Dynamics of rigid body motion

Sourendu Gupta

TIFR, Mumbai, India

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The Lagrangian

In the inertial space frame we found the kinetic energy of a rotating rigid body to be

$$T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}I_{ij}\omega_i\omega_j, \quad \text{where} \quad I_{ij} = \sum_{\alpha=1}^{N}(r^2\delta_{ij} - r_i r_j),$$

M is the mass of the system and I is the inertia tensor.
The inertia tensor is a 3×3 matrix. Its eigenvectors are special directions within the rigid body called the principal axes. The eigenvalues of the tensor, I_1, I_2 and I_3, are called the principal moments of inertia.
The Lagrangian is

$$L = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\boldsymbol{\omega}\cdot I\cdot\boldsymbol{\omega} - V(\mathbf{R},\phi),$$

where V is the external potential within which the body moves.

The angular momentum

The angular momentum $\mathbf{L}$ is the generalized momentum conjugate to the angular coordinates. As usual, we can write

$$L_i = \frac{\partial L}{\partial \omega_i}, \quad \text{which implies} \quad \mathbf{L} = I\boldsymbol{\omega}.$$

In general the angular momentum is not parallel to the axis of rotation of the body.
Since I defines three independent principal axes, $\hat{\mathbf{u}}_i$, one can decompose any vector into a linear sum of components alone each of these axes. So, using the decomposition $\boldsymbol{\omega} = \omega_i \hat{\mathbf{u}}_i$, we find that

$$\mathbf{L} = I\boldsymbol{\omega} = I_i \omega_i \hat{\mathbf{u}}_i.$$

In the special case when two of the ω_i vanish, *i.e.*, the angular velocity is initially in the direction of one of the principal axes, then $\mathbf{L}$ is parallel to the $\boldsymbol{\omega}$.

The equations of motion

In the inertial space frame, the Lagrangian is given by

$$L = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}I_{ij}\dot{\theta}_i\dot{\theta}_j - V(\mathbf{R}, \theta_i).$$

So the equations of motion reduce to one set of equations for the rate of change of $\dot{\mathbf{R}}$ and another for the rate of change of $\boldsymbol{\omega}$. We have earlier discussed why θ_i do not form the components of a vector, but $\omega_i = \dot{\theta}_i$ do.

The motion of the CM is governed by the usual equations

$$M\ddot{\mathbf{R}} = \mathbf{F} = -\nabla_R V(\mathbf{R}, \theta_i),$$

where the gradient contains partial derivatives with respect to the components of $\mathbf{R}$. In the absence of external forces, $\mathbf{F} = 0$. The remaining three equations of motion are

$$\dot{\mathbf{L}} = \mathbf{M} = -\nabla_\theta V(\mathbf{R}, \theta_i),$$

where $\mathbf{M}$ is the external torque acting on the body.

The external torque

Representing a rigid body by N particles fixed to each other,

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{x}_i \times \mathbf{p}_i = \sum_{i=1}^{N} m_i \mathbf{x}_i \times \dot{\mathbf{x}}_i.$$

Here the $\mathbf{x}$ are taken in an inertial frame. Then,

$$\dot{\mathbf{L}} = \sum_{i=1}^{N} \mathbf{x}_i \times \dot{\mathbf{p}}_i = \sum_{i=1}^{N} \mathbf{x}_i \times \mathbf{f}_i,$$

where $\mathbf{f}_i$ is the force on the i-th body. The forces of constraint satisfy $\mathbf{f}_{ij} = -\mathbf{f}_{ji}$. If there were only forces due to constraints, then

$$\sum_{i} \mathbf{x}_i \times \mathbf{f}_i = \sum_{ij} \mathbf{x}_i \times \mathbf{f}_{ij} = -\sum_{ij} \mathbf{x}_i \times \mathbf{f}_{ji} = -\sum_{ij} \mathbf{x}_i \times \mathbf{f}_{ij}.$$

As a result, this would be zero. Therefore, the net torque on the body is due to external forces only. When $\mathbf{F} = 0$, the torque is called a couple.

Free rotations

When the torque vanishes we have free rotations. The solution is that **L** is conserved. If **L** coincided with one of the principal axes, then this would imply that the angular speed of rotations about that axis is constant.

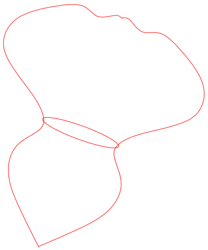
Spherical top

For a spherical top ($I_1 = I_2 = I_3$) this implies that $\boldsymbol{\omega}$ is constant, since we are free to choose the principal axis along the direction of **L**.

Rigid rotator

For a rotator ($I_1 = I_2$ and $I_3 = 0$), **L** is always orthogonal to the axis of symmetry. Hence the free motion is a rotation in this plane.

Precession and spin



For a symmetric top, $I_1 = I_2 \neq I_3$, the axis of symmetry is the principal axis $\hat{\mathbf{u}}_3$.

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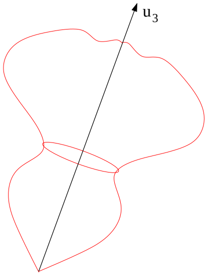

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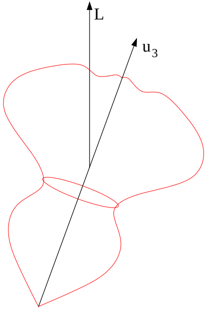

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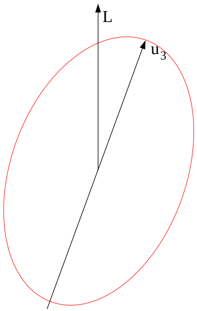

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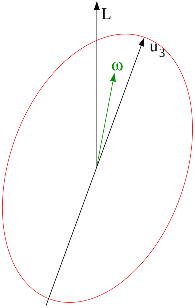

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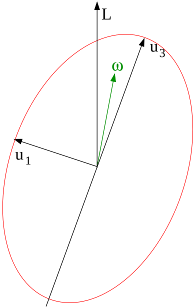

For a symmetric top, $I_1 = I_2 \neq I_3$, the axis of symmetry is the principal axis $\hat{\mathbf{u}}_3$. The direction of $\mathbf{L}$ and $\boldsymbol{\omega}$ do not coincide in general. Since all directions perpendicular to $\hat{\mathbf{u}}_3$ are equivalent, we can choose the $\hat{\mathbf{u}}_1$ directions at some time t so that $\mathbf{L}$ and $\boldsymbol{\omega}$ are in the 13-plane. Then, since $L_2 = I_2\omega_2$, we have $L_2 = \omega_2 = 0$. Since $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$, the velocity at any point in the $\hat{\mathbf{u}}_3$ axis is in the $\hat{\mathbf{u}}_2$ direction.

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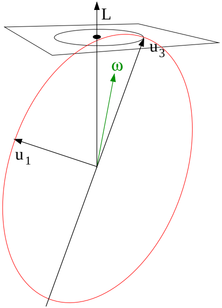

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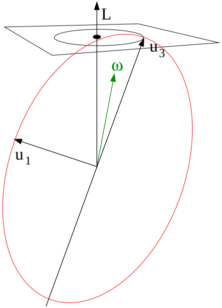
This is independent of t. So, the axis $\hat{\mathbf{u}}_3$ rotates around the fixed vector $\mathbf{L}$; this is called precession.

Precession and spin



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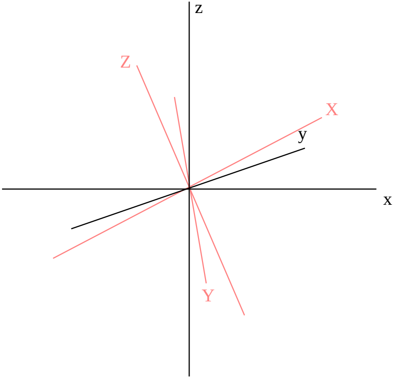
This is independent of t. So, the axis $\hat{\mathbf{u}}_3$ rotates around the fixed vector $\mathbf{L}$; this is called precession. The top also rotates around $\hat{\mathbf{u}}_3$; this is called spin. If $\hat{\mathbf{L}} \cdot \hat{\mathbf{u}}_3 = \cos\theta$, then the spin velocity is $\omega_3 = L_3/I_3 = L\cos\theta/I_3$. The precession velocity is the component of ω_1 parallel to $\mathbf{L}$, so $\omega_p = \omega_1/\sin\theta = M/I_1$.

Euler angles



Take the *xyz* axes to be the inertial space frame. Orient the body frame *XYZ* along the principal axes. If the spin velocity around $\hat{\mathbf{Z}}$, is $\dot{\gamma}$. The line of nodes, $\hat{\mathbf{n}}$, is orthogonal to $\hat{\mathbf{z}}$ and $\hat{\mathbf{Z}}$. The angular velocity along $\hat{\mathbf{n}}$ is $\dot{\beta}$. The angle between $\hat{\mathbf{x}}$ and $\hat{\mathbf{n}}$ is α.

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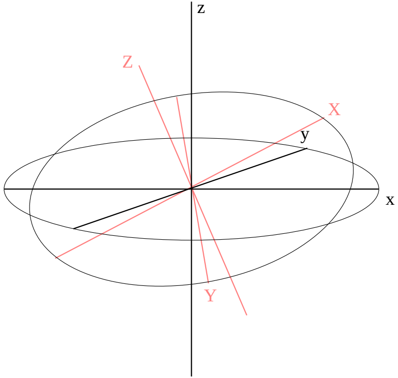

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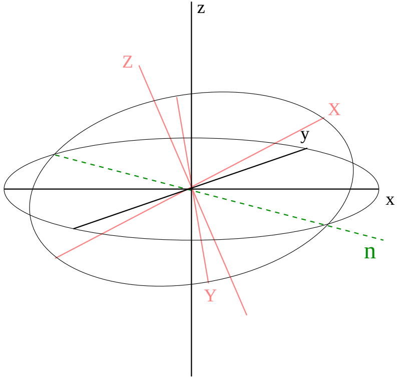

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Angular velocity

Problem 55: Angular velocity in Euler angles

Check that the angular velocity in the space frame, $\boldsymbol{\omega}$, is related to the rate of change of the Euler angles by

$$\begin{aligned}
\omega_x &= \dot{\alpha}\sin\beta\sin\gamma + \dot{\gamma}\cos\beta, \\
\omega_y &= \dot{\alpha}\sin\beta\cos\gamma - \dot{\gamma}\sin\beta, \\
\omega_z &= \dot{\alpha}\cos\beta + \dot{\beta}.
\end{aligned}$$

Construct the rotational term in the kinetic energy using these expressions. Specialize to the cases of the spherical top and the rigid rotator and check that the correct description of free rotations is obtained in both these cases.

Simple problems

Problem 56: Precession in Euler angles

For a symmetric top with $I_1 = I_2 \neq I_3$, with the symmetry axis along $\hat{\mathbf{Z}}$, choose one of the principal axes along $\hat{\mathbf{n}}$. Show that

$$T = \frac{I_1}{2}(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2) + \frac{I_3}{2}(\dot{\alpha}\cos\beta + \dot{\gamma})^2.$$

Describe free rotations and find the precession and spin angular velocities.

Problem 57: Symmetric top with a couple

Take a symmetric top with $I_1 = I_2 \neq I_3$ subject to the potential $V = M\cos\gamma$. Write the Lagrangian for this problem and find the conserved quantities and the cyclic coordinates. Using these, reduce the problem to one-dimension. Describe the general character of the motion, including nutations.

A trivial special case

In the special case when $\mathbf{F}$ and $\mathbf{M}$ are orthogonal, one can find a spatial vector $\mathbf{a}$ such that

$$\mathbf{M} = \mathbf{a} \times \mathbf{F}.$$

Now, for any vector $\mathbf{a}(\lambda) = \mathbf{a} + \lambda\mathbf{F}$, the above equation is also satisfied. Additionally, if one shifts the origin of coordinates to $\mathbf{x}' = \mathbf{x} - \mathbf{a}$, then the torque becomes $\mathbf{M}' = \mathbf{M} - \mathbf{a} \times \mathbf{F} = 0$. So, for any of these choices of coordinate systems, the external torque vanishes if in another system the force and torque are orthogonal.

Problem 58: A linear equation

Given vectors $\mathbf{F}$ and $\mathbf{M}$, one can find a vector $\mathbf{a}$ such that $\mathbf{M} = \mathbf{a} \times \mathbf{F}$, because these are three equations in 3 unknowns. Examine the linear equations in the components of $\mathbf{a}$ and find what is special about them when $\mathbf{F}$ and $\mathbf{M}$ are orthogonal.

The trivial case is realistic

When the force field is uniform, one can write each of the $\mathbf{f}_i = \mu_i \hat{\mathbf{u}}$. As a result, with the definition that μ is the sum over μ_i, the torque is

$$\mathbf{M} = -\hat{\mathbf{u}} \times \left\{ \sum_{i=1}^{N} \mu_i \mathbf{x}_i \right\} = \mu \mathbf{X} \times \hat{\mathbf{u}}, \quad \text{where} \qquad \mathbf{X} = \frac{1}{\mu} \sum_{i=1}^{N} \mu_i \mathbf{x}_i .$$

In the case where the forces are purely gravitational, $\mathbf{X} = \mathbf{R}$, and the torque about the center of mass vanishes, as it does along any line in the direction of the force along the center of mass.

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This is relevant to the problem discussed on September 12, and given in the mid-sem. For a satellite moving in an orbit far larger than the radius of the earth, the tidal forces may be neglected if the satellite is small. In this case the gravitational force on the satellite is exactly as above, and the motion can be analyzed as if there were no torque.

Euler equations

The simplest description of a rotating body is found in the body frame with axial directions chosen along the principal axes of the body. Clearly, for any vector

$$\left.\frac{d\mathbf{V}}{dt}\right|_i = \left.\frac{d\mathbf{V}}{dt}\right|_r + \mathbf{\Omega}\times\mathbf{V},$$

where the subscript i refers to the change in the inertial system and r to the change in a system rotating with angular velocity $\mathbf{\Omega}$. Using this we get Euler's equations for rigid bodies in the body frame

$$\dot{\mathbf{P}} + \mathbf{\Omega}\times\mathbf{P} = \mathbf{F}, \qquad \dot{\mathbf{L}} + \mathbf{\Omega}\times\mathbf{L} = \mathbf{M}.$$

Problem 59: Euler equations for free rotations

Solve Euler's equations for free rotations of a body, *i.e.*, for $\mathbf{M} = 0$. Check that these solutions are the same as the solutions obtained earlier for free rotations, but viewed from a non-inertial frame.

Keywords and References

Keywords

angular momentum, principal axes, space frame, torque, couple, free rotations, spherical top, rotator, symmetric top, precession, spin, Euler angles, line of nodes, nutations, Euler's equations

References

Goldstein, chapter 6
Landau, sections 33, 34, 35