Oscillations

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The continuum limit

The Lagrangian of the discrete (lattice) problem that we just solved was

$$L = \frac{m}{2}\sum_i \dot{\xi}_i^2 - \frac{KQ^2}{2}\sum_i \left(\frac{\xi_{i+1} - \xi_i}{Q^2}\right)^2,$$

where K is the spring constant, m the mass of the particles, and Q is the equilibrium distance for each oscillator. The equilibrium size of the system is QN. Now we take $N \to \infty$ keeping the total size fixed. The equilibrium position of the i-th oscillator is at $x = iQ/N$. In the limit as $N \to \infty$, we find

$$L = \int dx \left\{ \left[\frac{\partial\xi(x,t)}{\partial t}\right]^2 - \omega^2 \left[\frac{\partial\xi(x,t)}{\partial x}\right]^2 \right\} \equiv \int dx \mathcal{L}$$

The sum over i in the expression on top becomes the integral over x, and the difference between nearest neighbours becomes the spatial derivative. $\mathcal{L}$ is called the Lagrangian density.

The Lagrangian formalism

The degrees of freedom are the fields, $\xi(x,t)$: in this case a scalar field. If we start with oscillations in three spatial dimensions, it would be more natural to have a vector field of displacements of each particle in the three dimensional lattice, $\boldsymbol{\xi}(\mathbf{x},t)$.
The kinetic term is usually taken to be quadratic in the first derivative of the field—

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\boldsymbol{\xi}(\mathbf{x},t)}{\partial t}\right)^2 - V[\boldsymbol{\xi}(\mathbf{x},t)].$$

V could involve the fields at widely separated points. Since V is a scalar, this can always be systematically approximated by a few low order derivatives at a point—

$$V = P_n(\boldsymbol{\xi}) + \frac{\kappa}{2}\left(\nabla\boldsymbol{\xi}\right)^2 + \cdots$$

where P_n is a polynomial of order n.

Symmetries of the Lagrangian

The Lagrangian has a symmetry. If we make a transformation of the components of the field, $\boldsymbol{\xi} \to O\boldsymbol{\xi}$, where O is a 3×3 orthogonal matrix, then the kinetic term, $\sum_i \dot{\xi}_i^2$, is invariant under this transformation. So is the term $(\nabla\boldsymbol{\xi})^2$. If the terms in P_n are powers of $|\boldsymbol{\xi}|^2$, then the whole Lagrangian is symmetric under these orthogonal transformations. Then there must be conserved quantities.

Problem 65: The conserved charge

For the Lagrangian that we wrote, the conjugate momenta are

$$\boldsymbol{\pi}(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\xi}}(\mathbf{x}, t)} = \dot{\boldsymbol{\xi}}(\mathbf{x}, t)$$

Use Noether's theorem to find the conserved charge of this field theory. Also find the equations of motion.

Scaling symmetries of the Lagrangian

Examine the scaling of the field theory when the spatial distances are scaled as $\mathbf{x} \to \lambda^\alpha \mathbf{x}$ and the time is scaled as $t \to \lambda^\beta t$. Assume also that the field scales simultaneously as $\boldsymbol{\xi} \to \lambda^\gamma \boldsymbol{\xi}$. The integral measure d^3x scales as $\lambda^{3\alpha}$. So the contribution of the kinetic term to the Lagrangian scales as $\lambda^{3\alpha-2\beta+2\gamma}$.
The derivative term in the potential scales as $\lambda^{2\gamma+\alpha}$. If these are to scale in the same way, then one must have $\kappa \to \lambda^{2(\alpha-\beta)}\kappa$. If this is the scaling of the coupling constant then any solution, $\boldsymbol{\xi}$, to the equations of motion can be scaled to zero by appropriate choice of λ. These are the kinds of small oscillations problems which we have examined before.
In general, scaling constrains the form of the coupling constants which give rise to simple oscillations. When the scaling of the coupling constants change, one may not be able to shrink the field amplitudes to zero: giving rise to interesting solitonic solutions to the equations of motion.

Relativistic classical field theories

If the Lagrangian density is to be Lorentz invariant, then two simplifications occur.
First the symmetries dictate that each term in the Lagrangian density has to be a Lorentz scalar. For a scalar field ξ this places no restrictions on the polynomial terms in the potential. However, it implies that the term involving the first derivatives of the action should be $(\partial_\mu \xi)^2$, which relates the coefficients of the time and space derivatives. For a vector field ξ_ν the polynomial part is restricted to powers of $\xi_\nu \xi^\nu$ and the kinetic part determines the first derivative terms to be $\partial_\mu \xi_\nu \partial^\mu \xi^\nu$. Other constraints follow in a similar fashion. Noether's theorem gives the conserved charge and current.
Second, the scaling laws must have $\beta = \alpha$, so that the different parts of the term $\partial_\mu \xi_\nu \partial^\mu \xi^\nu$ scale similarly. The polynomial terms in the action then determine whether there are non-trivial scaling laws.

Oscillatory equations

The most general differential equations for an oscillator with time-independent coefficients is

$$\ddot{\xi} + 2K\dot{\xi} + \omega^2\xi = F(t),$$

where K is called the damping coefficient, ω is the oscillator frequency and $F(t)$ is the driving force. Since the equations are linear, it is useful to use a Fourier expansion

$$\xi(t) = \int_{-\infty}^{\infty} dz\tilde{\xi}(z)\mathrm{e}^{-izt}, \qquad \text{and} \qquad \tilde{\xi}(z) = \int_{-\infty}^{\infty} \frac{dt}{2\pi}\xi(t)\mathrm{e}^{izt}.$$

In the integral z takes real values. This integral transform reduces the differential equation into an algebraic equation in the frequency domain—

$$-z^2 - 2iKz + \omega^2 = \tilde{F}(z).$$

Discretized equations

Discretize time into equal steps: $t = jh$. Then the derivative operator can be approximated as

$$\dot{\xi} \simeq \frac{1}{h}\left[x_{i+1} - x_i\right], \qquad \text{and} \qquad \ddot{\xi} \simeq \frac{1}{h}\left[x_{i+1} - 2x_i + x_{i-1}\right].$$

Then, if one puts periodic boundary conditions in the time direction one obtains the matrix equation

$$\left[(-2 + A + A^\dagger) + 2K(A - A^\dagger) + \omega^2\right]\xi = F,$$

where the vector ξ has elements ξ_j, which are the values of $\xi(t)$ at times jh, and similarly for the vector F. We know how to solve this!

Problem 66: Lattice discretizations

Check that the derivative operator acting on the Fouier basis functions gives the same eigenvalues as does $A - A^\dagger$ in the limit $h \to 0$.

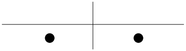
The homogenous equations

The homogenous equations have a solution for any amplitude whatsoever:

$$[-z^2-2iKz+\omega^2]\xi = 0, \quad \text{and} \quad [(2-A-A^\dagger)+2K(A-A^\dagger)+\omega^2]\xi = 0.$$

The only non-zero solutions occur when the determinant of the matrix vanishes. The eigenvalue equations are quadratic, so for each zero there is a two-fold degeneracy of solutions. These correspond to

$$z = -iK \pm \sqrt{\omega^2 - K^2}.$$



If ξ and ζ are the two vectors which corresponds to these solutions, then so is any linear combination. All initial conditions, $\xi(0)$ and $\dot{\xi}(0)$, therefore lead to solutions.
The eigenvalues lie in the lower half of the complex plane. Since the eigenvectors are $\exp(-izt)$, they are damped in the future. Anything else would violate the second law of thermodynamics.

The propagator

The simplest inhomogenous equation is

$$[z^2+2iKz-\omega^2]\xi = 1, \qquad \text{and} \qquad [(2-A-A^\dagger)+2K(A-A^\dagger)+\omega^2]\xi = 1.$$

This forcing function is the Fourier transform of the instantaneous impulse, $\delta(t)$ or δ_{t0}. The solution of this equation is called the propagator (or the Green's function):

$$G(z) = \frac{1}{z^2+2iKz-\omega^2}.$$

These diverge at the pole, *i.e.*, at the solutions of the homogenous equations. The general solution of the inhomogenous equation is a superposition of this with the solutions of the homogenous equations. When $K \neq 0$ the latter solutions decay after time $t \gg 1/K$, and therefore we will not write these terms for the steady state solution.

Steady state under harmonic forcing

If the forcing function is harmonic, with frequency Ω, then the solution in the Fourier domain is

$$\xi(z) = \frac{\delta(z-\omega)}{z^2 + 2iKz - \omega^2}.$$

Since Ω is real there are no divergences, as long as $K > 0$. The complex solution in the time domain is

$$\xi(t) = \frac{e^{-i\Omega t}}{\Omega^2 - \omega^2 + 2iK\Omega}$$

The real part of the solution is

$$\xi(t) = \frac{(\Omega^2 - \omega^2)\cos\Omega t + 2K\Omega\sin\Omega t}{(\Omega^2-\omega^2)^2 + 4K^2\Omega^2}.$$

The amplitude increases as $\Omega \to \omega$, becoming proportional to $1/K$ when $\Omega = \omega$. This is called resonance.

Resonance

Problem 67: Investigating the resonance

Investigate the shape of the complex response $\xi(t)$ due to a harmonic forcing function $F(t) = \exp(i\Omega t)$. Use Mathematica to draw graphs of the variation with Ω of (a) the square of the modulus, $|\xi(t)|^2$, (b) the phase of $\xi(t)$, and (c) the Argand diagram, *i.e.*, the path traced out in the complex plane by $\xi(t)$.

Problem 68: Causal response of an oscillator

When the forcing function is a unit pulse, $F(t) = 1$ when $0 \le t \le 1$, and vanishes at all other times. Find the response $\xi(t)$ due to this pulse, such that solution is causal, *i.e.*, the oscillator is at rest up to time $t = 0$. Does causality impose a relation between the real and imaginary parts of $\xi(t)$?

Causality

Problem 69: Causal restrictions on oscillators

Does causality place any restrictions on the order of the differential equation for an oscillator? Examine higher order differential equations and check whether they always give rise to damped oscillations in the presence of friction.

Problem 70: Causal restrictions on field theories

From your analysis of causal restrictions on the equations of an oscillator, check whether you can place any constraints on the equations of a field theory. Does this imply any restrictions on the Lagrangian density of a classical field theory?

Keywords and References

Keywords

Lagrangian density, scalar field, vector field, invariant, Noether's theorem, coupling constant, solitonic solutions, damping coefficient, oscillator frequency, driving force, Fourier expansion, integral transform, propagator, Green's function, resonance, Argand diagram, causality

References

Appropriate chapters of Goldstein, and Landau