# Hamilton-Jacobi Method

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# A clever canonical transformation

The motion of a mechanical system with $D$ degrees of freedom can be easily integrated if there are $D$ conserved quantities. Examples are: one dimensional systems with time-independent Hamiltonians, the Kepler problem, motion of rigid bodies, *etc.*.
One can exhibit this through a canonical transformation to a new set of coordinates and momenta which are constant. If they are constant, then

$$\dot{Q}_i = \frac{\partial \mathcal{H}}{\partial P_i} = 0, \qquad \text{and} \qquad \dot{P}_i = -\frac{\partial \mathcal{H}}{\partial Q_i} = 0.$$

It is sufficient to choose the transformed Hamiltonian, $\mathcal{H}$, to be zero. This implies that

$$\mathcal{H} = H(q_i, p_i, t) + \frac{\partial S(q_i, P_i, t)}{\partial t} = 0,$$

where $S$ is a generating function of the transformation.

# The Hamilton-Jacobi equation

Using the fact that $S$ generates $p_i$, this equation becomes

$$H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t} = 0, \qquad \text{since} \qquad p_i = \frac{\partial S}{\partial q_i}.$$

This is the Hamilton-Jacobi equation.

This first order differential equation has $D+1$ variables, and therefore, can be integrated with $D+1$ constants of integration to give the solution $S(q_i, \alpha_i, t)$. Since $S$ does not appear in the equations, but only its derivatives do, one of the constants of integration is an additive constant. This is immaterial to the dynamics, and hence can be dropped. Retaining $D$ constants of integration, $\alpha_i$, we can choose

$$P_i = \alpha_i, \quad Q_i = \frac{\partial S}{\partial \alpha_i} = \beta_i, \quad p_i = \frac{\partial S}{\partial q_i}.$$

The constants $\alpha_i$ and $\beta_i$ are related to the initial conditions. This solves the dynamics of the system.

## The generating function

Since

$$\frac{dS}{dt} = \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial t} = p_i\dot{q}_i - H = L.$$

In other words, the generating function is the action,

$$S = \int L dt.$$

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When $H$ is independent of time, then the only time dependence in

$$\frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}\right) = 0,$$

is in the time derivative. Hence the solution must be of the form $S = W - \alpha_1 t$. So the equation becomes

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) = \alpha_1.$$

The new and old Hamiltonians are the same, and $\alpha_1$ is its value.

# More on time-independent Hamiltonians

For a time-independent Hamiltonian, the Hamilton-Jacobi equation can be written in terms of $W$, which generates a quite different canonical transformation:

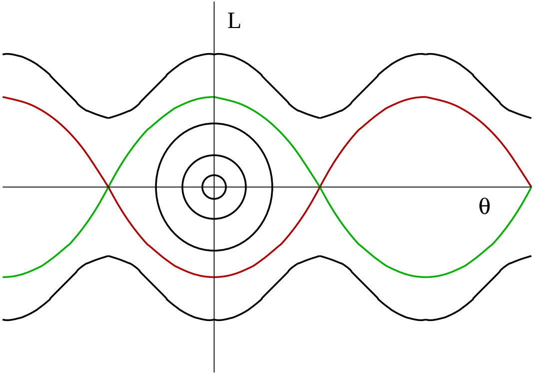
$$p_i = \frac{\partial W}{\partial q_i} \quad \text{and} \quad Q_i = \frac{\partial W}{\partial \alpha_i}, \quad \text{with} \quad H(p_i, q_i) = \alpha_1.$$

The canonical equations after transformation are

$$P_i = \alpha_i, \quad \dot{Q}_i = \delta_{i1}, \qquad Q_i = t\delta_{i1} + \beta_i.$$

One is at liberty to choose $P_i$ to be some functions of $\alpha_i$, which would also lead to conserved values of $P_i$. However, then the canonical equation would not lead to $\dot{Q}_i = 0$. Instead one would find the $Q_i$ to be cyclic coordinates; these are called action-angle variables. Phase space trajectories would then lie on tori. Such systems are said to be integrable.

# Tori in phase space



# Two problems

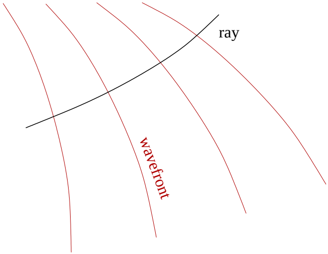
### Problem 71: Small oscillations

Investigate the harmonic oscillator in $D$ dimensions and show that it has $D$ constants of motion. Apply the Hamilton-Jacobi equations to solve this problem and hence show that small oscillations of non-rigid systems is an integrable problem.

### Problem 71: The Kepler problem

Solve the Kepler problem using the Hamilton Jacobi method. Is motion in a $1/r$ potential integrable in all dimensions of space? Investigate one, two, three and four dimensions and try to find the number of conserved quantities in each in order to answer this question.

# Analogy with optics




For time-independent Hamiltonians, if $S=\sigma$, then $W=\sigma-Et$, *i.e.*, the change in surfaces of constant $S$ lead to travelling wavefronts of constant $W$, with $dW=Edt$, since motion is periodic.

However, along the ray $dW=\nabla W ds$ so the phase velocity of the wave:

$$u=\frac{ds}{dt}=\frac{W}{\nabla W}=\frac{E}{\sqrt{2m(E-V)}}=\frac{E}{\sqrt{2mT}},$$

provided that the kinetic energy is quadratic in momenta.

# Keywords and References

## Keywords

Hamilton-Jacobi equation, action, action-angle variables, tori, integrable systems

## References

Appropriate chapters of Goldstein and Landau