The earth provides a non-inertial frame

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Inertial and non-inertial frames

In an inertial frame a particle moving without forces on it will obey the composition laws $\mathbf{v}' = \mathbf{v} + \mathbf{V}_f$ and $\mathbf{x}' = \mathbf{x} + \mathbf{v}'t$. Finding whether a frame is non-inertial involves testing these laws.

How close to inertial is an earth-fixed frame?

We know that the motion of a particle near the earth is strongly non-inertial in the vertical direction, since everything falls. So the question is refined to examining motion in the horizontal directions. Since the earth rotates rigidly around a north-south axis, an earth-fixed frame will have different east-west velicities at different latitudes. The question is about the level of accuracy needed to detect this.

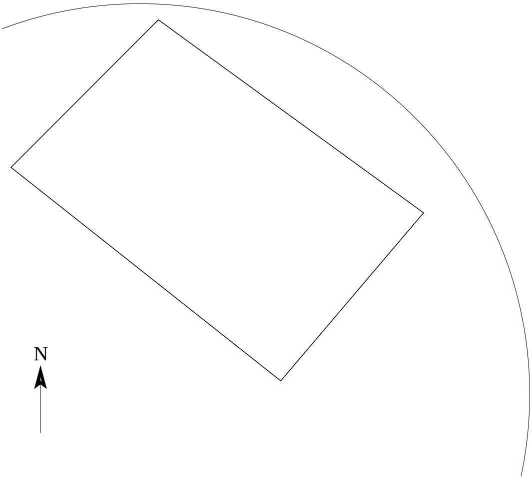
Motion near the earth's surface

Scales of the problem

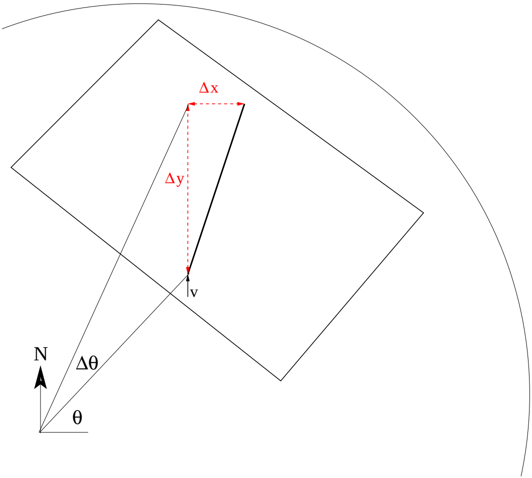
First restrict ourselves to displacements $|\mathbf{x}|$ which are much smaller than the radius of the earth, R_E. As long as $\delta = |\mathbf{x}|/R_E \ll 1$ the curvature of the earth's surface can be neglected, and the geometric and dynamic effects can be separated. We may take $|\mathbf{x}| \simeq 1$ Km, so that $\delta \simeq 10^{-4}$.
The earth's rotational speed is $v_E = 2\pi R_E \cos\theta / T_D$ where θ is the latitude and $T_D = 86400$ s is the length of the day (*i.e.*, $T_D \simeq 100$ Ks). The rotational speed at the equator is $V_E^E = 2\pi R_E / T_D \simeq 1$ Km/s.

North-south (NS) motion clearly shows non-inertial effects through Coriolis acceleration, *i.e.*, the apparent deflection of the particle in the east-west (EW) direction. EW motion at two latitudes can be compared to detect non-inertial effects.

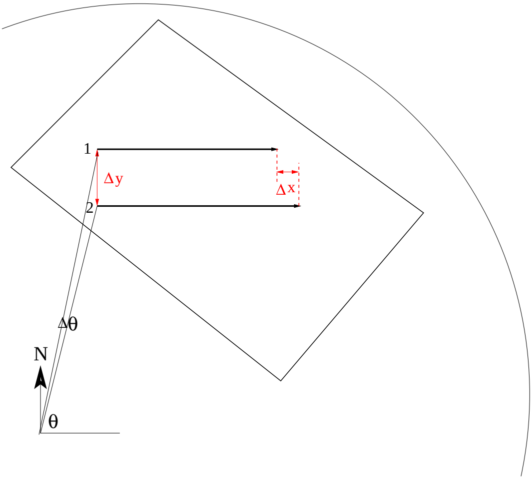
Experiments near the earth



Experiments near the earth



Experiments near the earth



East-West Motion

Particle 1 at latitude θ, sees the earth's tangential velocity as $v_1^E = V_E^E \cos\theta$; particle 2 at $\theta + \Delta\theta$ sees $v_2^E = v_1^E + V_E^E \Delta\theta \sin\theta$. The two particles start with the same initial velocities and separated only by a NS distance

$$\Delta y = R_E \Delta\theta \sin\theta = V_E^E T_D/(2\pi)\Delta\theta \sin\theta.$$

The EW distance between them changes as $\Delta x = t V_E^E \Delta\theta \sin\theta$. A dimensionless measure of how big the non-inertial effects are is given by $\epsilon = \Delta x/\Delta y = 2\pi t/T_D$. The only controllable parameter is the duration of the experiment, t.

Typically, if the vertical component of the particle's displacement is about 1 Km, then $t \simeq 10$ s, and $\epsilon \simeq 10^{-3}$. So, in motions covering about a Km, the fact that the earth is not an inertial frame changes the motion by about 1 m.

North-South Motion

A particle is thrown in a NS direction, with velocity v and travels a distance $\Delta y = vT \ll R_E$. The EW deflection in a time interval between t and $t + dt$ is $dt v_E(\theta)$, where v_E is the earth's tangential velocity at the latitude $\theta(t) = \theta_i + vt/R_E$. The net EW deflection is

$$\Delta x = \int_0^T dt v_E(\theta) = R_E \left(\frac{V_E^E}{v}\right) \int_{\theta_i}^{\theta_i + \Delta y/R_E} d\theta \cos\theta \simeq \frac{V_E^E}{v} \Delta y.$$

We can define $\epsilon = \Delta x/\Delta y = V_E^E/v$. This dimensionless number ϵ captures how non-inertial the frame is. Clearly, if the earth were not rotating, so that V_E^E vanished, so would ϵ.

Why does a slow moving particle show the earth's rotation better? To understand this, note that in the last expression in the displayed equation above, $\Delta y/v$ is just the time T over which we study the motion. When v decreases, T increases, and the rotation of the earth becomes more apparent.