Kinematics of rigid body motion

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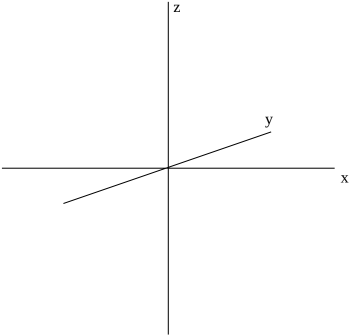
Counting

Model a rigid body by N particles, with positions $\mathbf{x}_i$ $(1 \le i \le N)$, such that the relative separations $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$ are fixed vectors. In an external inertial frame, called the space frame, one has to specify $3N$ coordinates. The number of constraints seems to be $3N(N-1)/2$, so there seem to be more constraints than variables. However, $\mathbf{x}_{ij} - \mathbf{x}_{jk} = \mathbf{x}_{ik}$, so only the constraints $\mathbf{x}_{1i}$ are independent. Hence there are $3(N-1)$ constraints. The number of degrees of freedom seem to be 3.

If there are no forces on the particle, then one can create another inertial frame in which $\mathbf{x}_{12}$ is parallel to $\hat{\mathbf{x}}$, and $\mathbf{x}_{13}$ lies in the xy plane. The $\hat{\mathbf{z}}$ direction is then automatically fixed. The orientation of this body frame with respect to the space frame requires 3 other degrees of freedom.

So, the dynamics of a rigid body is described by 6 degrees of freedom, and a 12 dimensional phase space.

Describing orientation



Standard conventions: use Euler angles

Other conventions: *zyz*, *xzx*, *etc.*.

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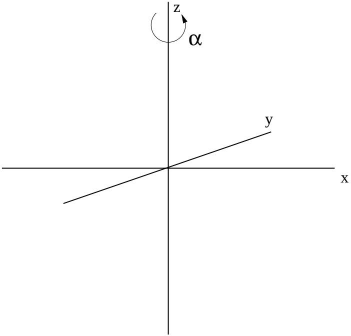

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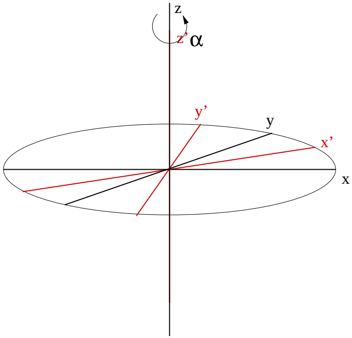

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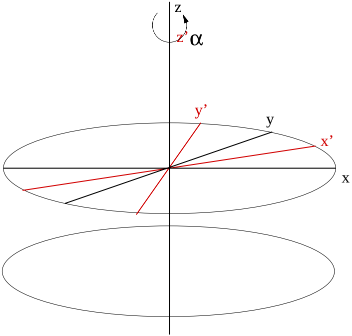

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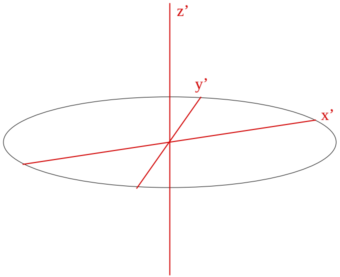

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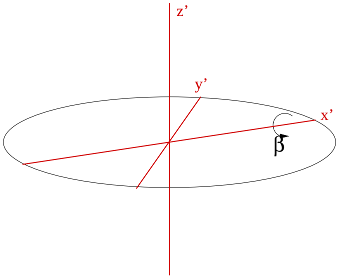

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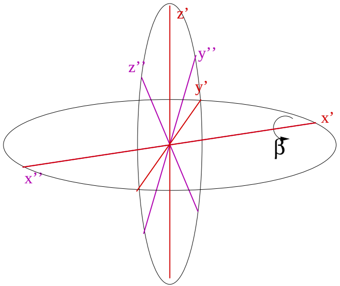

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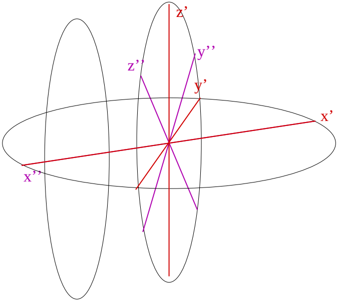

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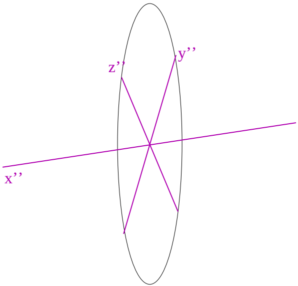

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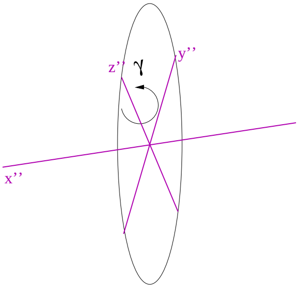

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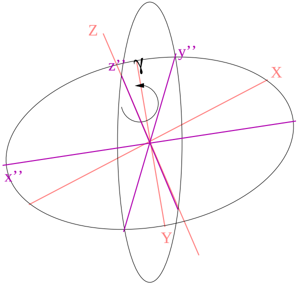

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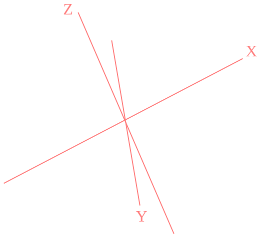

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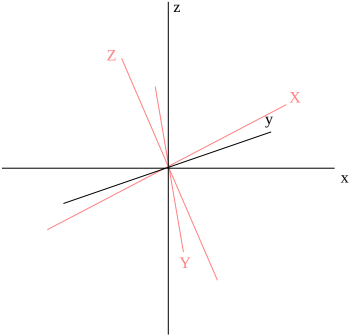

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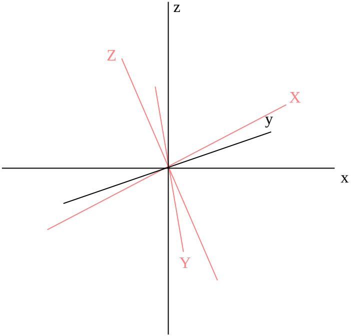

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Problem 49: Euler angles

In the *zxz* convention, find the Euler angles (α, β, γ) required to bring the axes to new orientations $\hat{\mathbf{X}} = \hat{\mathbf{z}}$, $\hat{\mathbf{Y}} = \hat{\mathbf{x}}$ and $\hat{\mathbf{Z}} = \hat{\mathbf{y}}$.

Infinitesimal rotations

An infinitesimal rotation by an angle δ about $\hat{\mathbf{z}}$ is given by

$$M_z(\delta) = I + \delta \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = I + \delta G_z.$$

An infinitesimal rotation by an angle δ about $\hat{\mathbf{x}}$ is given by

$$M_x(\delta) = I + \delta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = I + \delta G_x.$$

Similarly, the rotation about $\hat{\mathbf{y}}$ by infinitesimal angle δ can be written as $M_y(\delta) = I + \delta G_y$. A simple calculation gives the Lie bracket (also called the commutator)

$$G_z G_x - G_x G_z \equiv [G_z, G_x] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = G_y.$$

Vector representation of infinitesimal rotations

Clearly to linear order we have,

$$M_x(\delta_x)M_z(\delta_z)M_y(\delta_y) = I + \sum_k \delta_k G_k,$$

i.e., infinitesimal rotations commute. The components of G_k are given by ϵ_{ijk}. For the rotated vector $\mathbf{V}'$ we find $V_i' = V_i + \epsilon_{ijk} V_j \delta_k$, so that

$$\frac{\partial V_i}{\partial \phi_j} = -\epsilon_{ijk} V_k.$$

If the rotation angle changes with time, then one can write

$$\dot{V}_i = -\sum_j \dot{\phi}_j \frac{\partial V_i}{\partial \phi_j} = -\epsilon_{ijk} \dot{\phi}_j V_k \quad i.e. \quad \dot{\mathbf{V}} = -\omega \times \mathbf{V}.$$

Here ω is the angular velocity.

Transformation to non-inertial frames

If the frame A is inertial and B is non-inertial, and the origin is fixed in both, then one finds the transformation of velocity, $\mathbf{v}_B = \mathbf{V}_A + \omega \times \mathbf{x}$. Another derivative gives

$$\dot{\mathbf{v}}_B = \dot{\mathbf{v}}_A + \omega \times \mathbf{v}_B = \dot{\mathbf{v}}_A + 2\omega \times \mathbf{v}_A + \omega \times (\omega \times \mathbf{x}).$$

The last two terms are entirely due to changing to a non-inertial frame— the second term is the centripetal acceleration, the first the Coriolis acceleration

Problem 50: Centripetal acceleration

The earth's rotation period is approximately 24 hours, and its radius is 6371 Km. Find the ratio χ/g where χ is the centripetal acceleration and g is the acceleration due to gravity. Search for the rotation periods, masses and radii of the remaining 7 planets in the solar system and report χ/g for those bodies.

Some problems

Problem 51: The bathtub fallacy

Fill a basin with water before letting it drain out. What is the magnitude of Coriolis acceleration on the water. Look up the viscosity of water and check whether the Coriolis acceleration can be damped out by viscous forces alone.

Problem 52: Coriolis acceleration

A hurricane moves along the surface of the earth with velocity v initially in the north-south direction at 15 degrees latitude. Integrate the equations of motion in the earth-fixed frame assuming that there are no forces acting on the hurricane. How does the track change if it starts from the same position and same speed but the direction of initial velocity change? Is it possible for the hurricane to reverse direction?

The choice of origin

In a frame rotating with angular velocity ω (with respect to an inertial frame), take the point $\mathbf{x}$ which is moving with velocity be $\mathbf{v}$. Now take a new frame whose origin is shifted by $\mathbf{a}$. In this frame all quantities have values denoted by primed quantities— $\mathbf{x}'$, $\mathbf{v}'$ and ω'.

Clearly, $\mathbf{x} = \mathbf{x}' + \mathbf{a}$. If the velocity at that point in an inertial frame is $\mathbf{V}$, then $\mathbf{v} = \mathbf{V} + \omega \times \mathbf{a} + \omega \times \mathbf{x}'$. We also have $\mathbf{v}' = \mathbf{V} + \omega' \times \mathbf{x}'$. This gives

$$\omega' = \omega, \qquad \text{and} \qquad \mathbf{v}' = \mathbf{v} + \omega \times \mathbf{a}.$$

Since the velocity of a body depends on the choice of the origin of the body system of coordinates, we need to specify it before proceeding. We will choose to work with the origin of the body system fixed at the CM of the body.

The kinetic energy

The kinetic energy of a system of N rigidly connected particles of masses m_α is

$$T = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha \mathbf{v}_\alpha^2 = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha (\mathbf{V} + \omega \times \mathbf{r}_\alpha)^2,$$

where $\mathbf{V}$ is the velocity of the CM measured in the (inertial) space frame, and $\mathbf{r}_\alpha$ is the position of the particle in the body frame. The cross term is zero, since $\mathbf{V} \cdot \omega \times \mathbf{r}_\alpha = \mathbf{r}_\alpha \cdot \mathbf{V} \times \omega$. Since the last cross product is independent of α, it can be pulled outside the sum, giving the cross term $(\mathbf{V} \times \omega) \sum m_\alpha \mathbf{r}_\alpha$. The sum is zero, since the origin is at the CM.

The square of the first term gives the familiar result

$$T_{\rm lin} = \frac{1}{2} V^2 \sum_{\alpha=1}^{N} m_\alpha = \frac{1}{2} M V^2.$$

The inertia tensor

For the square of the cross product we use the identity

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jk}\delta_{lm}.$$

This allows us to write the second term in the kinetic energy as

$$T_{\rm rot} = \frac{1}{2} I_{km}\omega_k\omega_m, \quad \text{where} \quad I_{km} = \sum_{\alpha=1}^{N} (r^2\delta_{km} - r_k r_m).$$

I is called the inertia tensor.

The inertia tensor is a 3×3 matrix. Its eigenvectors are special directions within the rigid body called the principal axes. The eigenvalues of the tensor, I_1, I_2 and I_3, are called the principal moments of inertia.

The Lagrangian in the body frame is

$$L = \frac{1}{2} MV^2 + \frac{1}{2} I_{ij}\omega_i\omega_j - U(\mathbf{R}, \phi),$$

where U is the external potential within which the body moves.

Some problems

Problem 53: A rigid system of particles

Take a system of three particles fixed rigidly to each other. In one body frame they have coordinates $\mathbf{r}_1 = (0,0,0)$, $\mathbf{r}_2 = (1,0,0)$ and $\mathbf{r}_3 = (0,1,1)/\sqrt{2}$. Assume that they have equal masses, which we take to be the unit of mass in the problem. Find the inertia tensor in the CM of the particles, the principal moments of inertia and the principal axes.

Problem 54: Moments of inertia

Find the inertia tensor of a cone (height L, opening angle ψ) and cylinder (height L, radius r). Find the principal moments and axes. Using only the dynamics of these objects, is it possible to distinguish the shape?

Forces on a rigid body

If the parts of a rigid body always have a fixed relation to each other, *i.e.*, $\mathbf{x}_{ij}$ are constant, then we must assume that the only forces between them are forces of constraints. The forces of constraint have to act instantaneously, otherwise there could be temporary changes in $\mathbf{x}_{ij}$. Hence, in special relativity there can be no rigid bodies!

The dynamics of a rigid body without external forces acting on it is very simple. When $U = 0$, the CM moves in a straight line, $\mathbf{R}(t) = \mathbf{R}(0) + \mathbf{V}t$, where $\mathbf{R}(0)$ is the initial position of the CM, and $\mathbf{V}$ is its initial velocity. The orientational coordinates are also cyclic and change periodically with time, $\phi_i(t) = \phi_i(0) + \omega_i t$, where $\phi_i(0)$ is the initial value of one of these angles and ω_i is the corresponding angular velocities. Also, $\mathbf{V}$ and ω_i are constant.

The angular momentum

The angular momentum $\mathbf{L}$ is the momentum conjugate to the angular coordinates. As usual, we can write

$$L_i = \frac{\partial L}{\partial \omega_i}, \quad \text{which implies} \quad \mathbf{L} = I\omega.$$

In general the angular momentum is not parallel to the axis of rotation of the body.
Since I defines three independent principal axes, $\hat{\mathbf{u}}_i$, one can decompose any vector into a linear sum of components alone each of these axes. So, using the decomposition $\omega = \omega_i \hat{\mathbf{u}}_i$, we find that

$$\mathbf{L} = I_i \omega_i \hat{\mathbf{u}}_i.$$

In the special case when two of the ω_i vanish, *i.e.*, the angular velocity is initially in the direction of one of the principal axes, then $\mathbf{L}$ is parallel to the ω.

Keywords and References

Keywords

space frame, body frame, Euler angles, infinitesimal rotation, Lie bracket, commutator, centripetal acceleration, Coriolis acceleration, inertia tensor, principal axes, moments of inertia, angular momentum

References

Goldstein, chapter 5

Landau, sections 31, 32, 33