Topics in rigid body motion

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Euler equations

The simplest description of a rotating body is found in the body frame with axial directions chosen along the principal axes of the body. Clearly, for any vector

$$\left.\frac{d\mathbf{V}}{dt}\right|_i = \left.\frac{d\mathbf{V}}{dt}\right|_r + \boldsymbol{\omega} \times \mathbf{V},$$

where the subscript i refers to the change in the inertial system and r to the change in a system rotating with angular velocity $\boldsymbol{\omega}$. Using this we get Euler's equations for rigid bodies in the body frame

$$\dot{\mathbf{P}} + \boldsymbol{\omega} \times \mathbf{P} = \mathbf{F}, \qquad \dot{\mathbf{L}} + \boldsymbol{\omega} \times \mathbf{L} = \mathbf{M}.$$

Euler's equations are most useful in a body frame oriented along the principal axes of the rigid body. In this frame the inertia ellipsoid is diagonal. Although $\mathbf{L}$ and $\boldsymbol{\omega}$ point in different directions, $L_i = I_i\omega_i$.

Components of Euler's equations

Component-wise the second of Euler's equations can be written as

$$I_i \dot{L}_i + \epsilon_{ijk}\omega_j\omega_k I_k = M_i.$$

It is worth displaying the component equations

$$\begin{aligned}
\dot{\omega}_1 &= \frac{M_1}{I_1} + \frac{(I_2 - I_3)}{I_1}\omega_2\omega_3, \\
\dot{\omega}_2 &= \frac{M_2}{I_2} + \frac{(I_3 - I_1)}{I_2}\omega_3\omega_1, \\
\dot{\omega}_3 &= \frac{M_3}{I_3} + \frac{(I_1 - I_2)}{I_3}\omega_1\omega_2.
\end{aligned}$$

The second term on the left is nonzero only when all the I_k are non-zero.

Two problems

Problem 60: Free motion of a spherical top

For a spherical top, we have $I_1 = I_2 = I_3 = I$. When this is in free motion then all the Euler equations reduce to $\dot{\omega}_i = 0$. What is the path of a point on the surface of the top, as seen from body and space frames?

Problem 61: Spherical top

When a spherical top has a couple acting on it, choose axes so that $M_1 = M_2 = 0$. Then the equations of motion are $\dot{\omega}_{1,2} = 0$ and $\dot{\omega}_3 = M/I$. Solve the equations of motion. What is the path of a point on the surface of the top, as seen from body and space frames?

Free motion of a symmetric top

We set $\mathbf{M} = 0$. For the symmetric top we take $I_1 = I_2$. Then the 3-component of Euler's equations gives $\dot{\omega} = 0$. Since ω_3 is a constant, we introduce a new constant frequency $\omega = |I_3 - I_1|\omega_3/I_1$. Then the remaining Euler's equations become

$$\dot{\omega}_1 = \pm\omega\omega_2, \qquad \text{and} \qquad \dot{\omega}_2 = \mp\omega\omega_1.$$

When $I_3 > I_1$ then the first equation has a plus sign, and minus otherwise. In either case, they give $\ddot{\omega}_{1,2} = -\omega^2\omega_{1,2}$.
So each of these two components of ω change harmonically with time. This describes the precession of the vector **L** around the 3-direction in the body frame. In the space frame since **L** is fixes, this describes the poinsession of the body around the direction of **L**, exactly as seen before.

A problem

Problem 62: Symmetric top with couple

Assume that $I_1 = I_2 \neq I_3$ and a couple acts on this symmetric top with $M_1 = M_2 = 0$. Then, for ω_3 we find $\dot{\omega}_3 = M_3/I_3$, exactly as for the spherical top. Introducing the notation $\upsilon = M_3/I_3$, we can write the solution as

$$\omega_3(t) = \omega_3^0 + \upsilon t,$$

where the constant of integration, $\omega_3^0 = \omega_3(0)$. With the notation $r = (I_3 - I_1)/I_1$ and $\omega = |r|\omega_3^0$, the remaining equations are

$$\dot{\omega}_1 = \pm\omega\omega_2 + r\upsilon t, \qquad \text{and} \qquad \dot{\omega}_2 = \mp\omega\omega_2 - r\upsilon t.$$

Solve these equations and find what nutation looks like in the body frame.

The free motion of a rigid body

The Lagrangian for the free motion of a rigid body is

$$T = \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \boldsymbol{\omega} \cdot I \cdot \boldsymbol{\omega}.$$

In an inertial frame co-moving with the CM, the conserved quantities are the angular momenta $\mathbf{L} = I\boldsymbol{\omega}$ and the energy. With six dimensions of phase space and four constraints, the motion can be reduced to two dimensions of phase space.

We can write the conservation relations as

$$\begin{aligned} 2E &= \frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3}, \\ L^2 &= L_1^2 + L_2^2 + L_3^2. \end{aligned}$$

We take $I_1 \leq I_2 \leq I_3$. Then these equations describe the intersection of a sphere of radius L with an ellipsoid with semi-axes given by the I_k. Intersections are guaranteed.

Solving Euler's equations

The conservation equations are

$$2E = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2, \qquad L^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2.$$

They allow us to eliminate two of the variables by writing

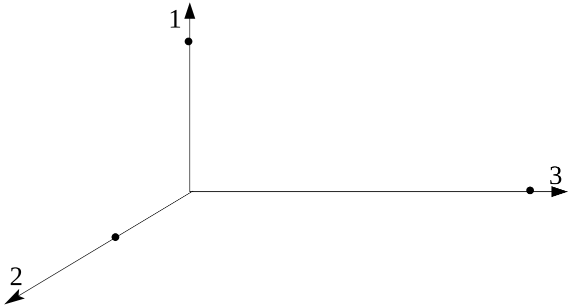
$$\begin{aligned} \omega_1^2 &= \frac{1}{I_1(I_3 - I_1)}\left[2EI_3 - L^2 - I_2(I_3 - I_2)\omega^2\right], \\ \omega_3^2 &= \frac{1}{I_3(I_3 - I_1)}\left[L^2 - 2EI_1 - I_2(I_2 - I_1)\omega^2\right] \end{aligned}$$

The equation for ω_2 is of the form

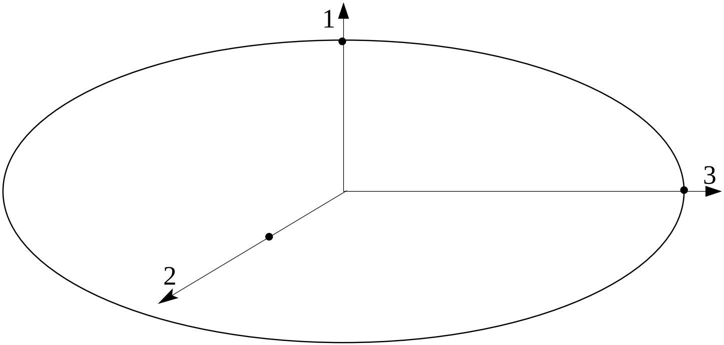
$$\dot{\omega}_2 = \frac{1}{I_2\sqrt{I_1 I_3}}\sqrt{(a_1 - b_1\omega_2^2)(a_2 - b_2\omega_2^2)}.$$

This can be integrated using elliptic functions. Do it

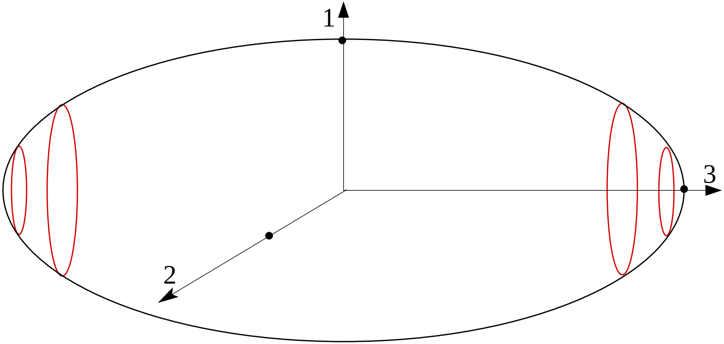
Poinsot's analysis



Poinsot's analysis

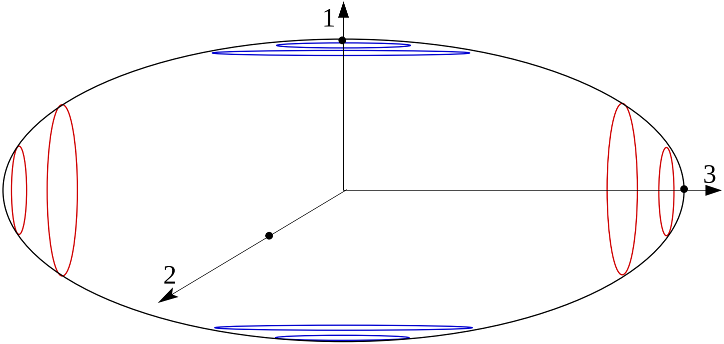


Poinsot's analysis



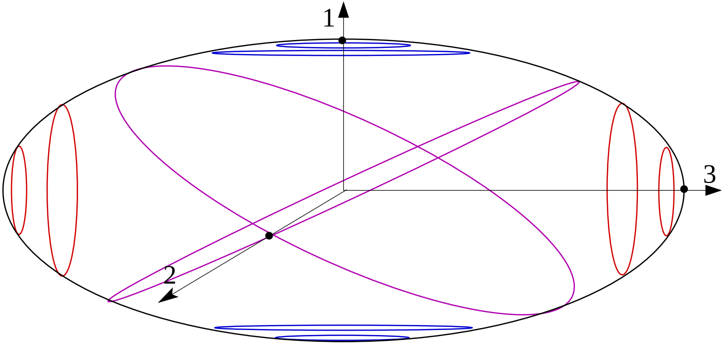
$L^2 \simeq 2EI_3,$

Poinsot's analysis



$L^2 \simeq 2EI_3$, $L^2 \simeq 2EI_1$,

Poinsot's analysis



$L^2 \simeq 2EI_3$, $L^2 \simeq 2EI_1$, and $L^2 \simeq 2EI_2$.

Rolling of rigid bodies on each other

If a rigid body moves over another such that the point of contact is instantaneously at rest, then it is said to roll without slipping. This gives a constraint on the velocities: $c_{\alpha i}\dot{q}_i = 0$ (where $1 \leq \alpha \leq C$ labels the constraint equation.

Sphere rolling on a plane

For example, for a ball rolling on a horizontal table, one has $\mathbf{v} = R\omega \times \hat{\mathbf{z}}$ where $\mathbf{v}$ is the velocity of the CM and R is the radius of the sphere. Although $\mathbf{v} = \dot{\mathbf{x}}$, where $\mathbf{x}$ are the coordinates of the CM, $\boldsymbol{\omega}$ are not the total time derivatives of a vector of coordinates. So the constraint on velocities cannot be integrated into a constraint on coordinates.

Since, the constraint equations cannot be used to elimiate some coordinates, these are non-holonomic constraints. Disk? We have to extend the Lagrangian formalism to deal with this.

Lagrange multipliers

Use the coordinates counted without the constraint of rolling without slipping to write the kinetic and pontetial energies for the D degrees of freedom. Neglecting the constraints, the variation of the action is the usual expression

$$\delta S' = \int dt \sum_i \delta q_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right].$$

However, the constraints modify this expression. One utilizes the method of Lagrange multipliers which allows us to treat all the coordinates as independent. This modifies the equations to give

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \sum_\alpha \lambda_\alpha c_{\alpha i}, \quad \text{and} \quad \sum_i c_{\alpha i} \dot{q}_i = 0.$$

The total number of equations is $D + C$ which is equal to the total number of unknowns (the D different q_i and the C different λ_α). $\lambda_\alpha c_{\alpha i}$ are forces of reaction.

Keywords and References

Keywords

Euler's equations, roll without slipping, non-holonomic constraints, Lagrange multipliers, forces of reaction.

References

Goldstein, chapter 5, section 2-6
Landau, sections 36, 37, 38