

Perturbation theory

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Exact extension of the Hamilton-Jacobi method

Write the Hamiltonian of the problem at hand, H , as the sum of the Hamiltonian of an **integrable system**, H_0 , and the remaining terms,

$$H(q_i, p_i, t) = H_0(q_i, p_i, t) + \Delta H(q_i, p_i, t).$$

All three Hamiltonians, H , H_0 and ΔH act on the same phase space. We know a canonical transformation $\{q_i, p_i\} \rightarrow \{\beta_i, \alpha_i\}$ generated by $S(q_i, \alpha_i, t)$ which solves the dynamics of H_0 .

The transformation generated by S is canonical irrespective of the Hamiltonian, so the Hamiltonian of the perturbed problem is

$$K(\beta_i, \alpha_i, t) = H_0 + \frac{\partial S}{\partial t} + \Delta H = \Delta H(\beta_i, \alpha_i, t).$$

As a result, the equations of motion of the new coordinates and momenta are

$$\dot{\alpha}_i = -\frac{\partial \Delta H}{\partial \beta_i}, \quad \text{and} \quad \dot{\beta}_i = \frac{\partial \Delta H}{\partial \alpha_i}.$$

A simple example: first step

For the harmonic oscillator problem, use the free particle Hamiltonian as the integrable system, *i.e.*, $H_0 = p^2/2m$. So the **Hamilton-Jacobi equation** for this problem is

$$\frac{1}{2m} \left(\frac{\partial S}{\partial x} \right)^2 + \frac{\partial S}{\partial t} = 0,$$

which, we know, has the solution $S = \alpha x - \alpha^2 t/(2m)$. The transformed momentum is α . So the new coordinate is

$$\beta = \frac{\partial S}{\partial \alpha} = x - \frac{\alpha t}{m}.$$

As we already know, the new coordinates are precisely the initial conditions on p and x . We can exhibit this by inverting the equation for β to get

$$x = \beta + \frac{\alpha t}{m}.$$

A simple example: second step

The perturbation Hamiltonian is $V = m\omega^2 x^2/2$. In terms of the new coordinates, we have

$$\Delta H(\beta, \alpha, t) = \frac{m\omega^2}{2} \left(\beta + \frac{\alpha t}{m} \right)^2.$$

As a result, the equations of motion now become

$$\dot{\alpha} = -m\omega^2 \left(\beta + \frac{\alpha t}{m} \right), \quad \text{and} \quad \dot{\beta} = \omega^2 t \left(\beta + \frac{\alpha t}{m} \right).$$

These equations are exact, and can be solved exactly. In order to do that note first that

$$\dot{\beta} + \frac{\dot{\alpha} t}{m} = 0.$$

As a result, one obtains

$$\ddot{\alpha} = -\omega^2 \alpha.$$

So the solution for α is exactly the simple harmonic motion that one expects. Prove that the solution for x is also harmonic.

Perturbation theory

It often turns out to be hard to solve the exact equations

$$\dot{\alpha}_i = -\frac{\partial \Delta H}{\partial \beta_i}, \quad \text{and} \quad \dot{\beta}_i = \frac{\partial \Delta H}{\partial \alpha_i}.$$

Instead we want to develop an approximation for the case when ΔH is small. This seems to be straightforward when we can write $\Delta H = \lambda h$, where $\lambda \ll 1$, then we can try the series expansions

$$\alpha_i = \sum_{j=0}^{\infty} \lambda^j \alpha_i^{(j)}, \quad \text{and} \quad \beta_i = \sum_{j=0}^{\infty} \lambda^j \beta_i^{(j)}.$$

The terms $\alpha_i^{(0)}$ and $\beta_i^{(0)}$ are solution of the dynamics of H_0 , and the series is called a **perturbation series**. We need to insert the series expansion into the expression for ΔH and equate equal powers of λ in the canonical equations.

The same simple example by perturbation

Defining $\lambda = m\omega^2$, we get $\Delta H = \lambda(\beta + \alpha t/m)^2$. Now inserting the series expansions for β and α into this, we get

$$\Delta H = \frac{\lambda}{2} \left(\beta^{(0)} + \frac{\alpha^{(0)}t}{m} \right)^2 + \mathcal{O}(\lambda^2).$$

So the leading order perturbative equations are $\dot{\alpha}^{(0)} = \dot{\beta}^{(0)} = 0$, as expected. The next order equations are

$$\dot{\alpha}^{(1)} = - \left(\beta^{(0)} + \frac{\alpha^{(0)}t}{m} \right), \quad \text{and} \quad \dot{\beta}^{(1)} = \frac{t}{m} \left(\beta^{(0)} + \frac{\alpha^{(0)}t}{m} \right).$$

Choosing, for simplicity, the initial condition $\beta^{(0)} = 0$, these integrate to the leading terms in the series expansions of circular functions, *i.e.*,

$$\alpha^{(1)} = -\frac{\alpha^{(0)}t^2}{2m}, \quad \text{and} \quad \beta^{(1)} = \frac{\alpha^{(0)}t^3}{3m^2}.$$

Secular terms

The example of the harmonic oscillator is technically simple, and we know from the exact solution that the problem is well behaved. However, the perturbative solution illustrates one of the main technical difficulties with perturbation theory. Note that both $\alpha^{(1)}$ and $\beta^{(1)}$ increase unboundedly with t . Terms in the perturbative solution which grow with t are called **secular terms**.



The appearance of secular terms put into question our initial assumption that $\alpha^{(0)}$ and $\beta^{(0)}$ provide good initial approximations to the solution. KAM theory, to be discussed later, tells us whether the series expansion in λ merely distorts the tori or destroys them.

Perturbation of orbits

One may be more interested in the perturbation of the elements of an orbit than in the orbit itself: for example, how the eccentricity of a Keplerian orbit changes due to a perturbation. The elements of the orbit are functions on phase space, $\zeta_k(\beta_i, \alpha_i)$. So their time dependence is given exactly by

$$\dot{\zeta}_k = [\zeta_k, \Delta H] = \nabla_{\xi} \zeta_k J \nabla_{\xi} \Delta H = \nabla_{\xi} \zeta_k J \nabla_{\xi} \zeta_l \frac{\partial \Delta H}{\partial \zeta_l} = [\zeta_k, \zeta_l] \frac{\partial \Delta H}{\partial \zeta_l}.$$

A perturbation theory can be set up, as before, by making series expansions in λ and matching terms in equal powers of λ .

Problem 73: Precession of perihelia

The perihelia of Keplerian orbits are given by the Runge-Lenz vector. For perturbations of Newton's law of gravity by a term in $1/r^n$ (fixed n) set up the equation for the **motion of perihelia**. Find the first order perturbative solution for $n = 2, 3$ and 4 .

Variation of parameters

Consider a system in which some parameter, a , is slowly varying: such as the length of a pendulum. For constant a , we can find action-angle variables $\{\beta, \alpha\}$ such that the Hamiltonian is $H(\alpha, a)$. Assume that this is obtained by a canonical transformation $\mathcal{S}(q, \beta, a)$ instead of the usual $S(q, \alpha, a)$. When a varies, then the appropriate Hamiltonian is

$$K(\beta, \alpha, a) = H(\alpha, a) + \frac{\partial \mathcal{S}}{\partial t} = H(\alpha, a) + \dot{a} \frac{\partial \mathcal{S}}{\partial a}.$$

Clearly, then the variation of α is governed by

$$\dot{\alpha} = -\frac{\partial K}{\partial \beta} = -\dot{a} \frac{\partial}{\partial \beta} \frac{\partial \mathcal{S}}{\partial a}.$$

If a varies slowly, one can take it to be constant over the original period of motion, T , and average this over that interval to get

$$\bar{\dot{\alpha}} = -\frac{\dot{a}}{T} \int_0^T dt \frac{\partial}{\partial \beta} \frac{\partial \mathcal{S}}{\partial a} + \mathcal{O}(\dot{a}^2, \ddot{a}).$$

Adiabatic invariants

Since S is the action, it increases by α in time T . Because S and $\dot{S}$ are related by Legendre transforms, so does $\dot{S}$. Hence $\dot{S}$ and its derivatives can be expanded in a Fourier series over multiples of the frequency $2\pi/T$. They have no constant coefficient, and the integral vanishes. This is therefore an **adiabatic invariant**.

Problem 74: Plasma confinement

Assume that a charged particle moves in a constant and nearly uniform magnetic field, $\mathbf{H}(\mathbf{x})$. Also assume that it has initially a very small velocity in the direction of the field. Show that when it enters a region where the field strength is larger than it encountered initially, the particle can be reflected. Use this phenomenon to design a **plasma bottle**.

Matching of Tori

Assume that there is an invariant torus T in phase space which contains the motion of the system. If the angle variable is x , then in time t under the action of the integrable Hamiltonian, H_0 , the system moves to a new point $P_0(x) = x + 2\pi t$. It is useful to work in an extended phase space labelled by $\{x, t\}$ and define $P_0(x, t) = \{x + 2\pi t, t\}$. Also assume that under the perturbed Hamiltonian the new point is $P_\lambda(x, t) = \{x + 2\pi t + \epsilon f(x, t), t\}$, where λ is small.

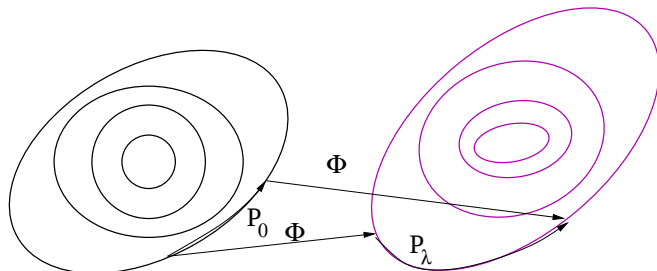
If we try to match the trajectories of the original and unperturbed system then we may have to change the coordinates as well as the time. This involves a function

$$\Phi_\lambda(x, t) = \{x + \lambda \xi(x, t), t + \lambda \tau(t)\}.$$

If such a mapping preserves the tori, then one must find that

$$P_\lambda(\Phi_\lambda(x, t)) = \Phi_\lambda(P_0(x, t)).$$

The commutative diagram



The diagram above illustrates the meaning of the desired equality

$$P_\lambda(\Phi_\lambda(x, t)) = \Phi_\lambda(P_0(x, t)).$$

Matching conditions

The left hand side of the desired equality is

$$P_\lambda(\Phi_\lambda(x, t)) = \{x + \lambda\xi(x, t) + 2\pi t + 2\pi\lambda\tau(t) \\ + \lambda f(x + \lambda\xi(x, t), t + \lambda\tau(t)), t + \lambda\tau(t)\}.$$

The right hand side gives

$$P_0(\Phi_\lambda(x, t)) = \{x + 2\pi t + \lambda\xi(x + 2\pi t, t), +\lambda f(x, t), t + \lambda\tau(t)\}.$$

Equating them reduces the desired equation into

$$\xi(x + 2\pi t, t) - \xi(x, t) = 2\pi\tau(t) + f(x + \lambda\xi(t), t + \lambda\tau(t)).$$

This can be simplified by expanding each function in a series in λ and equating equal powers, as before. The result is

$$\xi_0(x + 2\pi t, t) - \xi_0(x, t) = 2\pi\tau_0(t) + f_0(x, t).$$

Large denominators

This can be solved by Fourier transforming in the angle variable—

$$\xi_0(x, t) = \sum_k \xi_0(k, t)e^{ikt}, \quad \text{and} \quad f_0(x, t) = \sum_k f_0(k, t)e^{ikt},$$

if the perturbation has **commensurate frequencies**. Since $\tau_0(t)$ is independent of x , it is a term in $k = 0$. The equation

$$\xi_0(x + 2\pi t, t) - \xi_0(x, t) = 2\pi\tau_0(t) + f_0(x, t)$$

then gives the solution

$$\xi_0(k, t) = \frac{f_0(k, t) + 2\pi\tau_0(t)\delta_{k0}}{\exp(2\pi ikt) - 1}.$$

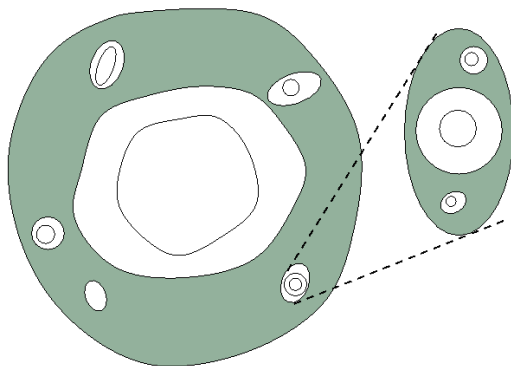
Clearly, when the perturbing function f_0 is small the solution is small except when the denominator is large. **Large denominators** occur when kt is integer. It seems possible that these can destroy invariant tori.

The Kolmogorov-Arnold-Moser Theorem

In general each invariant torus corresponds to motion with a certain frequency (and its harmonics). On the other hand, the perturbation Hamiltonian will, in general, have only a few fixed frequencies. The **KAM theorem** states that only those tori which correspond to motion at rational harmonics of the perturbation will be destroyed. The intuition gained in the above analysis is correct. This has interesting testable implications for the solar system—

- ➊ Unless the earth-moon-sun system is unstable, the length of the day and the year must be incommensurate. Similarly, the lengths of the lunar month and the year have to be incommensurate.
- ➋ No moons of any planet will have rational periods with respect to the orbital period of the parent planet.
- ➌ No asteroids will be observed with periods which are rational multiples of the orbital period of Jupiter.

Cantori



Some of the initial tori are distorted by perturbation, others are destroyed. The width of the unstable region increases with λ . The structure in the unstable region is fractal, like a **Cantor set**, hence sometimes called **cantori**.

Keywords and References

Keywords

integrable system, Hamilton-Jacobi equations, perturbation series, secular terms, precession of perihelia, adiabatic invariant, plasma bottle, commensurate frequencies, large denominators, KAM theorem, Cantor set, cantori

References

Appropriate chapters of Goldstein and Landau