Hamilton's principle and Symmetries

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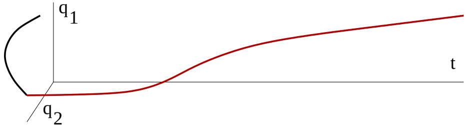
Classical Mechanics 2011
August 12, 2011

Action

For a mechanical system with generalized coordinates q_1, q_2, in motion between times t_1 and t_2, the action is defined as the integral

$$S[t_1, t_2] = \int_{t_1}^{t_2} dt L(q_1, q_2, \cdots, \dot{q}_1, \dot{q}_2, \cdots, t)$$

The value of the action depends on the world line of the particle $\{q_1(t), q_2(t), \cdots\}$: it is a functional of the world line.



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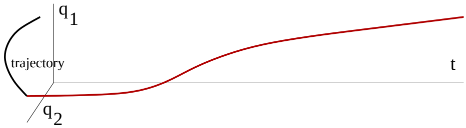

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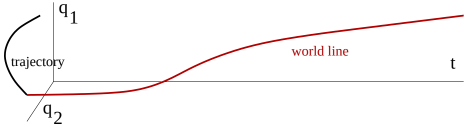

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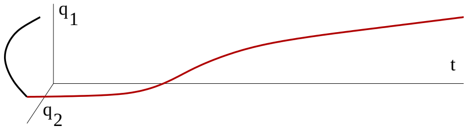


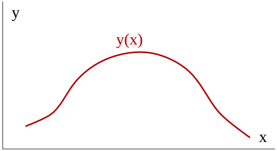
Hamilton's principle

The particle's actual trajectory is the one that minimizes the action subject to the boundary conditions imposed on it.

Functionals: easy as π

An example of a functional of a curve: the area under a curve!

$$A[y] = \int_{x_i}^{x_f} dx y(x).$$



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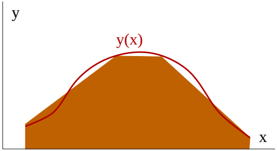

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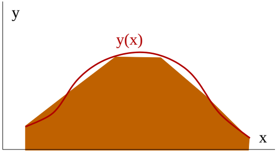

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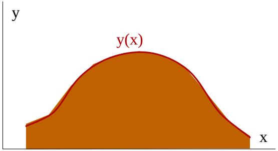

The area is a functional of the curve. Use Riemann sums to approximate the area in terms of N variables $y(x_1)$, $y(x_2)$, *etc.*,

$$A[y] = h_N \sum_{i=1}^{N} y(x_i), \qquad \text{where} \qquad h_N = \frac{x_f - x_i}{N}.$$

Finding variations of the area with the curve is just calculus of many variables.

Calculus of variations

Finding an extremum of a function involves setting its derivatives to zero and then solving the resulting equations. Checking whether an extremum is a maximum or minimum involves checking the sign of the second derivative. The minimum of a functional is similar.

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$$\delta L = \sum_k \delta q_k \frac{\partial L}{\partial q_k} + \delta \dot{q}_k \frac{\partial L}{\partial \dot{q}_k}$$

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$$\delta L = \sum_k \delta q_k \frac{\partial L}{\partial q_k} + \frac{d}{dt}\left(\delta q_k \frac{\partial L}{\partial \dot{q}_k}\right) - \delta q_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)$$

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Now we put this inside the integral to evaluate δS. Since

$$\int_{t_1}^{t_2} dt \frac{d}{dt}\left(\delta q_k \frac{\partial L}{\partial \dot{q}_k}\right) = \delta q_k \frac{\partial L}{\partial \dot{q}_k}\Bigg|_{t_1}^{t_2} = 0,$$

we recover the Euler-Lagrange equations.

Two examples

	Quartic oscillator	Physical pendulum
L	$\frac{1}{2}m\dot{q}^2 - \frac{1}{4}V_0q^4$	$\frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1-\cos\theta)$
Q	V_0q^3	$-mg\ell\sin\theta$
p	$m\dot{q}$	$m\ell^2\dot{\theta}$
EoM	$m\ddot{q} + V_0q^3 = 0$	$m\ell^2\ddot{\theta} + mg\ell\sin\theta = 0$
	$\ddot{q} + \zeta^2q^3 = 0$	$\ddot{\theta} + \Omega^2\sin\theta = 0$
	where $\zeta^2 = \frac{V_0}{m}$	where $\Omega^2 = \sqrt{\frac{g}{L}}$
$t \to \Omega t$		$\ddot{\theta} + \sin\theta = 0$
$q \to q/\zeta$	$\ddot{q} + q^3 = 0$	

Numerical solutions

Minimization of the action can be carried out numerically very simply using a lattice discretization. Work with D degrees of freedom, $q_1, q_2, \cdots q_D$. Divide the time interval into N equal pieces. At times $t_j = t_0 + jh_N$ use the notation $q_k^j = q_k(t_j)$, so that the action is a function of $D(N-1)$ variables—

$$\begin{aligned} S &= \frac{m_k}{2h_N^2} \sum_{k=1,j=1}^{D,N} \left(q_k^j - q_k^{j-1} \right)^2 - \sum_{j=1}^{N} V(q_1^j, q_2^j, \cdots, q_D^j) \\ &= N^2 \sum_{j=1}^{N} \left[\sum_{k=1}^{D} q_k^j q_k^{j-1} - \overline{V}(q_1^j, q_2^j, \cdots, q_D^j) \right]. \end{aligned}$$

In the second line we have assumed that units have been chosen so that $m_k = 1$ and $t_N - t_0 = 1$. Since V depends only the q_k at one fixed time, we have defined $\overline{V} = V - N^2 \sum_k (q_k^j)^2$. The only connection between different times is the hopping term $q_k^j q_k^{j-1}$.

An idiotically simple algorithm for minimization

There are many ways to minimize a function of many variables. A mindlessly simple algorithm is:

1. Choose a stopping criterion ϵ.
2. Start with a trajectory $\{q_k^j\}$. The corresponding value of the action is S.
3. Select a random set of values $\{q'^j_k\}$ and compute the corresponding value of the action, S'.
4. $|S - S'| < \epsilon$ then stop. The trajectory is now $\{q_k^j\}$.
5. If $S' < S$ set $q_k^j = q'^j_k$. Repeat from step 3.

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Minimization of a function of many variables

Can you think of other ways of minimizing the action which do not use the Euler-Lagrange equations?

Two Problems

Problem 6: Simple harmonic motion

The Lagrangian for a simple harmonic oscillator becomes

$$L = \frac{1}{2}\dot{\theta}^2 - \frac{1}{2}\theta^2, \qquad \theta(0) = \frac{1}{2},\ \theta(1) = -\frac{1}{2}.$$

Find the trajectory by numerical minimization.

Problem 7: The physical pendulum

The Lagrangian for the physical pendulum is

$$L = \frac{1}{2}\dot{\theta}^2 - (1 - \cos\theta), \qquad \theta(0) = \frac{1}{2},\ \theta(1) = -\frac{1}{2}.$$

Find the trajectory by numerical minimization.

Integrals of motion

Some constraints on the solutions of the equations of motion can be found without solving the full problem, *i.e.*, without giving q's as functions of time. One example is of a free particle— in this case the integral of motion is the conserved value of the momentum. Similarly for a freely rotating body the conserved angular momentum is an integral of motion.

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First integrals of motion

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They are called first integrals because the equations of motion involve $\ddot{q}_k$, *i.e.*, differential equations of the second order. However, these conditions "integrate" the equations once, so that they provide differential equations of first order.

Symmetries of a system

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Symmetries of mechanical systems

These notions of symmetry permit immediate generalization. We say that a system posseses a symmetry (or symmetries) if one (or more) of the generalized coordinates do not appear in the Lagrangian.

Symmetries and integrals of motion

Write the Euler-Lagrange equations in the form

$$\dot{p}_k = \frac{\partial L}{\partial q_k}.$$

If a Lagrangian possesses a symmetry with respect to one of the generalized coordinates, q_k, then the right hand side vanishes. The symmetry leads to a first integral of motion, the conservation of the corresponding momentum.

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Noether's Theorem

Every symmetry leads to a conservation law for the corresponding momentum. Generalized coordinates associated with such integrals of motion are called cyclic coordinates.

Motion of a charged particle

Problem 8: Motion of a charged particle

What is the canonical momentum of a charged particle in a EM field? If the external fields are time-independent, then L does not depend explicitly on the time. In this case check whether $T+V$ is conserved. system?

Problem 9: A charged particle in a constant magnetic field

Are there any conserved quantities when a charged particle moves in a constant magnetic field and zero electric field? What does your analysis show about the general character of the trajectories?

Problem 10: A charged particle around a magnetic monopole

A magnetic monopole is any source which produces a radial magnetic field. For a charged particle moving in the field of a fixed magnetic monopole, what are the conserved quantities?

Scaling symmetries

Under a scaling $L \to \lambda L$ the equations of motion and its solutions remain unchanged.

Assume that under a scaling of coordinates $q_k \to \lambda q_k$, the potential scales as $V \to \lambda^\alpha V$. Scale the time as $t \to \lambda^\beta t$; the kinetic energy scales as $T \to \lambda^{2(1-\beta)} T$. For the Lagrangian to scale uniformly, one must have

$$2(1-\beta) = \alpha, \qquad i.e. \qquad \beta = 1 - \frac{\alpha}{2}.$$

As a result, two trajectories related by scaling must obey the law

$$\frac{t'}{t} = \left(\frac{l'}{l}\right)^{(1-\alpha/2)} .$$

Then Kepler's third law that the cube of the distance of a planet from the sun is proportional to the square of its period of revolution implies $\alpha = -1$, *i.e.*, Newton's law of gravity.

Keywords and References

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action, world line, functional, lattice discretization, hopping term, algorithm, stopping criterion, first integral of motion, symmetry, conservation law, cyclic coordinates, Noether's theorem, scaling, Kepler's third law, Newton's law of gravity

References

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Landau, Chapter II