Phase space flows and chaos

Sourendu Gupta

TIFR, Mumbai, India

Classical Mechanics 2011
October 27, 2011

Vector fields and flows

A system of first order differential equations

$$\dot{\boldsymbol{\xi}} = \mathbf{f}(\boldsymbol{\xi}),$$

defines a vector field on the space of variables $\boldsymbol{\xi}$. Any vector field on a space is said to define a flow on the space: *i.e.*, a mapping of the space on itself.

Fixed points of a flow, $\boldsymbol{\xi}^*$, are defined by the condition $\mathbf{f}(\boldsymbol{\xi}^*) = 0$. This means that at a fixed point $\dot{\boldsymbol{\xi}} = 0$. A fixed point is said to be attractive if all nearby points flow into the fixed point as $t \to \infty$. A repulsive fixed point occurs when all nearby points flow into it as $t \to -\infty$. A fixed point can attract in some directions and repel in others.

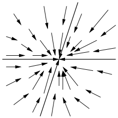
Vector fields and flows

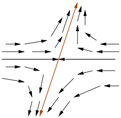
A system of first order differential equations

$$\dot{\boldsymbol{\xi}} = \mathbf{f}(\boldsymbol{\xi}),$$

defines a vector field on the space of variables $\boldsymbol{\xi}$. Any vector field on a space is said to define a flow on the space: *i.e.*, a mapping of the space on itself.

Fixed points of a flow, $\boldsymbol{\xi}^*$, are defined by the condition $\mathbf{f}(\boldsymbol{\xi}^*) = 0$. This means that at a fixed point $\dot{\boldsymbol{\xi}} = 0$. A fixed point is said to be attractive if all nearby points flow into the fixed point as $t \to \infty$. A repulsive fixed point occurs when all nearby points flow into it as $t \to -\infty$. A fixed point can attract in some directions and repel in others.





Hamiltonian flows

The canonical equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

define a Hamiltonian flow on phase space, *i.e.*, a map $\{q_i, p_i\} \to \{q_i, p_i\}$. Liouville's theorem implies that fixed points of Hamiltonian flows can be neither attractive nor repulsive. Interesting structures arise when the energy in not conserved, *i.e.*, for systems which are either driven (energy is added), or for systems which are dissipative (energy is lost) or for driven dissipative systems. In this case it is interesting to consider an extended phase space: (q_i, p_i, t).

Problem 75: Extended phase space

Draw the trajectories of a simple harmonic oscillators and the physical pendulum in extended phase space.

Poincaré sections

If the driving force of a driven system has period T, then we can take sections of the extended phase space at times t, $t+T$, $t+2T$, *etc.*. Such stroboscopic pictures of the evolution of the system are called Poincaré sections through extended phase space.

Problem 76: A driven system

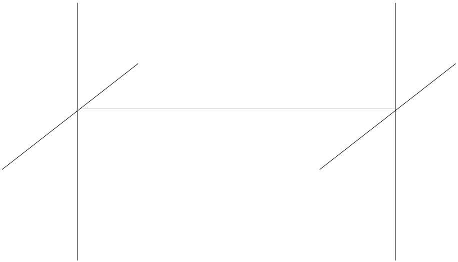
Consider a ball bouncing elastically inside a box, one wall of which is slowly driven in. Construct adiabatic invariants and draw the trajectories in extended phase space.

Problem 77: A periodically driven system

In the above problem, if the wall is driven in for time $T/2$ and then drawn out in time $T/2$ so that after time nT the wall is always back at the same position, then draw the trajectories in extended phase space. Draw the Poincaré sections.

Simple facts about Poincaré sections

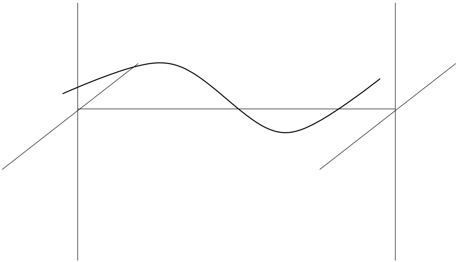
Consider the system of equations $\dot{\xi} = \mathbf{f}(\boldsymbol{\xi})$. The Poincaré section is a map from phase space to itself: $A : \boldsymbol{\xi}(t) \to \boldsymbol{\xi}(t+T)$.



$\boldsymbol{\xi}^*$ is a fixed point if it corresponds to a periodic solution with period T. If the system $\dot{\boldsymbol{\xi}} = \mathbf{f}(\boldsymbol{\xi})$ is linear, then A is linear, *i.e.*, $\boldsymbol{\xi}(t+T) = A\boldsymbol{\xi}(t)$. If the system is Hamiltonian then $\det A = 1$.

Simple facts about Poincaré sections

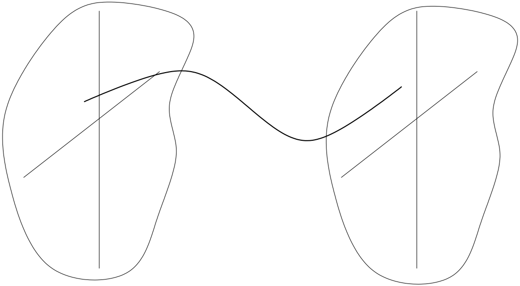
Consider the system of equations $\dot{\xi} = \mathbf{f}(\xi)$. The Poincaré section is a map from phase space to itself: $A : \xi(t) \to \xi(t + T)$.



ξ^* is a fixed point if it corresponds to a periodic solution with period T. If the system $\dot{\xi} = \mathbf{f}(\xi)$ is linear, then A is linear, *i.e.*, $\xi(t + T) = A\xi(t)$. If the system is Hamiltonian then $\det A = 1$.

Simple facts about Poincaré sections

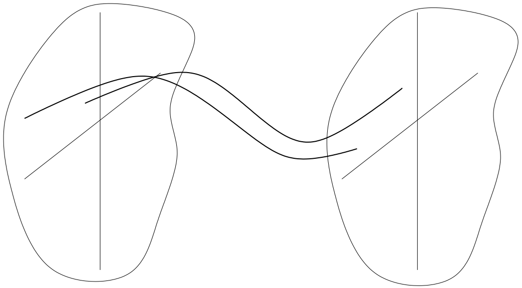
Consider the system of equations $\dot{\xi} = \mathbf{f}(\xi)$. The Poincaré section is a map from phase space to itself: $A : \boldsymbol{\xi}(t) \to \boldsymbol{\xi}(t+T)$.



$\boldsymbol{\xi}^*$ is a fixed point if it corresponds to a periodic solution with period T. If the system $\dot{\boldsymbol{\xi}} = \mathbf{f}(\boldsymbol{\xi})$ is linear, then A is linear, *i.e.*, $\boldsymbol{\xi}(t+T) = A\boldsymbol{\xi}(t)$. If the system is Hamiltonian then $\det A = 1$.

Simple facts about Poincaré sections

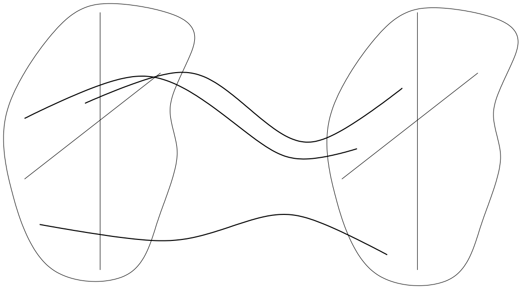
Consider the system of equations $\dot{\xi} = \mathbf{f}(\boldsymbol{\xi})$. The Poincaré section is a map from phase space to itself: $A : \boldsymbol{\xi}(t) \to \boldsymbol{\xi}(t+T)$.



$\boldsymbol{\xi}^*$ is a fixed point if it corresponds to a periodic solution with period T. If the system $\dot{\boldsymbol{\xi}} = \mathbf{f}(\boldsymbol{\xi})$ is linear, then A is linear, *i.e.*, $\boldsymbol{\xi}(t+T) = A\boldsymbol{\xi}(t)$. If the system is Hamiltonian then $\det A = 1$.

Simple facts about Poincaré sections

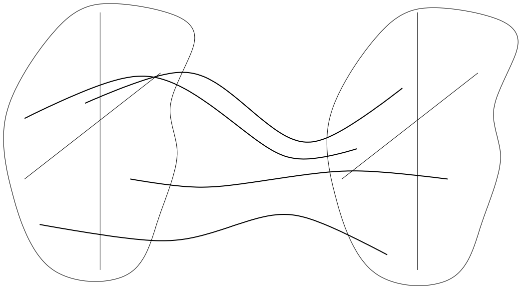
Consider the system of equations $\dot{\xi} = \mathbf{f}(\xi)$. The Poincaré section is a map from phase space to itself: $A : \boldsymbol{\xi}(t) \to \boldsymbol{\xi}(t+T)$.



$\boldsymbol{\xi}^*$ is a fixed point if it corresponds to a periodic solution with period T. If the system $\dot{\boldsymbol{\xi}} = \mathbf{f}(\boldsymbol{\xi})$ is linear, then A is linear, *i.e.*, $\boldsymbol{\xi}(t+T) = A\boldsymbol{\xi}(t)$. If the system is Hamiltonian then $\det A = 1$.

Simple facts about Poincaré sections

Consider the system of equations $\dot{\xi} = \mathbf{f}(\xi)$. The Poincaré section is a map from phase space to itself: $A : \xi(t) \to \xi(t+T)$.



ξ^* is a fixed point if it corresponds to a periodic solution with period T. If the system $\dot{\xi} = \mathbf{f}(\xi)$ is linear, then A is linear, *i.e.*, $\xi(t+T) = A\xi(t)$. If the system is Hamiltonian then $\det A = 1$.

Parametric resonance

Consider the system of equations

$$\ddot{x} + \omega^2(t)x = 0, \quad \text{with} \quad \omega^2(t+T) = \omega^2(t),$$

and let $\boldsymbol{\xi} = (x, \dot{x})$. In 2d phase space the condition that $\det A = 1$ implies that the product of eigenvalues $\lambda_1\lambda_2 = 1$. Clearly $\mathrm{Tr}\, A$ is real. If $\lambda_{1,2}$ are real then $\mathrm{Tr}\, A \geq 2$ and if they are complex then $\mathrm{Tr}\, A \leq 2$.

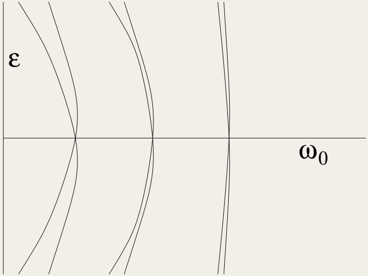
The solutions of the equations of motion are periodic if the Poincaré section has fixed points, *i.e.*, if $\mathrm{Tr}\, A \leq 2$. The limits of stability are given by the condition $\mathrm{Tr}\, A = 2$.

If $\omega^2(t) = \omega_0^2[1 + \epsilon f(t)]$. For $\epsilon = 0$ one can easily compute A. The initial conditions $\boldsymbol{\xi}_1(0) = (1, 0)$ gives the solution $\boldsymbol{\xi}_1(t) = (\cos\omega_0 t, -\omega_0 \sin\omega_0 t) = A\boldsymbol{\xi}_1(0)$. The orthogonal initial condition $\boldsymbol{\xi})_2(0) = (0, 1)$ gives $\boldsymbol{\xi}_2(t) = A\boldsymbol{\xi}_2(0) = (\sin\omega_0 t, \omega_0 \cos\omega_0 t)/\omega_0$. As a result, we find that $\mathrm{Tr}\, A = 2\cos 2\pi\omega_0$. The solution is unstable at $\cos 2\pi\omega_0 = 1$.

Parametric resonance

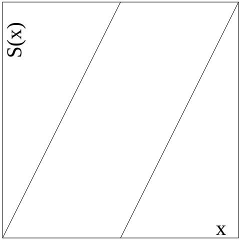
Problem 78: Parametric resonance

Take the simpler periodic forcing function $\omega(t) = \omega_0 + \epsilon$ for $0 \le t \le \pi$ and $\omega(t) = \omega_0 - \epsilon$ for $\pi \le t \le 2\pi$. For this function find the matrix $A(\epsilon)$ and show that the loci of the edge of periodicity are as shown in the figure below.



Simple models of dynamics

Simple models of driven dissipative systems are maps from the interval $U = [0,1]$ to itself: $M : U \to U$. These are models of the Poincare section for some flow.

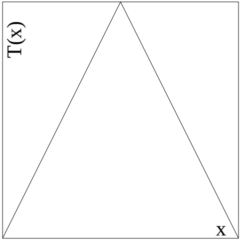


The map $S : U \to U$ is given by

$$S_a(x) = \begin{cases} ax & (x < 1/2) \\ ax - 1 & (x > 1/2) \end{cases},$$

with $a = 2$. Express the number $x \in U$ in binary $x = 0 \cdot b_1 b_2 b_3 \cdots$, where $b_i \in \{0, 1\}$. Clearly $S_2(x)$ corresponds to dropping the most significant bit of the expansion. S_2 exhibits sensitive dependence on initial conditions, for if $|x - y| \leq 2^{-m}$ then after m iterations of S_2, the points are arbitrarily far apart. Since every rational number has terminating or periodic binary representation, each rational point is either attracted to zero or is an member of a periodic orbit.

Another simple map



Using the binary representation $x = 0 \cdot b_1 b_2 b_3 \cdots$, examine

$$T_a(x) = \begin{cases} ax & (x < 1/2) \\ a(1-x) & (x > 1/2) \end{cases}$$

With the definition that for any bit, $\overline{b} = 1 - b$, clearly

$$T_2(x) = \begin{cases} 0 \cdot b_2 b_3 \cdots & (x < 1/2) \\ 0 \cdot \overline{b}_2 \overline{b}_3 \cdots & (x > 1/2) \end{cases},$$

As a result, $T_2(T_2(x)) = T_2(S_2(x))$ and $T^{n+1} = TS^n$. Using these one can demonstrate that T_2 shows sensitivity to initial conditions, and that rational points are periodic or attracted to zero.

The logistics map and chaos

Examine the logistics map $L_a(x) = ax(1-x)$. Define an auxiliary function $h(x) = \sin^2(\pi x/2)$ and let $y = h(x)$. Then

$$L_4(y) = 4\sin^2\left(\frac{\pi x}{2}\right)\cos^2\left(\frac{\pi x}{2}\right) = \sin^2(\pi x) = h(T_2(x)).$$

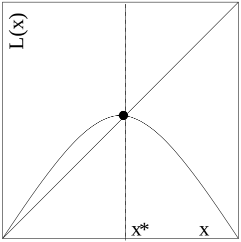
Since $L_4 h = h T_2$, results on periodicity and sensitivity to initial conditions also follow for L_4.

Chaos

Dynamics of a mechanical system is said to be chaotic when there is

1. an infinite number of periodic points of all periodicity and
2. extreme sensitivity to intial conditions otherwise.

Bifurcations



If $x^*(a) \neq 0$ is a fixed point of the logistics map then

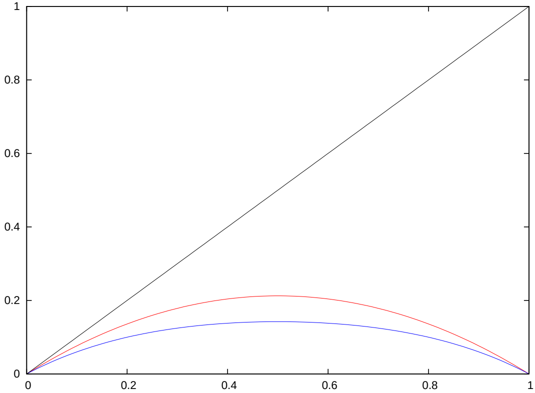
$$x^*(a) = ax^*(a)[1 - x^*(a)], x^*(a) = 1 - \frac{1}{a}.$$

Let $x = x^* + \delta$. Then $L_a(x) = x^* + \delta L'(x^*)$. As long as $|L'(a)| < 1$ the fixed point is attractive. Since $L'(x^*) = 2 - a$, the fixed point is attractive for $1 \leq a \leq 3$.
The first iteration of the map is

$$L_a^2(x) = L_a(L_a(x)) = a^2 x(1 - x)\{1 - ax(1 - x)\}.$$

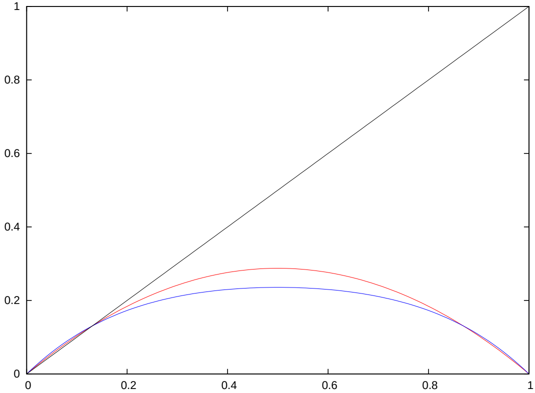
This map has a fixed point at $x = 0$ and at most three more. When does it have 1,2 or 4 fixed points? Which of these are stable?

Iterations of the logistics map



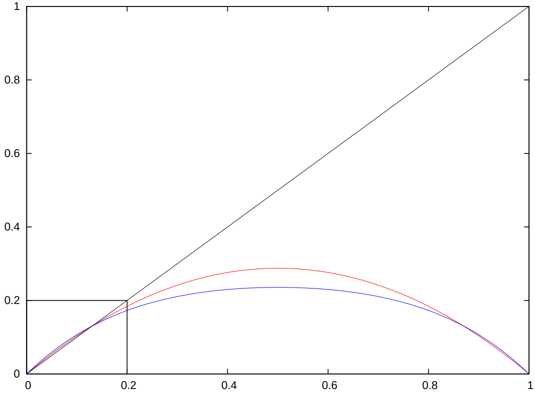
Attractive FP: $1 \leq a \leq 3$. The iterated map must also have the same attractive FP.

Iterations of the logistics map

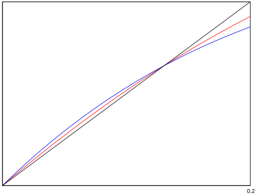


Attractive FP:
$1 \leq a \leq 3$. The iterated map must also have the same attractive FP.

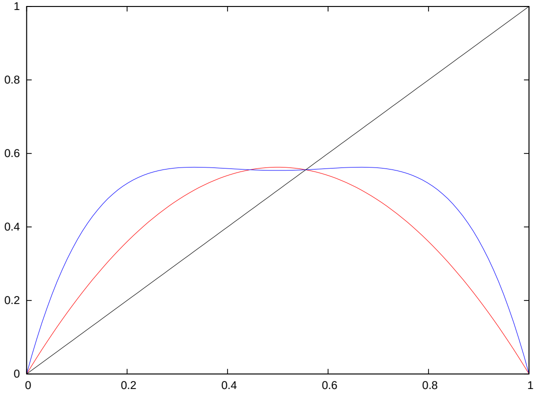
Iterations of the logistics map



Attractive FP: $1 \leq a \leq 3$. The iterated map must also have the same attractive FP.

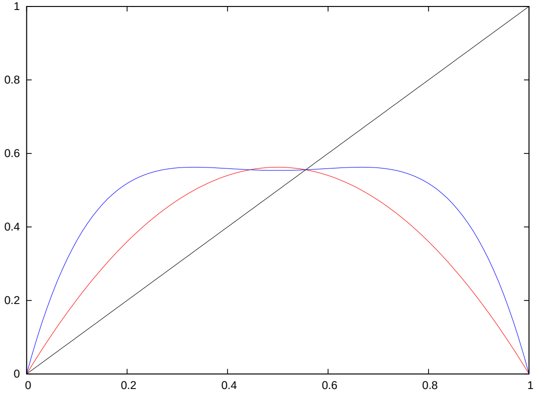


Iterations of the logistics map

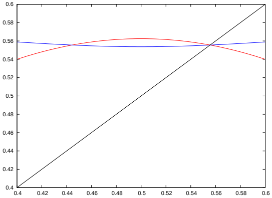


Attractive FP: $1 \leq a \leq 3$. The iterated map must also have the same attractive FP.

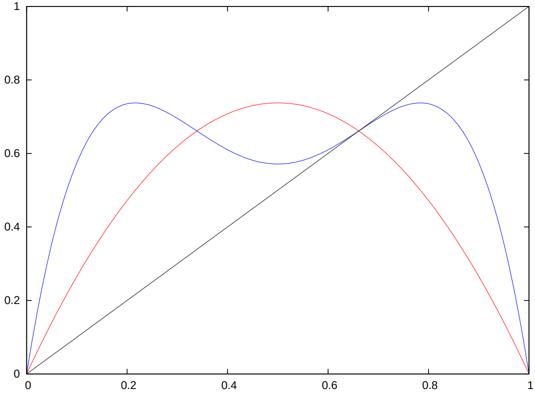
Iterations of the logistics map



Attractive FP: $1 \leq a \leq 3$. The iterated map must also have the same attractive FP.

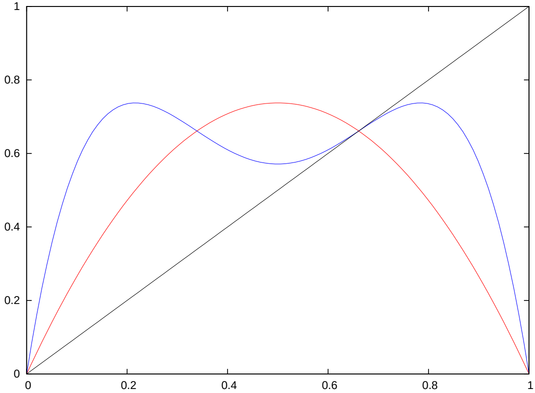


Iterations of the logistics map

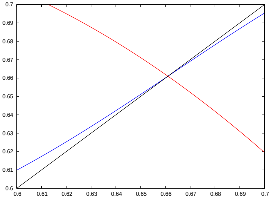


Attractive FP:
$1 \leq a \leq 3$. The iterated map must also have the same attractive FP.

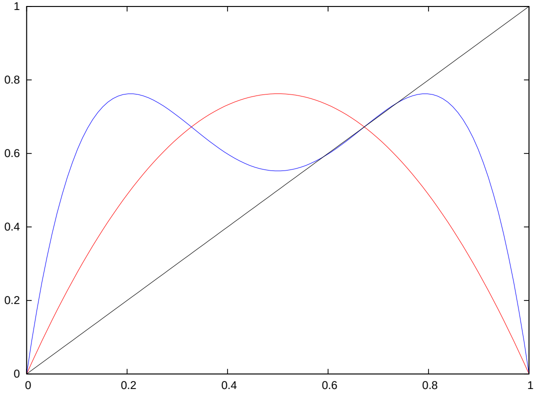
Iterations of the logistics map



Attractive FP: $1 \leq a \leq 3$. The iterated map must also have the same attractive FP.

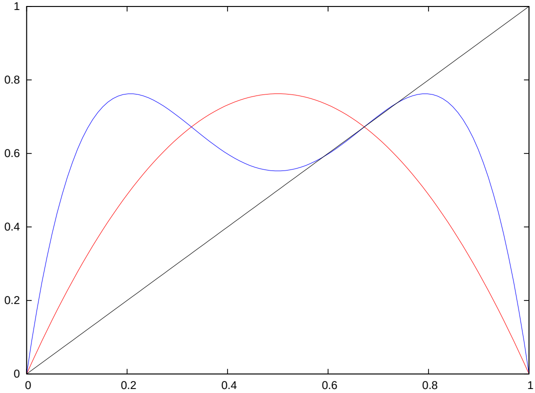


Iterations of the logistics map

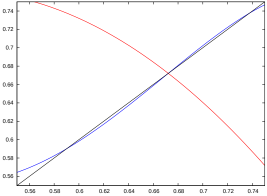


Attractive FP: $1 \leq a \leq 3$. The iterated map must also have the same attractive FP.

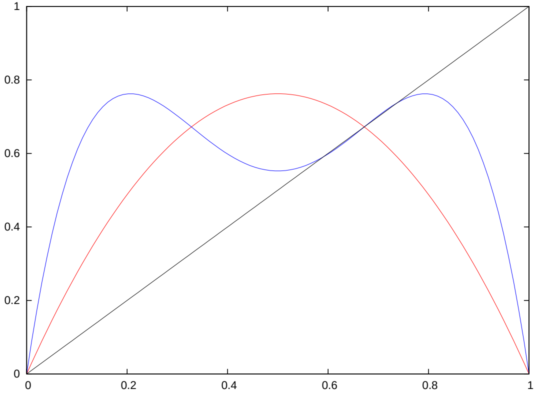
Iterations of the logistics map



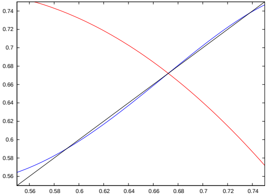
Attractive FP:
$1 \leq a \leq 3$. The iterated map must also have the same attractive FP.



Iterations of the logistics map

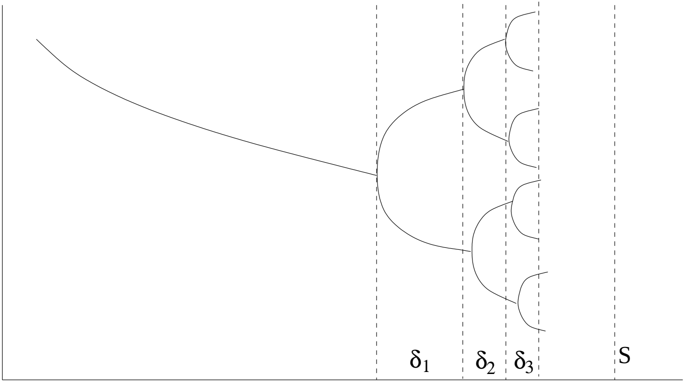


Attractive FP:
$1 \leq a \leq 3$. The iterated map must also have the same attractive FP.



Bifurcation of the FP!

Period doubling route to chaos



$$F = \lim_{k\to\infty} \frac{\delta_k}{\delta_{k+1}} = 4.6692016091029\cdots \qquad S_\infty = 3.5699456$$

The Feigenbaum number F is universal, S_∞ depends on the map.

Investigating chaos

Problem 79: Numerical investigations

1. Find the fixed points of $L_a^2(x)$ in closed form as a function of a, and find the range of a over which there are four real fixed points. Analyze whether these are attractive fixed points. Use your results to obtain a first approximation to the Feigenbaum number.
2. Numerically find the fixed points of $L_a^4(x)$ as a function of a, and find the range of a over which there are 8 real fixed points. Which of these are attractive? Use your results to obtain a second approximation to the Feigenbaum number.
3. How many CPU seconds does your program (along the lines of the above questions) take to compute correctly the first 5 digits of the Feigenbaum number?

Keywords and References

Keywords

vector field, flow, Fixed points, attractive fixed point, repulsive fixed point, Hamiltonian flow, extended phase space, Poincaré sections, sensitive dependence on initial conditions, logistics map, chaos, bifurcations, Feigenbaum number, period doubling route to chaos

References

Appropriate sections of Landau. Arnold, section 25.
http://en.wikipedia.org/wiki/Logistic_map