Particle collisions and decays

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A particle decaying into two particles

Assume that there is a reaction A→B+C in which external forces and torques may be neglected. Then the energy, momentum and angular momentum are conserved. The complete description of motion consists of predicting the positions and momenta of B and C given the position and momentum of A. The number of variables to be determined is 12. However, there are 7 conservation equations, so there are 5 remaining free variables.

It is simplest to choose an inertial frame in which A is at rest. So $\mathbf{x}_A = 0$ and $\mathbf{p}_A = 0$. Also, then, $\mathbf{L}_A = \mathbf{x}_A \times \mathbf{p}_A = 0$. Conservation of momenta requires $\mathbf{p}_C = -\mathbf{p}_B$. Also, $\mathbf{x}_B(t) = \mathbf{x}_B(0) + \mathbf{p}_B t$ and $\mathbf{x}_C(t) = \mathbf{x}_C(0) - \mathbf{p}_B t$. The direction of $\mathbf{p}_B$ can be taken to define the x-axis. Since there is no intrinsic direction to the problem, if one observes many such decays, the final particles will be emitted isotropically on average.

Other conservation laws

Conservation of angular momentum gives

$$\mathbf{L} = (\mathbf{x}_B - \mathbf{x}_C) \times \mathbf{p}_B = [\mathbf{x}_B(0) - \mathbf{x}_C(0)] \times \mathbf{p}_B = \mathbf{x}_{BC}(0) \times \mathbf{p}_B = 0$$

implies either that the initial separation between the two particles vanishes or is parallel to $\mathbf{p}_B$.

Choose the direction of $\mathbf{p}_B$ to be the x-axis. Then energy conservation gives

$$E_A = \frac{|\mathbf{p}_B|^2}{2}\left(\frac{1}{m_B} + \frac{1}{m_C}\right) = \frac{|\mathbf{p}_B|^2}{2m},$$

where this defines the reduced mass m. Clearly, if the initial energy of A is only its kinetic energy, then $E_A = 0$ and the final state particles must be at rest. However, a particle can spontaneously break up only if it has internal energy which is larger than the final internal energies of the daughters. Non-vanishing values for $|\mathbf{p}_B|$ result from counting this difference in E_A.

Internal structure

If internal energy is taken into account, then there is no reason not to have other internal structure. This may cause the particle to have a net initial angular momentum $\mathbf{L}$. Take the direction of $\mathbf{L}$ to define the z direction. Since $\mathbf{x}_{BC}(0) \times \mathbf{p}_B = \mathbf{L}$, one finds $\mathbf{L} \cdot \mathbf{p}_B = 0$, and one can take $\mathbf{p}_B$ to define the x-direction as before. On observing many such decays, one must find the momentum direction to be orthogonal to $\mathbf{L}$, but not having any other directionality. Furthermore, $\mathbf{x}_{BC}(0)$ lies in the xy-plane and y coordinate is constrained by angular momentum conservation. Although the momenta $\mathbf{p}_B$ and $\mathbf{p}_C$ are collinear, the positions of B and C are not.

The conservation of $\mathbf{P}$ reduced the 6 unknown components $\mathbf{p}_B$ and $\mathbf{p}_C$ to 3. Of these, 2 were removed by a choice of coordinates, and the third obtained by energy conservation. The conservation of $\mathbf{L}$ reduced the 6 unknowns from the initial values of the position to 3.

Three body decays

Problem 21: Transformation of frames

Transform all the results obtained here into an inertial frame in which A moves with momentum $\mathbf{P}$ and has mass M. Is there a kinematic correlation between $\mathbf{P}$ and $\mathbf{p}_B$?

We found that the dynamics of a particle decaying into two particles is almost completely determined by conservation laws.

Problem 22: Decay of a particle into three particles

Assume that a particle A decays into three, A$\rightarrow$B+C+D when all forces and torques on the system can be neglected. Count the number of variables to be determined and check whether the kinematics is totally determined. Construct the kinematics in the rest fram of A and solve the system as far as possible. Show that whatever variables remain cannot be obtained from the initial data.

Collisions in a gas of molecules

When studying collisions of particles, for example molecules in a gas, often we concentrate on the collisions of two particles. The reason is that the probability of three particle collisions is relatively small.

Assume that there are N molecules, each a sphere of radius r in a volume V of a gas. The volume of each molecule is $\omega = 4\pi r^3/3$. The volume fraction occupied by the molecules is $f = N\omega/V$. In a typical gas, one has $r = 10^{-10}$ m, $N = 6 \times 10^{23}$ and $V = 0.02\ \mathrm{m}^3$, so that $f \simeq 10^{-4}$.

What is the rate of two-body collisions? Assume that the average speed with which a molecule moves is v. The calculations of the rates of multi-body collisions proceed as follows: each molecule sweeps out a cylinder which searches for some configuration, C, of other molecules. If this search volume swept out in unit time is ν_C, then the probability of collisions is $p_C = (N\nu_C/V) \times f_C$ where f_C is the volume fraction occupied by C.

Frequency of collisions

For counting two-particle collisions, the radius of the search cylinder is $2r$. Then the search volume per unit time for a single other molecule, $\nu_1 = 8\pi r^2 v$. Introduce the microscopic time scale for one molecule to travel by another: $\tau = 2r/v$. Then $\nu_1 = 16\pi r^3/\tau = 12\omega/\tau$. The two-body collision rate is

$$r_2 = \left(\frac{N\nu_1}{V}\right) f \propto \frac{f^2}{\tau}.$$

The search cylinder for two molecules in collision will have radius $4r$. The corresponding search volume per unit time is $\nu_2 = 48\omega/\tau$. The three body collision rate is therefore

$$r_3 = \left(\frac{N\nu_2}{V}\right)(\tau r_2) \propto \frac{f^3}{\tau}.$$

In general, N-body collision rates are f^N/τ. This proportionality depends only on the fact that f and τ exist.

Two-body collisions

When two particles collide and we want to predict the future trajectories of the particles, we know the positions of the particles at the time of collisions, and need to compute the momenta. The two momenta require a specification of 6 variables. Momentum conservation gives 3 constraints, energy conservation gives another. This leaves two unknowns, one of which can be absorbed into the choice of axes. Thus the problem of two-body collisions in the absence of external forces and torques is almost completely solvable using only first integrals of motion.

If the masses of the two particles are m_1 and m_2, and momenta are $\mathbf{p}_1$ and $\mathbf{p}_2$, then one can boost to the center of mass (CM) frame. The center of mass moves with the momentum $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$, *i.e.*, with velocity $\mathbf{V} = (\mathbf{p}_1 + \mathbf{p}_2)/(m_1 + m_2)$. In the CM frame, the initial momenta of the particles are directed opposite each other, *i.e.*, m_1 moves with momentum $\mathbf{p}$ and m_2 with $-\mathbf{p}$.

Solving for two-body collisions

In the CM frame choose the x-axis to be the direction of $\mathbf{p}$. After collision, suppose that the momentum of m_1 is $\mathbf{k}$; as a result that of m_2 is $-\mathbf{k}$. Conservation of energy gives

$$E = \frac{|\mathbf{p}|^2}{2m} = \frac{|\mathbf{k}|^2}{2m}, \qquad ie, \qquad |\mathbf{p}| = |\mathbf{k}|,$$

where m is the reduced mass. Choose the initial and final momenta to lie in the xy plane, then the scattering angle, χ is given by the expression

$$\cos\chi = \frac{\mathbf{p}\cdot\mathbf{k}}{2mE}.$$

This scattering angle is specified by the force between the two particles.

Cross sections

Scattering is studied by shooting beams of particles at each other. Particles pass each other at different distances (called the impact parameter ρ), and therefore feel different magnitudes of forces. Clearly, $\rho(\chi)$, since the larger the distance of approach, the smaller is the force and hence the scattering angle.

If the number of particles crossing unit area of the beam per unit time is n and the number of particles observed per unit time in a small angle $d\chi$ around χ is dN, then the cross section is defined through the formula

$$d\sigma = \frac{dN}{n} = 2\pi\rho d\rho = 2\pi\rho(\chi)\left|\frac{d\rho}{d\chi}\right| d\chi,$$

where the second equality comes from the fact that the only particles which are scattered into the cone of observation come from the ring of width $d\rho$ around the radius $\rho(\chi)$ in the initial beam, *i.e.*, $dN = 2\pi\rho d\rho n$.

Two particles going to many

For the kinematics of 2 particles going to N particles in the absence of external forces and torques, there are $3N$ unknown, since there are N final momenta. Since there are 7 constraint equations, there are $3N - 7$ undetermined variables.

Problem 23: Counting unknowns

The formula above does not agree with the counting which we did earlier for the problem of elastic scattering of two particles. Why? How do we change the argument here to include that case?

Problem 24: $2 \to N$ scattering

Solve the kinematics of the scattering of 2 particles giving N in the final state by converting the problem into that of an elastic scattering followed by an appropriate number of decays (for $N = 3$ and 4). For $N = 4$ does it matter how you chain the decays?

Keywords and References

Keywords

inertial frame, decay, reduced mass, search volume, two-body collisions, center of mass, scattering angle, scattering angle, beams of particles, impact parameter, cross section

References

Landau, Sections 16, 17, 18.