0,T) \, | \alpha_0 \rangle = |\alpha_0^0|^2 e^{-iE_0 T} + |\alpha_0^1|^2 e^{-iE_1 T} + \cdots$$

Transforming to **Euclidean time**, $it \to t$, one finds

$$\langle \alpha_0 | \, U(0,T) \, | \alpha_0 \rangle = |\alpha_0^0|^2 e^{-E_0 T} + |\alpha_0^1|^2 e^{-E_1 T} + \cdots$$

When $T \gg 1/(E_1 - E_0)$, only the leading term survives. This gives a numerical method for finding the eigenvalues.

The transfer matrix

The Euclidean problem over one lattice step in time is now phrased in terms of the **transfer matrix** —

$$\mathbf{T}(\delta t) = U(0, -i\delta t) = V^\dagger \begin{pmatrix} e^{-E_0\delta t} & 0 & 0 & \cdots \\ 0 & e^{-E_1\delta t} & 0 & \cdots \\ 0 & 0 & e^{-E_2\delta t} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} V.$$

Since $\mathbf{T}$ and H commute, they have the same eigenvectors. If the eigenvalues of $\mathbf{T}$ are called λ_i, then the eigenvalues of the Hamiltonian are $E_i = -(\log \lambda_i)/\delta t$. **Continuum limit**: what is the value of E_i when $\delta t \to 0$.

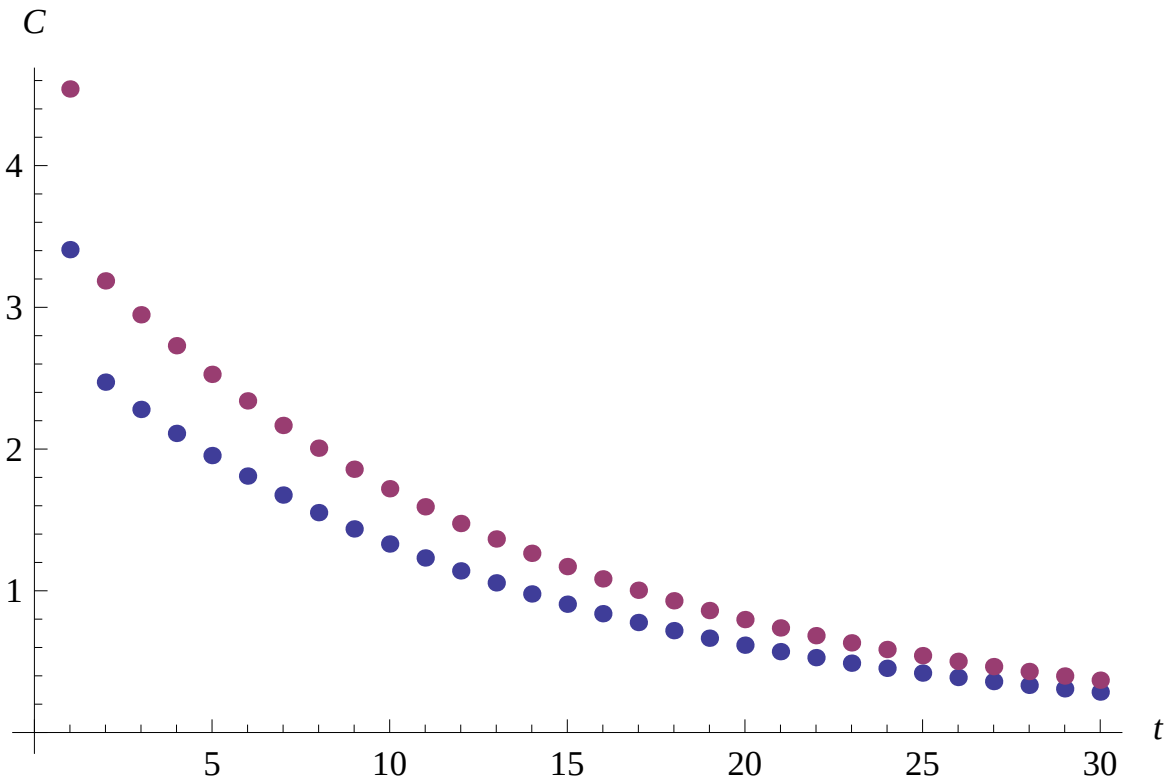
If E_i is finite when $\delta t \to 0$, then $\log \lambda$ must go to zero. The **correlation length** in the problem is $\xi = 1/\log \lambda$, so this must diverge in order to give finite E_i. Therefore the continuum limit corresponds to a **critical point**.

Algorithm for computing energies

For a new formulation of quantum mechanics we have a trivial algorithm for computing the energy. It exploits the simple fact that given a randomly chosen unit vector $|\phi\rangle$, the matrix element $\langle\phi|\,\mathbf{T}^n\,|\phi\rangle$ tends to λ_0^n as $n \to \infty$.

1. Choose a **source**. At one time slice construct a random linear combination of basis states: $|\phi_0\rangle$.
2. Choose a path **configuration**, *i.e.*, a random $|\phi_j\rangle$ on each lattice site $(j\delta t)$ with probability given by $\mathbf{T}$. Construct a measurement of the **correlation function** $C_{0j} = \langle\phi_j\phi_k\rangle$.
3. Repeat step 2 as many times as feasible and construct the mean $\langle C_{0j}\rangle = \langle\phi_0|\,\mathbf{T}^j\,|\phi_0\rangle$ (since $|\phi_j\rangle$ are chosen with appropriate weight, the mean suffices).
4. Plot $log\,\langle C_{0j}\rangle$ against j. At sufficiently large j the slope gives $-E_0\delta t$. Alternatively, find a plateau in the **local masses** $m_j = \log(\langle C_{0,j+1}\rangle\,/\,\langle C_{0j}\rangle)$.

A measurement of a correlation function



Unitarity of U corresponds to a property of **T** called **reflection positivity**. This implies that correlation functions decrease monotonically.

Outline

Quantum field theory

Quantum mechanics of a single particle is a 1-dimensional field theory. The (Euclidean) Feynman path integral is

$$Z = \int \mathcal{D}\mathbf{x} \exp\left[-S[\mathbf{x}]\right],$$

where S is the action. The variable of integration is the quantum state of the system; since we chose to work with a basis of position eigenstates, this appears to be an integral over the position $\mathbf{x}$. This integral is **regularized** by discretizing time.

We extend this to a quantum field theory in dimension D. If the space-time points are labelled by x, and the fields are $\phi(x)$, then the Euclidean partition function is

$$Z = \int \mathcal{D}\phi \exp\left[-S(\phi)\right],$$

where S is the action density and the integrals may be regulated by discretizing space-time.

The lattice and the reciprocal lattice

In the usual perturbative approach to field theory, the computation of any n-point function involves loop integrals which diverge. These are regulated by putting a cutoff Λ on the 4-momentum. When space-time is regulated by discretization, then the lattice spacing a provides the cutoff $\Lambda = 1/a$.
We will take the discretization of space-time to be a regular hypercubic lattice, with sites denoted by a vector of integers $x = a\mathbf{j}$. When fields are placed on such a lattice, $\phi(x)$, the momenta are no longer continuous, but form a reciprocal lattice.

$$k = \frac{2\pi}{a}\mathbf{l},$$

where the $\mathbf{l}$ are integers. The physics at all points on the reciprocal lattice are equivalent.

Fourier transforms and the Brillouin zone

In practice our lattice will not be infinite, but a finite hypercube with, say, N^D sites. At each site, x, on the lattice, let us put a complex number $\phi(x)$. We can put periodic boundary conditions, $\phi(x) = \phi(x + \hat{\mu}N)$. Next, we can make Fourier transforms

$$\phi(k) = \sum_x \phi(x) \mathrm{e}^{ik\cdot x}, \quad \phi(x) = \frac{1}{N^D} \sum_k \phi(k) \mathrm{e}^{-ik\cdot x},$$

where $k = 2\pi \mathbf{l}/N$, and $\mathbf{l}$ have components taking integer values between 0 and N or $-N/2$ and $N/2$. One sees that the boundary condition allows components only inside the **Brillouin zone**, *i.e.*, the region $-\pi/a \leq k_\mu \leq \pi/a$. Any k outside this is mapped back inside by the periodicity of the lattice.

The completeness of the Fourier basis implies

$$\frac{1}{N^D} \sum_x \mathrm{e}^{-iq\cdot x} = \delta_{0q}.$$

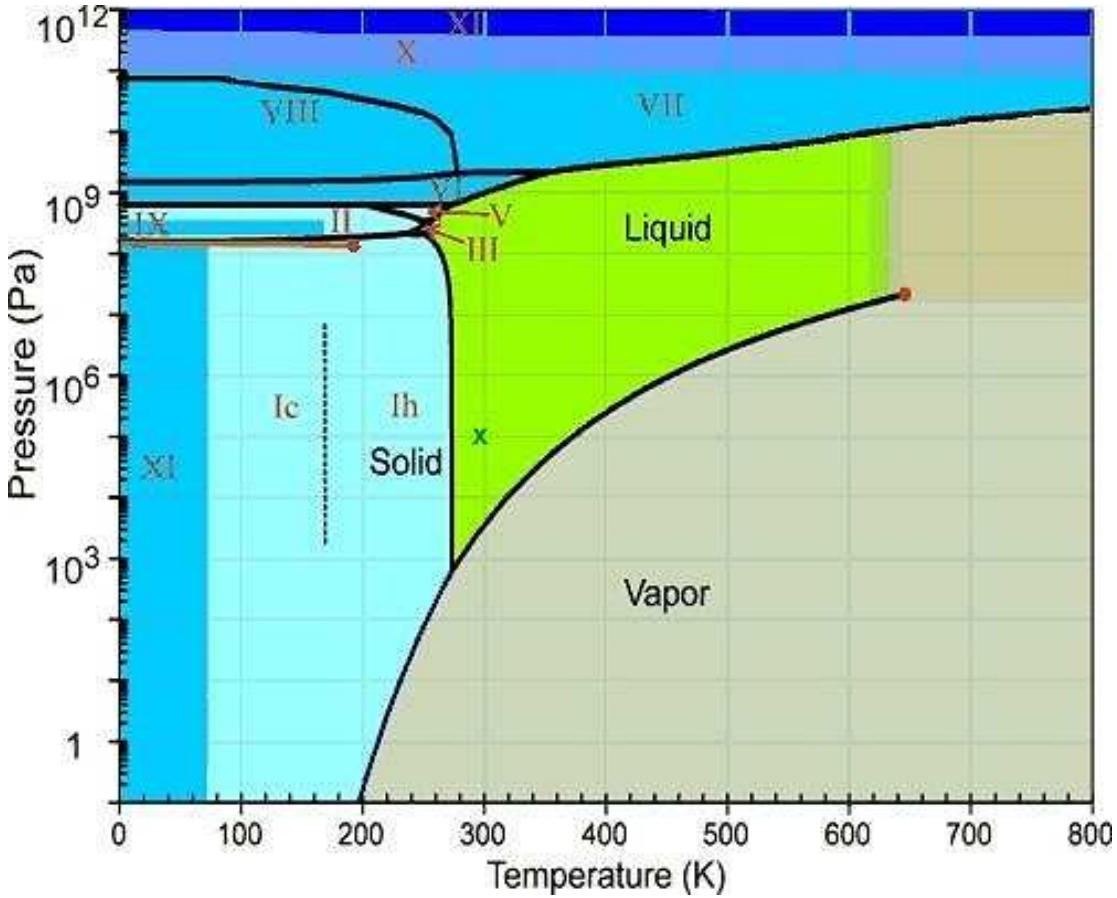
The great unification

The renormalization procedure will be to take the continuum limit $a \to 0$ (*i.e.*, $\Lambda \to \infty$) keeping some physical quantity fixed, such as a mass, m. If this is fixed in physical units, then in lattice units it must diverge as $a \to 0$. This corresponds to a second order phase transition on the lattice.

A lattice field theory in Euclidean time and D dimensions of space is exactly the same as a statistical mechanics on a $D+1$ dimensional lattice. Here is the precise analogy—

Action	$\leftrightarrow$	Transfer matrix
Path integral	$\leftrightarrow$	Partition function
2-point function	$\leftrightarrow$	Correlation function
Continuum limit	$\leftrightarrow$	2nd order phase transition
Unitarity	$\leftrightarrow$	Reflection positivity

Phase transitions



Normal single phase behaviour, two-phase coexistence (first order phase transitions), three-phase coexistence (triple points), critical point (second order phase transitions).

Divergences and critical exponents

Scaling of free energy at a critical point (T_c, P_c)

$$F(T,P) = p^a f\left(\frac{t}{p^b}\right) \qquad p = P - P_c,\ t = T - T_c.$$

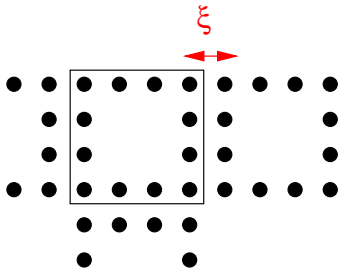
The scaling form implies power law divergence of the specific heat ($t^{-\alpha}$), order parameter ($t^{-\beta}$ and $p^{-\delta}$) and order parameter susceptibility ($t^{-\gamma}$) at the critical point. There are various relations between these **critical exponents** since the scaling form contains only two intensive variables (see [3]).
Also there is scaling of the correlation function—

$$G(r, T, p=0) = r^{(2-d-\eta)} g\left(\frac{r}{t^{-\nu}}\right).$$

At the critical point the correlation length diverges. The scaling form implies that

$$\xi \propto t^{-\nu}, \qquad G(r, t=0, p=0) \propto \frac{1}{r^{\eta+d-2}}.$$

Coarse graining and the Renormalization Group



If the correlation length of a system is ξ, then one can try to define **coarse grained** variables by summing over blocks of sites. When the block size becomes larger than ξ, the problem simplifies.

A renormalization group (RG) transformation is the following—

1. Coarse grain by summing the field over a block of size ζa, and scale the sum to the same range as the original fields. This changes $a \to \zeta a$.
2. Find the Hamiltonian of the coarse grained field which reproduces the thermodynamics of the original system. The couplings in the Hamiltonians "flow" $g(a) \to g(\zeta a) = g'$.
3. The flow follows the Callan-Symanzik **beta-function**

$$B(g) = -\frac{\partial g}{\partial \zeta},$$

(note the sign). A **fixed point** of the RG has $B(g) = 0$.

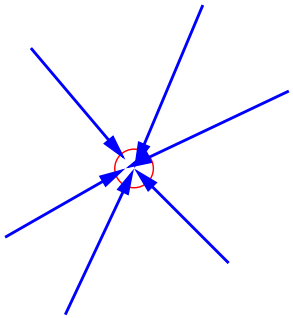
Linearized Renormalization Group transformation

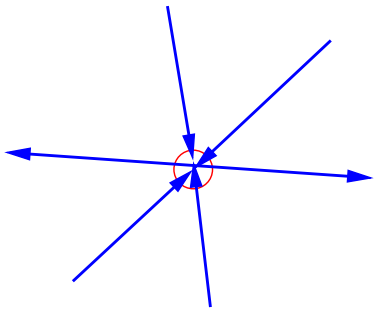
Assume that there are multiple couplings G_i with beta-functions B_i. At the critical point the values are G_i^c. Define $g_i = G_i - G_i^c$. Then,

$$B_i(G_1, G_2, \cdots) = \sum_j B_{ij} g_j + \mathcal{O}(g^2).$$

Diagonalize the matrix B whose elements are B_{ij}. In cases of interest the eigenvalues, y, turn out to be real. Under an RG transformation by a scaling factor ζ an eigenvector of B scales as $v \to \zeta^{-y} v$ The eigenvalues are called **anomalous dimensions**. Eigenvectors corresponding to positive eigenvalues scale away to zero under RG, and so are called **irrelevant operators**. For negative eigenvalues, the eigenvectors are called **relevant operators**. those with zero eigenvalues are called **marginal operators**. Marginal operators correspond to logarithmic scaling.

Renormalization Group trajectories





Fixed points: $\xi = 0$ (stable) or $\xi = \infty$ (unstable).

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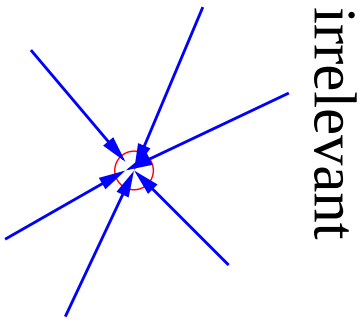

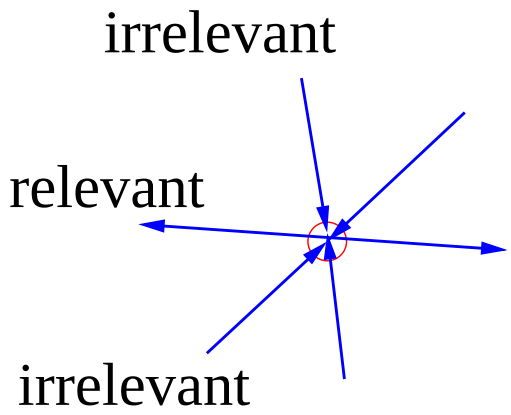

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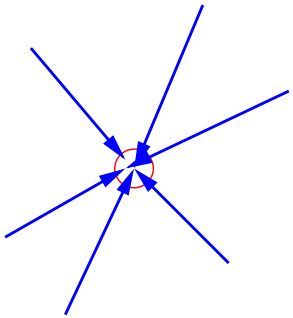

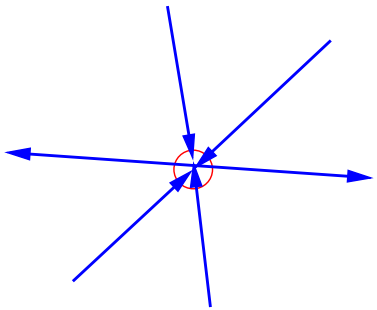

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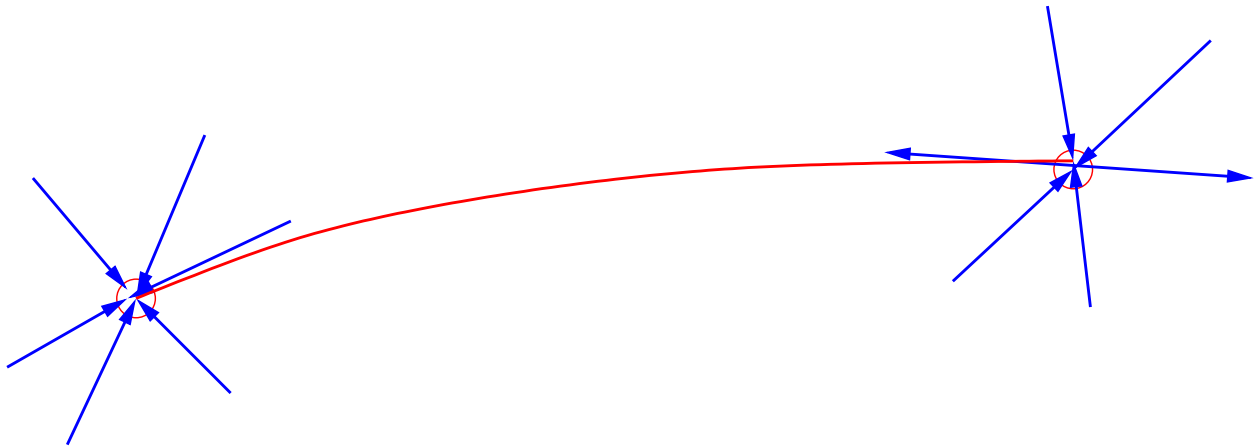

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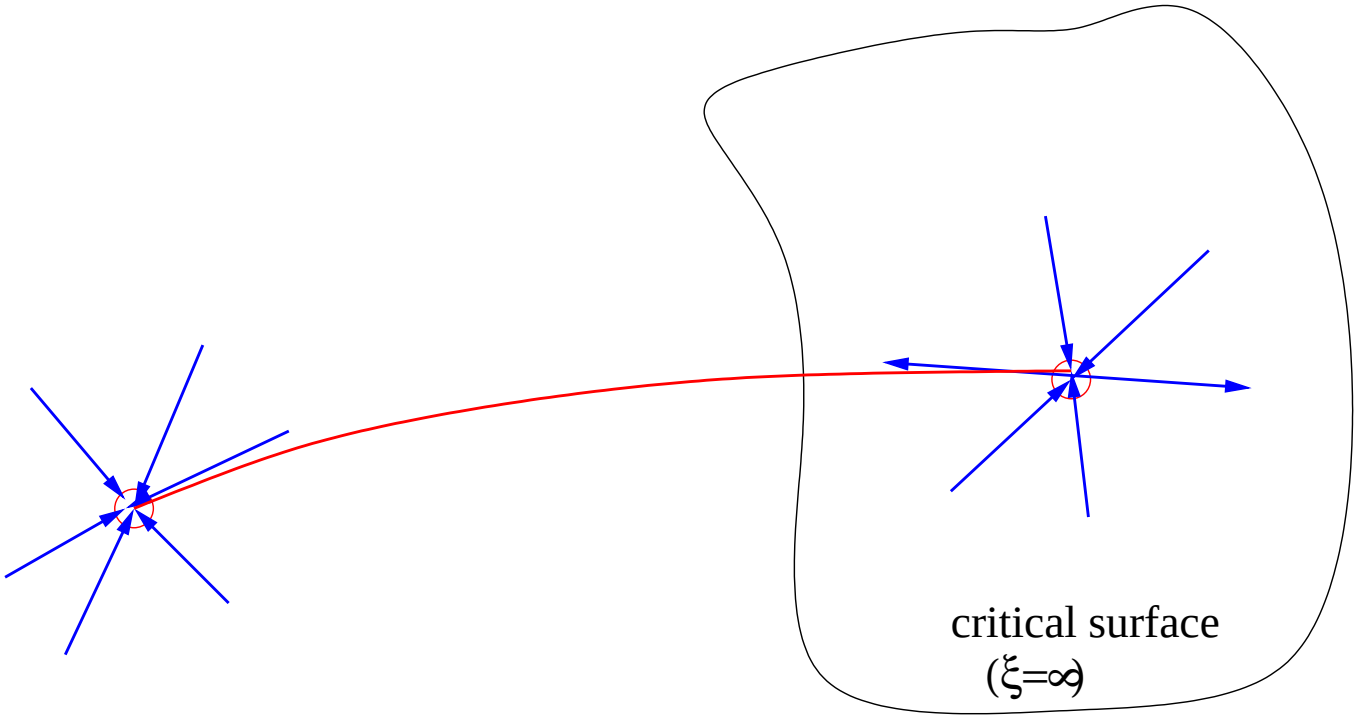

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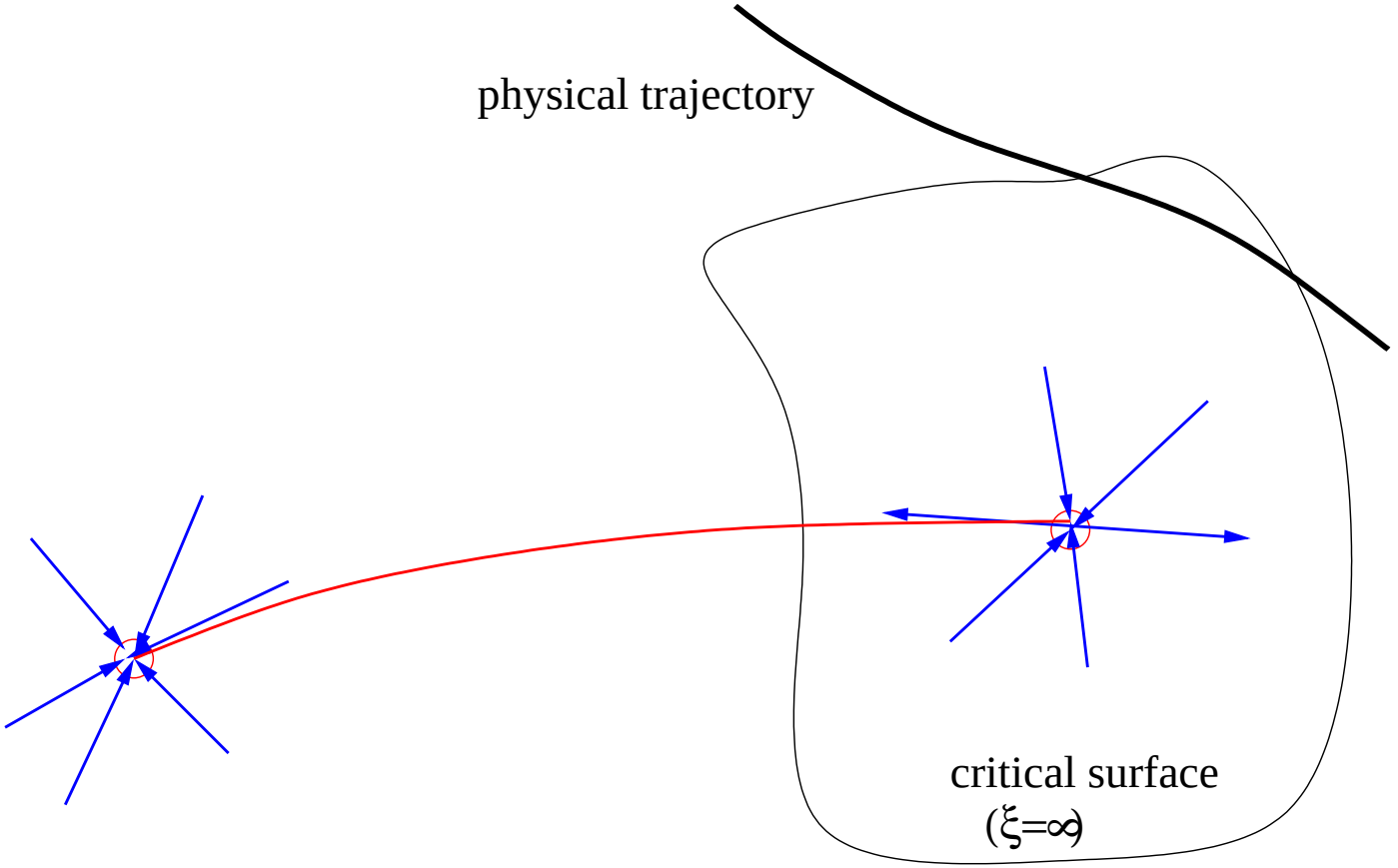

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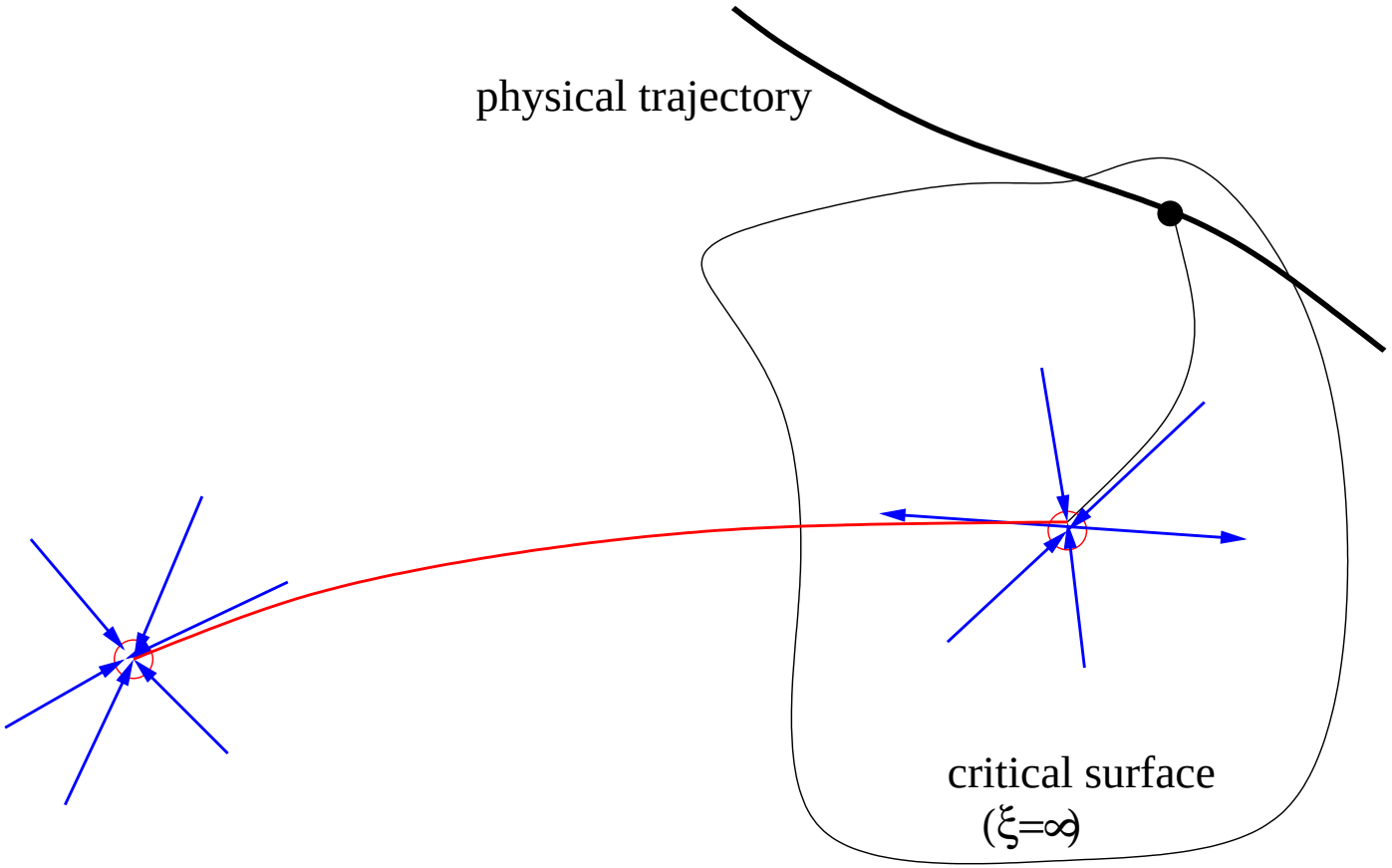

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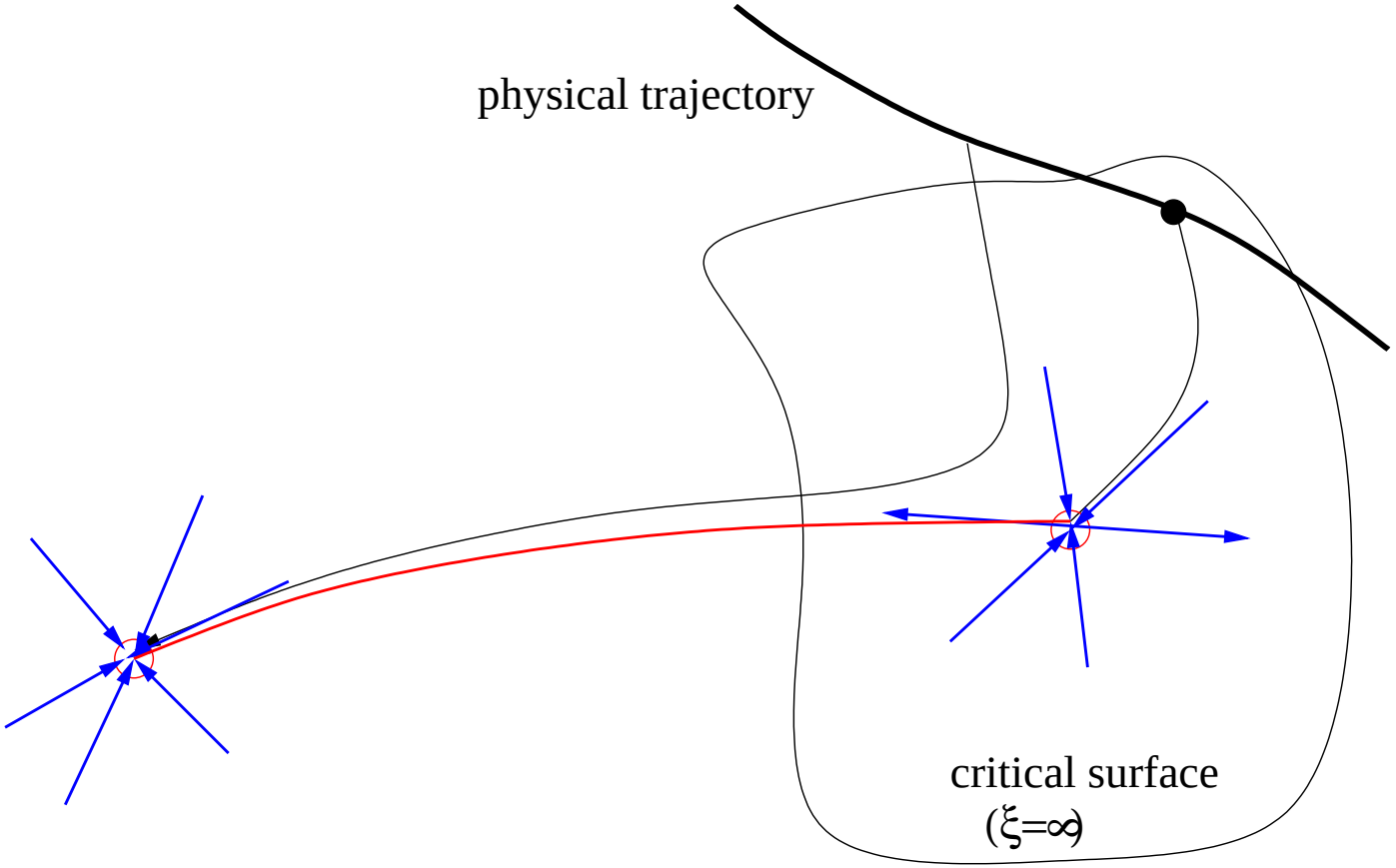

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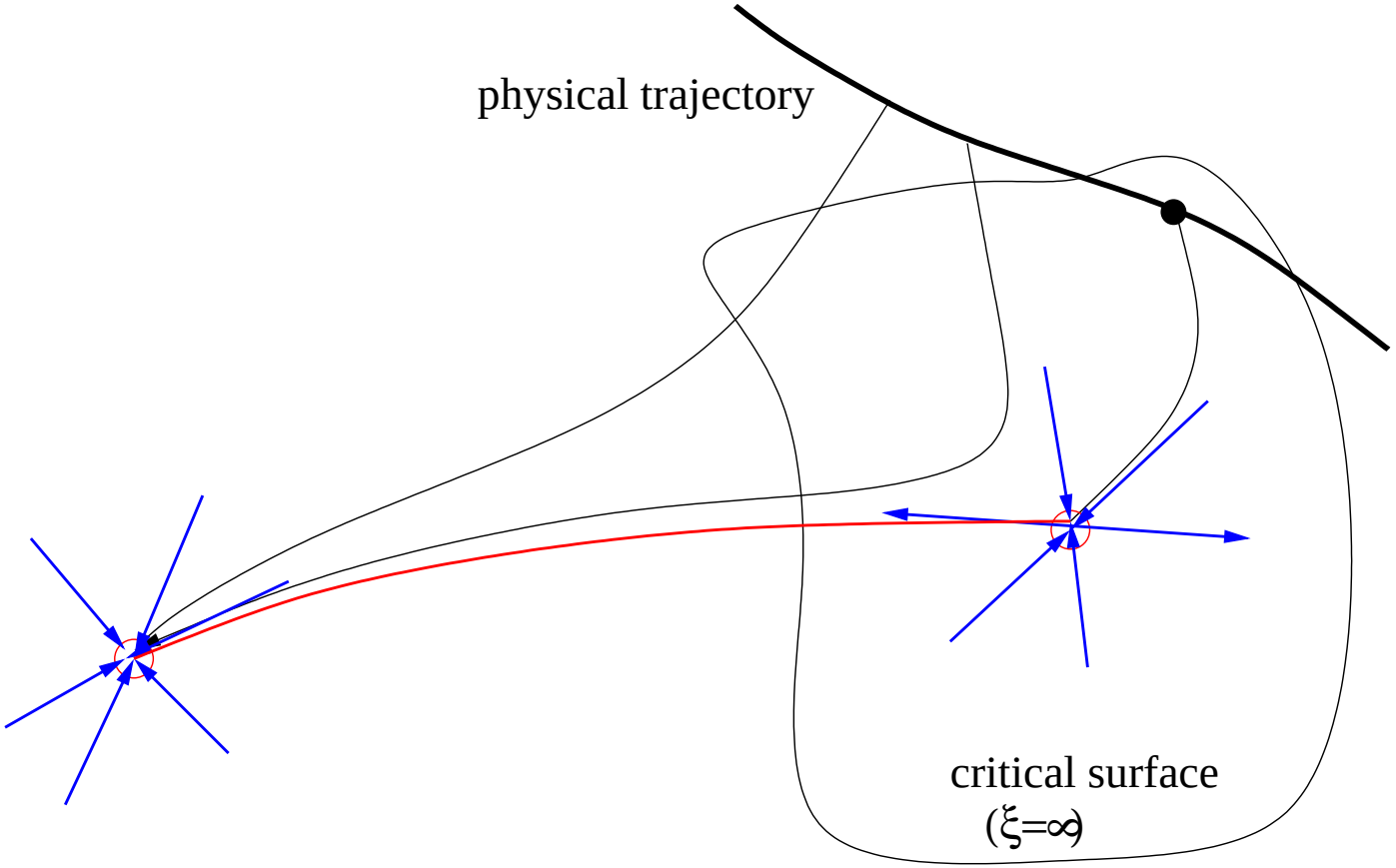
Renormalization Group trajectories



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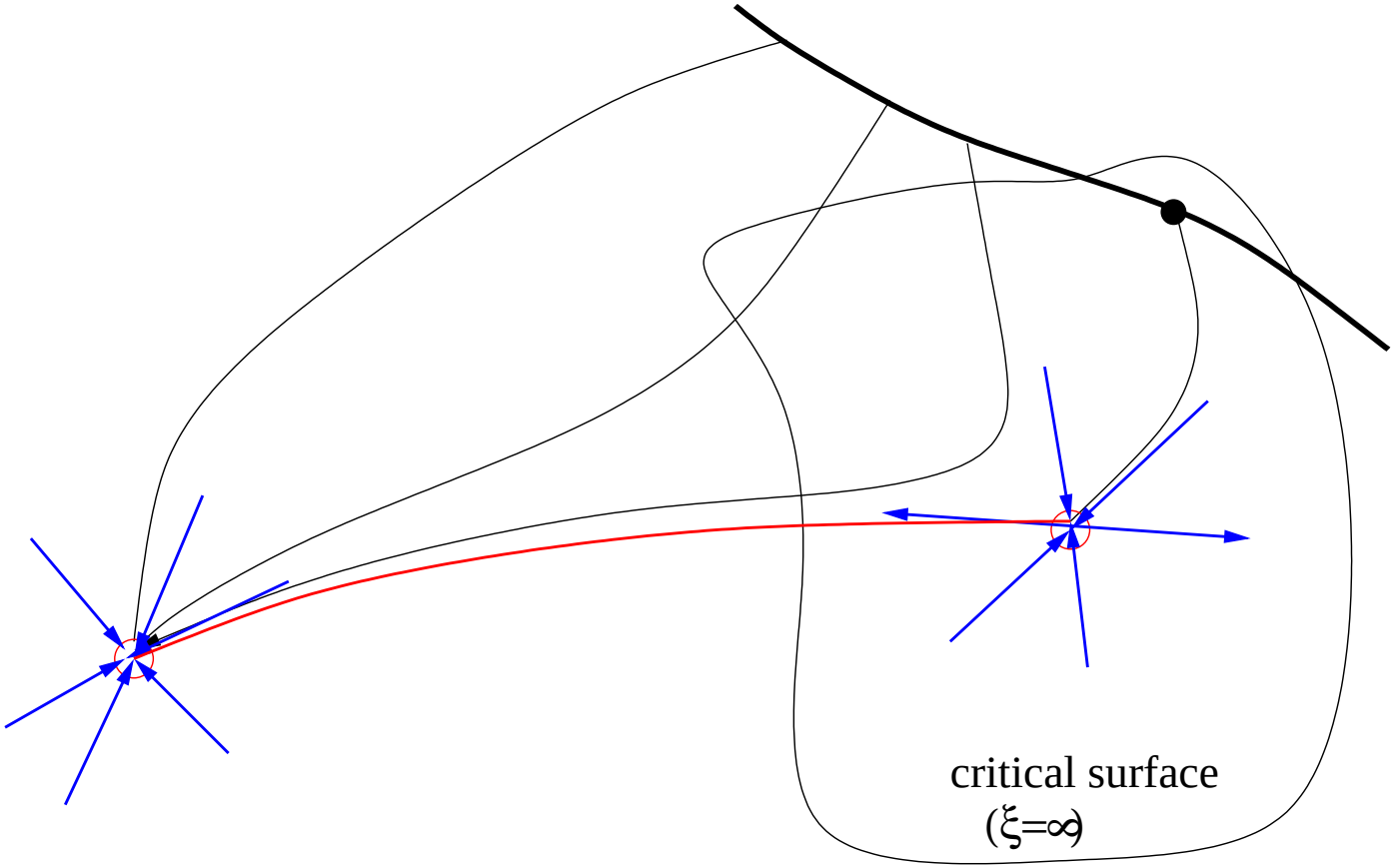
Renormalization Group trajectories



Fixed points: $\xi = 0$ (stable) or $\xi = \infty$ (unstable).

Outline

Probability theory as a trivial case of field theory

Generate random variables x with a probability distribution $P(x)$.
We can always shift our definition of x so that $\langle x \rangle = 0$.
It is useful to introduce the moment generating function

$$Z(j) = \sum_n \langle x^n \rangle \frac{j^n}{n!} = \int dx e^{xj} P(x).$$

The derivatives of $Z(j)$ give the moments. Now define the **characteristic function** $F(j) = \log Z(j)$. The derivatives give cumulants. We will use the notation

$$[x^2] = \left. \frac{d^2 F(j)}{dj^2} \right|_{j=0} = \langle x^2 \rangle - \langle x \rangle^2 = \sigma^2.$$

The Hamiltonian of statistical mechanics is analogous to $h(x) = \log P(x)$. Then $Z(j)$ is the partition function, and $F(j)$ the free energy. The derivatives of F give expectations of connected parts; these are the cumulants.

Coarse graining and the RG

Take a group of N random numbers, x_i, and define their mean

$$\overline{x} = \frac{1}{N}\sum_i x_i.$$

The $\overline{x}$ are **coarse grained** random variables. A standard question in probability theory is the distribution of these coarse grained variables. Clearly this is a question in RG.
We need to compute the coarse grained characteristic function $F_N(j)$. First,

$$\begin{aligned} Z_N(j) &= \int d\overline{x}\, \mathrm{e}^{\overline{x}j}\delta\left(\overline{x} - \frac{1}{N}\sum_{i=1}^{N} x_i\right)\prod_{i=1}^{N} dx_i \mathrm{e}^{h(x_i)} \\ &= [Z(j/N)]^N. \qquad \text{implies} \\ F_N(j) &= NF\left(\frac{j}{N}\right). \end{aligned}$$

The central limit theorem: a fixed point theorem

Since

$$F(j) = \sigma^2 \frac{j^2}{2!} + [x^3]\frac{j^3}{3!} + [x^4]\frac{j^4}{4!} + \cdots,$$

we find the RG flow gives

$$NF\left(\frac{j}{N}\right) = \frac{\sigma^2}{N}\,\frac{j^2}{2!} + \frac{[x^3]}{N^2}\,\frac{j^3}{3!} + \frac{[x^4]}{N^3}\,\frac{j^4}{4!} + \cdots.$$

In the limit, since all the higher cumulants scale to zero much faster, we find that the RG flows to the $\mathrm{Gaussian\ fixed\ point}$ $F_N(j) = \sigma^2 j^2/(2N)$. This is the content of the **central limit theorem**.

Subtleties may occur if $\sigma^2 = 0$, with extensions to the case when all the cumulants up to some order are zero. Other subtleties arise when the distributions are fat-tailed and all the cumulants diverge. Other RG methods are needed for these special cases.

Generating functions in field theory

Any integral with a non-negative integrand can be treated as a $D = 0$ field theory. Some of the tricks one plays with integrals can be generalized to field theories.
In any field theory it is useful to extend the path integral to a generating functional of correlation functions—

$$Z[J] = \int \mathcal{D}\phi \exp\left[- \int d^D x \; S(\phi) + J(x)\phi(x) \right],$$

The connected parts of correlation functions are recovered as usual by taking functional derivatives—

$$C(z, z') = \frac{1}{Z[J]} \frac{\delta^2 Z[J]}{\delta J(z) \delta J(z')} \bigg|_{J=0}.$$

These are clear generalization of the notions of the moment generating function and the characteristic function.

Outline

The Ising model

The Ising model on a one-dimensional lattice contains a "spin" variable, $\sigma_i = \pm 1$ at each site, i, of the lattice. The Hamiltonian is

$$H = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1}.$$

We may put periodic boundary conditions on the lattice through the condition that $\sigma_{N+1} = \sigma_1$. We write $\beta = J/T$.
This can be solved by introducing the transfer matrix [2]

$$T(\beta) = \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}.$$

Since $Z(\beta) = \mathrm{Tr}\, T^N$, the eigenvalues of the transfer matrix completely specify the solution. We find

$$Z(\beta) = 2^N \left[(\cosh \beta)^N + (\sinh \beta)^N \right].$$

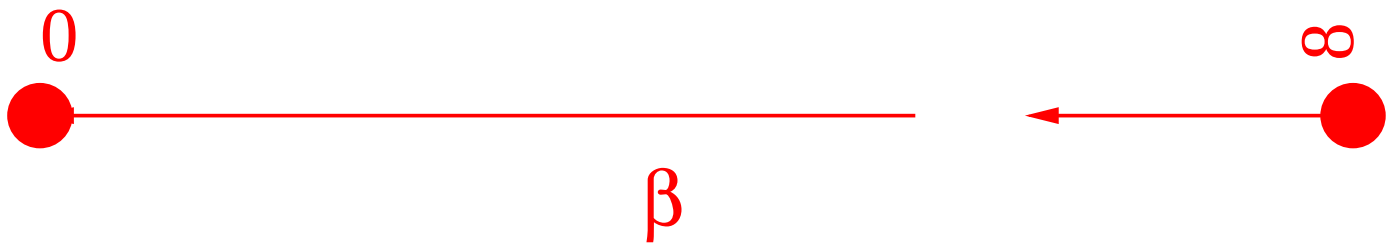
The system becomes ordered only in the limit $\beta \to \infty$.

Coarse graining and fixed points

However, we can also perform a coarse graining with $\zeta = 2$. Since

$$T^2(\beta) = \begin{pmatrix} 2\cosh 2\beta & 2 \\ 2 & 2\cosh 2\beta \end{pmatrix} = 2\sqrt{\cosh 2\beta} \begin{pmatrix} z & 1/z \\ 1/z & z \end{pmatrix},$$

where $z = \sqrt{(\cosh 2\beta)}$. This transfer matrix must be reproduced by the new Hamiltonian (with coupling β'). So the Callan-Symanzik equation is $\beta' = \log z$. The fixed points are solutions of $\beta = \log z$. There are 2 solutions: $\beta = 0$ and ∞. $\beta = 0$ corresponds to $J = 0$, *i.e.*, the free theory. This is an attractive fixed point. $\beta = \infty$ is a repulsive fixed point, because any finite β is attracted to $\beta = 0$.



The RG flow for the 1-d Ising model is particularly simple.

Outline

Power counting

Consider the relativistic quantum field theory of a single real scalar field ϕ. The Lagrangian density, $\mathcal{L}$, can be written as a polynomial in the field and its derivatives. One usually encounters the terms

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}M^2\phi^2 + \frac{g_3}{3!}\phi^3 + \frac{g_4}{4!}\phi^4 + \cdots$$

Let us count the mass dimensions of the fields in units of a length L or a momentum Λ. Since the action is dimensionless, $[\mathcal{L}] = L^{-D} = \Lambda^D$. The kinetic term shows that

$$[\phi] = L^{1-D/2} = \Lambda^{D/2-1}.$$

The couplings have dimensions

$$\begin{aligned} [M^2] &= L^{-2} = \Lambda^2, \\ [g_3] &= L^{(D-6)/2} = \Lambda^{(6-D)/2}, \\ [g_4] &= L^{D-4} = \Lambda^{4-D}. \end{aligned}$$

The upper critical dimension

For each operator in the theory there is a certain dimension at which the coupling is marginal. This is called the **upper critical dimension**, D_u. The coupling g_r corresponding to the operator ϕ^r has

$$D_u = \frac{2r}{r-2}.$$

The mass is a relevant coupling in all dimensions, g_3 is relevant below $D_u = 6$, g_4 below $D_u = 4$. All other operators are irrelevant in $D = 4$. Derivative couplings are relevant (the kinetic term is marginal) in all dimensions.

Bogoliubov and Shirkov [5] set out power counting rules for divergences of loop integrals. It turns out that for $D > D_u$ an operator is unrenormalizable; at $D = D_c$ the operator gives a renormalizable contribution, and for $D < D_u$ the theory is super-renormalizable.

Field theory is not exactly statistical mechanics

Divergences in statistical mechanics are due to long-distance physics. In field theory they are due to short distance physics. Therefore, in statistical mechanics it is the power of L which counts. For field theory, it is instead the power of Λ which determines which terms are important.
This is also reflected in the differences in the physical meaning of RG transformations in the two cases. The critical point in statistical mechanics is a point in the phase diagram where the correlation length actually becomes infinite. In field theory the critical point can be reached for any mass of the particle by scaling the lattice spacing to zero (momentum cutoff to infinity).

$$\begin{aligned} \text{irrelevant} &\leftrightarrow \text{un-renormalizable} \\ \text{marginal} &\leftrightarrow \text{renormalizable} \\ \text{relevant} &\leftrightarrow \text{super-renormalizable} \end{aligned}$$

Outline

Scalar Field Theory

The continuum Lagrangian for a single component real scalar field theory can be easily written for the lattice

$$\begin{aligned} S &= a^D \sum_x \frac{1}{2a^2} \sum_\mu [\phi(x+\hat{\mu}a) - \phi(x)]^2 + \frac{1}{2} m^2 \phi^2(x) + V(\phi) \\ &= \sum_x M^2 \phi^2(x) - \sum_\mu [\phi(x)\phi(x+\hat{\mu})] + V(\phi). \end{aligned}$$

In the first line we have replaced derivatives by the forward difference, ∇, on the lattice, and kept dimensional variables explicit. The notation is that x denotes a lattice site, μ one of the D directions, $\hat{\mu}$ an unit vector in that direction and a the lattice spacing. In the second line we have absorbed appropriate powers of a into every variable, written out the expressions in dimensionless units and then set $a = 1$. Note that $M^2 = D + m^2 a^2/2$.
By the earlier power-counting, it suffices to take $V(\phi) = g_4 \phi^4/4!$ in $D \geq 4$.

Notation for lattice theories

1. The lattice spacing will always be written as a except when we use units where $a = 1$.
2. We will use the notation x, y, *etc.*, to denote either a point in continuum space-time, or on the lattice.
3. Fourier transforms on N^D lattices are

$$\phi(k) = \sum_x \phi(x) \mathrm{e}^{ik\cdot x}, \quad \phi(x) = \frac{1}{N^D} \sum_k \phi(k) \mathrm{e}^{-ik\cdot x}, \quad k = 2\pi \mathbf{i}/N,$$

where reciprocal lattice points $\mathbf{i}$ have components taking values between 0 and N or $-N/2$ and $N/2$. In other words, the Brillouin zone contains momenta between $\pm\pi$. The completeness of the Fourier basis implies

$$\frac{1}{N^D} \sum_x \mathrm{e}^{-iq\cdot x} = \delta_{0q}.$$

Free scalar field theory

On a lattice of size N^D the free theory, $V = 0$, can be completely solved by Fourier transformation. The action becomes

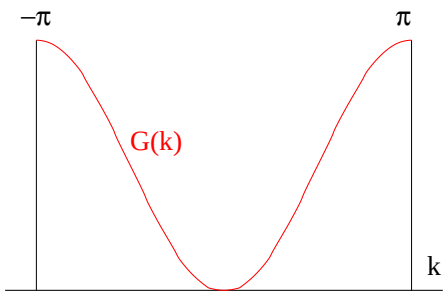
$$S = \sum_k \frac{1}{2} \left[m^2 - \sum_\mu (1 - \cos k_\mu) \right] \phi^2(k).$$

Since the Fourier transform is an unitary transformation of fields, the Jacobian for going from $\phi(x)$ to $\phi(k)$ is unity. Therefore, the Fourier transformation gives a set of decoupled Gaussian integrals, and

$$\begin{aligned} Z[\beta] &= \int \prod_k d\phi(k) \mathrm{e}^{-\beta S} = \prod_k \left[\frac{\beta}{2} (m^2 + \sum_\mu \sin^2 \frac{k_\mu}{2}) \right]^{-1/2} \\ &= \frac{1}{\sqrt{\det \frac{\beta}{2} (\nabla^2 + m^2)}}, \end{aligned}$$

where ∇ is the forward difference operator.

Low energy modes and Symanzik improvement



When $m = 0$ the two-point function of scalar field theory, G, vanishes at the center of the Brillouin zone and is maximum at the edges. Inside the Brillouin zone there is only one long distance mode when $a \to 0$.

At small k one has $G \simeq k^2[1 + \mathcal{O}(k^2a^2)]$. **Symanzik improvement** consists of improving the a-dependence at tree level at finite lattice spacing by adding irrelevant terms to the lattice action. For a scalar field, one can write

$$G(k) = \frac{4}{3}(1 - \cos k_\mu) - \frac{1}{12}(1 - \cos 2k_\mu) = \frac{1}{2}k^2 + \mathcal{O}(k^6).$$

Hence, by removing the k^4 terms, one has an improved action. Clearly this is achieved by taking the forward difference and the two-step forward difference with appropriate coefficients.

The interacting theory

The standard form of the action for the scalar theory is

$$S = \sum_x \left[V(\phi) - \kappa \sum_\mu \phi(x)\phi(x+\hat{\mu}) \right], \quad V(\phi) = \lambda(\phi^2-1)^2 - \phi^2.$$

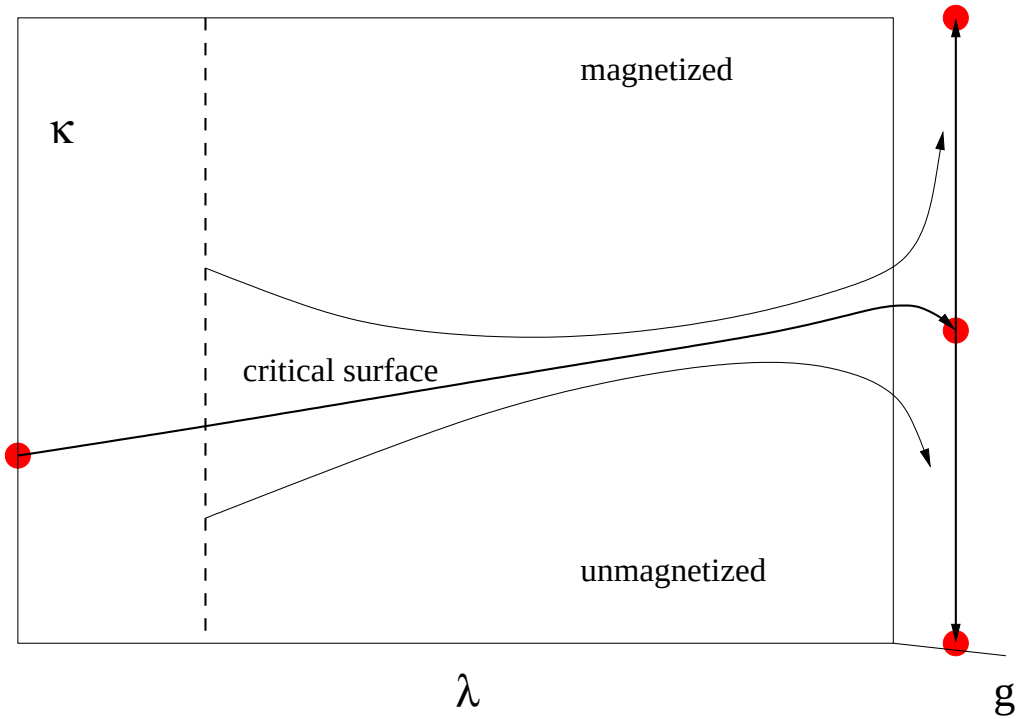
When the **hopping parameter** κ is large we may expand around the free field limit. This is lattice perturbation theory. In the limit when $\kappa \to 0$ we may make a hopping parameter expansion around a solution in which the sites are decoupled.
When $\lambda \to \infty$ the field at the scale of the cutoff must sit at the minimum of the potential, so the model looks like the Ising model. From our earlier discussion, we expect that the critical exponents of scalar field theory must be the same as that of the Ising model, *i.e.*, the two are in the same **universality class**.

Monte Carlo simulations

In general the theory is investigated by Monte Carlo simulations. The algorithm is the following—

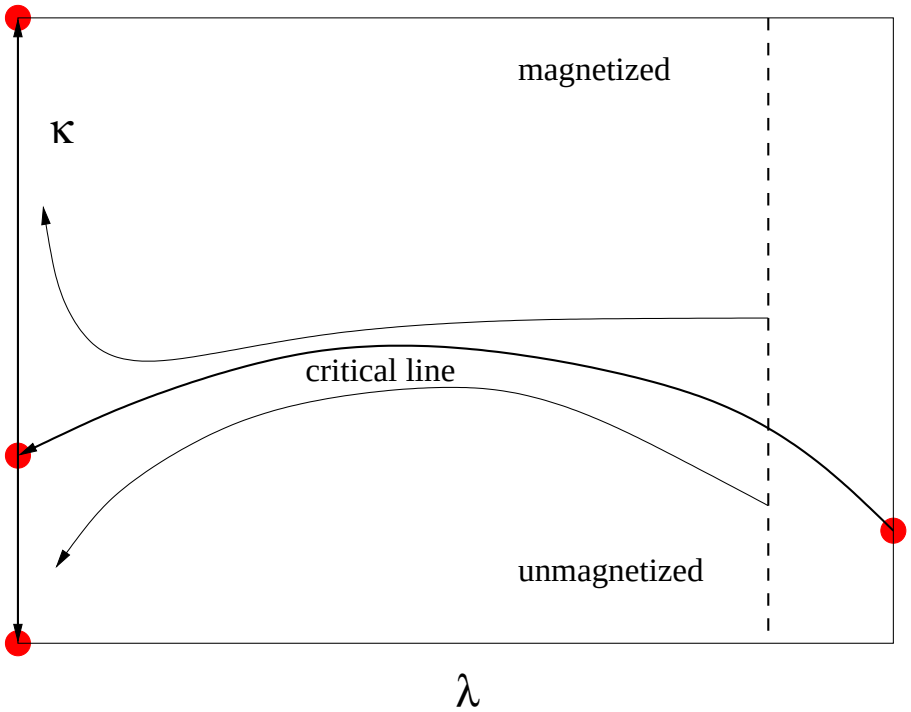
1. Start from a randomly generated **configuration of fields**, $\phi(x)$, on the lattice.
2. At one lattice site, x, make a random suggestion for a new value of the field, $\phi'(x)$.
3. Make a **Metropolis choice** as follows. If the change in the action, ΔS, due to the change in the field is negative, or $\exp(-\beta\Delta S)$ is smaller than a random number r (uniformly distributed between 0 and 1) then accept the suggestion. Otherwise reject it.
4. **Sweep** through every site of the lattice repeating steps 2, 3.
5. At the end of each sweep make **measurements** of the moments of the field variables.
6. Repeat from step 2 as many times as the computational budget allows.

Bosons in $D = 2$



The theory of interacting bosons in $D = 2$ has a non-trivial critical point corresponding to the Ising model. RG trajectories lying anywhere on the critical surface are attracted to this. Since the scalar field in $D = 2$ is dimensionless, an infinite number of couplings, g, in addition to κ and λ, need to be tuned to get to it.

Triviality of the Higgs in $D = 4$



In $D = 4$ the only attractive point on the critical surface has $\lambda = 0$. Since all RG trajectories are attracted to $\lambda = 0$, close to the continuum limit perturbation theory can be used to examine the beta-function. (**M. Luscher and P. Weisz, *Nucl. Phys.*, B 290, 25, 1987).**

Outline

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