The Kepler Problem

Sourendu Gupta

TIFR, Mumbai, India

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Angular motion

Since the angular term in the Lagrangian is $L_\phi = mr^2\dot{\phi}^2/2$, the coordinate ϕ is cyclic. We have

$$p_\phi = \frac{\partial L_\phi}{\partial \dot{\phi}} = mr^2\dot{\phi} = |\mathbf{L}|, \quad \text{and} \quad 0 = \frac{\dot{p}_\phi}{2m} = \frac{d}{dt}\left(\frac{1}{2}r^2\dot{\phi}\right) = \dot{A},$$

where $\dot{A}$ is the rate at which an area element of the orbit is swept out. This is Kepler's second law of planetary motion. It is independent of the form of the potential, and is just an expression of the conservation of angular momentum, *i.e.*, of rotational invariance in space.

The polar angle traversed as a function of time is given by the expression

$$\phi = \frac{|\mathbf{L}|}{m}\int_0^t \frac{dt}{r^2}, \quad \text{and} \quad 2\pi = \frac{|\mathbf{L}|}{m}\int_0^T \frac{dt}{r^2},$$

where T is the time period of the angular motion.

The Virial of Clausius

For N bodies moving only under their mutual interactions, consider the quantity called the virial, *i.e.*,

$$G = \sum_{i=1}^{N} \mathbf{p}_i \cdot \mathbf{x}_i.$$

Then

$$\dot{G} = \sum_{i=1}^{N} \dot{\mathbf{p}}_i \cdot \mathbf{x}_i + \mathbf{p}_i \cdot \dot{\mathbf{x}}_i = \sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{x}_i + m v_i^2.$$

The second term is twice the kinetic energy, $2T$. Averaging this over a very long time, $\mathcal{T}$, one finds

$$\mathop{\rm Lim}_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \int \dot{G}\, dt = \overline{2T} + \overline{\sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{x}_i}.$$

The Virial Theorem and its applications

Now, the left hand side is $[G(0) - G(\mathcal{T})]/\mathcal{T}$. Now if the motion is periodic (or bounded) then this expression vanishes in the limit $\mathcal{T} \to \infty$. This is the content of the Virial Theorem

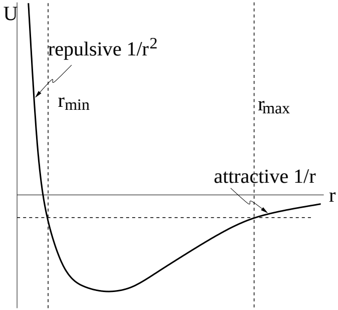
$$\overline{T} = -\frac{1}{2} \overline{\sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{x}_i} = \frac{1}{2} \overline{\sum_{i=1}^{N} \mathbf{x}_i \cdot \nabla_i V}.$$

If this is satisfied by a system of particles, then they must be in a stable bound configuration.

Problem 34: The classical atom

Take a classical atom with $Z = 8$, *i.e.*, a nucleus of charge $-8e$ with 8 electrons orbiting it. If this is in a stable bound configuration with typical size of 10^{-10} m, then what is the typical speed of the electrons?

Radial motion



1. $H = p^2/2m + U(r)$ with $U(r) = -\kappa/r + \lambda^2/r^2$ where $\kappa = GMm$ and $\lambda^2 = |\mathbf{L}|^2/(2m)$.
2. $t = \int dr[2\{E - U(r)\}]^{-1/2}$
3. Centrifugal term must be repulsive
4. When $E > 0$ $r_{\rm max} \to \infty$
5. Apses are points on the orbit where $\dot{r} = 0$. $r_{\rm min}$ and $r_{\rm max}$ called apsidal distances.
6. Circular orbit at the minimum of the potential

Equation of the orbit

The orbit is the path traced in space: $r(\phi)$. It has less information than the trajectory, *i.e.*, $\mathbf{x}(t)$. To find the orbit we note that

$$\dot{\phi} = \sqrt{\frac{2}{m}}\frac{\lambda}{r^2}, \qquad \text{and} \qquad \dot{r} = \sqrt{\frac{2}{m}[E - U(r)]}.$$

λ, and E are fixed for each trajectory. Using these two equations together we find that

$$\frac{d\phi}{dr} = \frac{\lambda}{r^2\sqrt{E-U}}, \quad \text{so} \quad \phi = \phi_0 + \int_{r_0}^{r} \frac{\lambda dr/(\sqrt{E}r^2)}{\sqrt{1-U/E}}.$$

The subscript zero on any variable denotes its initial value. Now introduce the notation $u = \lambda/(\sqrt{E}r)$, so that $U/E = u^2 - \gamma u$ where $\gamma = \kappa/(\lambda\sqrt{E})$ and $du = -\lambda dr/(\sqrt{E}r^2)$.

The orbit

The orbit then becomes

$$\phi = \phi_0 - \int_{u_0}^{u} \frac{du}{\sqrt{1 + \gamma u - u^2}}.$$

Using the fact that $1 + \gamma u - u^2 = 1 + \gamma^2/4 - (u - \gamma/2)^2$, and defining $\cos z = (u - \gamma/2)/\sqrt{1 + \gamma^2/4}$, we have

$$\phi - \phi_0 = z - z_0.$$

If we choose the initial condition such that $z_0 = 0$, then

$$u = \frac{\gamma}{2} + \sqrt{1 + \frac{\gamma^2}{4}} \cos(\phi - \phi_0).$$

Rewriting this in terms of r, we get the equation for the orbit

$$\frac{1}{r} = \frac{\kappa}{2\lambda^2}[1 + e\cos(\phi - \phi_0)], \quad \text{where} \quad e = \sqrt{1 + \frac{4E\lambda^2}{\kappa^2}}.$$

Note that when $E < 0$ then $e < 1$ and for $E > 0$ one has $e > 1$.

Conic sections

Choose the unit of distance to be $R = 2\lambda^2/\kappa$. Make the further choice that the polar angle is measured from the distance of closest approach of the particles. Since this is a choice of orientation of the axes, it can be made independently of the initial conditions. The resulting equation for the orbit is $1/r = 1 + e\cos\phi$.
Now we show that this is the equation of a conic section by rewriting this in Cartesian coordinates. Since $r^2 = x^2 + y^2$ and $\cos\phi = x/r$. One finds $1 - ex = r$.
Two special cases arise. For $e = 0$ one clearly has $r = 1$, *i.e.*, a circle. For $e = 1$ one obtains the equation $y^2 = 1 - 2x$, which is the equation of a parabola. More generally one finds $(1 - e^2)x^2 + 2ex + y^2 = 1$, which gives

$$(1 - e^2)\left(x + \frac{e}{1 - e^2}\right)^2 + y^2 = \frac{1}{1 - e^2}.$$

This is an ellipse for $e < 1$ and a hyperbola for $e > 1$.

Circular orbits

Since circular orbits occur when $e = 0$, one has,

$$\frac{4E\lambda^2}{\kappa^2} = -1, \quad \text{so} \quad E = -\frac{\kappa^2}{4\lambda^2}.$$

The radius of this orbit is $R = 2\lambda^2/\kappa$. So, the energy can be written in terms of R in the form

$$E = -\frac{\kappa}{2R},$$

which is half the potential energy. The remainder is in the kinetic energy, as is expected from the virial theorem.

The angular speed of rotation is $2\pi/T$, where T is the period of the circular orbit. As a result, $T = \pi R^2\sqrt{2m}/\lambda$. One can write this in the forms

$$T = 2\pi R\sqrt{\frac{mR}{\kappa}} = 2\pi R\sqrt{\frac{m}{-2E}}.$$

Elliptic orbits

The apsidal distances are easily computed from the energy equation, since the velocities vanish at these points. The equation to be solved is $E = U(r)$. This gives

$$r^2 = \frac{\lambda^2}{E} - \left(\frac{\kappa}{E}\right) r.$$

The coefficient of the linear term is the sum of the apsidal distances and hence equal to twice the semi-major axis of the ellipse. Therefore one has the expressions

$$a = -\frac{\kappa}{2E}, \qquad \text{and} \qquad e = \sqrt{1 - \frac{2\lambda^2}{\kappa a}}.$$

The equation for the orbit can then be written in terms of a and e in the form

$$\frac{1}{r} = \frac{1 + e\cos\phi}{a(1-e^2)}.$$

This gives the apsidal distances $r_{\rm min} = a(1-e)$ and $r_{\rm max} = a(1+e)$, these distances being reached when $\phi = 0$ and π respectively.

The trajectory

We had chosen an initial condition $z_0 = 0$. From the discussion above, we find that this corresponds to choosing $r = r_{\min}$ at time $t = 0$. Now we can find the trajectory. From the angular velocity one has

$$t = \frac{m}{|\mathbf{L}|} \int_0^t r^2 d\phi = \frac{|\mathbf{L}|^3}{m\kappa^2} \int \frac{d\phi}{(1 + e\cos\phi)^2}.$$

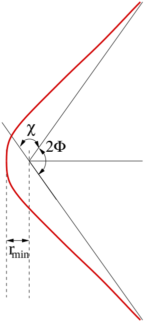
Vestiges of medieval astronomy

Starting instead from the radial velocity and introducing the eccentric anomaly, ψ, through $r = a(1 - e\cos\psi)$ where $0 \le \psi \le 2\pi$, one finds

$$t = \sqrt{\frac{ma^3}{\kappa}} \int_0^\psi d\psi(1 - \cos\psi), \qquad T = 2\pi\sqrt{\frac{ma^3}{\kappa}},$$

where the period T clearly obeys Kepler's third law of planetary motion. The integral gives Kepler's equation $\omega t = \psi - e\sin\psi$ where $\omega = 2\pi/T$ and ωt is called the mean anomaly. We get the trajectory by equating the expressions for r in terms of ϕ and ψ.

Hyperbolic orbits



For a hyperbolic orbit, one may write the orbit equation in the form

$$\frac{1}{r} = \frac{1}{R}(1 + e\cos\phi), \quad \text{with} \quad R = \frac{2\lambda^2}{\kappa}.$$

The scattering angle, χ, is related to the angle between the two asymptotes, 2Φ, by the expression $2\Phi + \chi = \pi$. Since one must have $1/r = 0$ asymptotically, $-\cos\Phi = \sin(\chi/2) = 1/e$. The impact parameter, ρ, determines the angular momentum $|\mathbf{L}| = \rho\sqrt{2mE}$, *i.e.*, $\lambda = \rho\sqrt{E}$. This gives

$$e^2 = 1 + \frac{4E\lambda^2}{\kappa^2} = 1 + \left(\frac{2E\rho}{\kappa}\right)^2, \quad \text{and} \quad \rho = \left(\frac{\kappa}{2E}\right)\cot\frac{\chi}{2}.$$

Rutherford scattering

Rutherford scattering is the result of two electric charges (of magnitude Ze and $Z'e$) projected towards each other. In this case $\kappa = ZZ'e^2$. The cross section is

$$\frac{d\sigma}{d\chi} = \frac{2\pi}{\sin\chi}\rho(\chi)\left|\frac{d\rho}{d\chi}\right| = \frac{\pi}{4}\left(\frac{\kappa}{E}\right)^2 \operatorname{cosec}^4\frac{\chi}{2}.$$

The same result is obtained in quantum mechanics. Since $\operatorname{cosec}\chi/2$ is infinite in the forward direction (*i.e.*, for $\chi \to 0$) one expects that most often the particles are barely scattered. These correspond to particles with large impact parameter. Very few particles are scattered at large angles, and these have small impact parameters.

Problem 35: Repulsive Coulomb forces

Solve the central forces problem with repulsive $1/r$ potential and show that all orbits are open and hyperpolic. Obtain the scattering cross section in this potential.

The Laplace Hamilton Gibbs Runge Lenz vector

2-particles moving under their mutual interactions generally evolve in a 4d phase space. But, in the Kepler problem, once r is specified, ϕ is also completely specified. So the phase space is effectively 2-dimensional. Hence there must be other conserved quantities.

Construct the vector $\mathbf{V} = \mathbf{p} \times \mathbf{L}$. Since $\dot{\mathbf{p}} = -\kappa\hat{\mathbf{r}}/r^2$, we can write

$$\dot{\mathbf{V}} = \dot{\mathbf{p}} \times \mathbf{L} = -\frac{m\kappa}{r^2}\hat{\mathbf{r}} \times (\mathbf{x} \times \dot{\mathbf{x}}) = \frac{m\kappa}{r^2}[r\dot{\mathbf{x}} - \hat{\mathbf{r}}(r\dot{r})],$$

using $\mathbf{x} = r\hat{\mathbf{r}}$ and $2\mathbf{x} \cdot \dot{\mathbf{x}} = d(r^2)/dt = 2r\dot{r}$. As a result

$$\dot{\mathbf{V}} = m\kappa\left[\frac{\dot{\mathbf{x}}}{r} - \frac{\mathbf{x}}{r}\frac{\dot{r}}{r}\right] = m\kappa\frac{d}{dt}\left(\frac{\mathbf{x}}{r}\right).$$

So the vector $\mathbf{A} = \mathbf{p} \times \mathbf{L} - m\kappa\hat{\mathbf{r}}$ is conserved. This is the Laplace-Hamilton-Gibbs-Runge-Lenz vector.

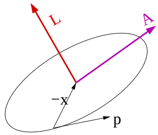
Since $\mathbf{A} \cdot \mathbf{L} = 0$, at most two components of $\mathbf{A}$ are non-vanishing. Next we show that the norm A is also known.

Equation of the orbit

Define the angle between $\mathbf{x}$ and $\mathbf{A}$ to be θ. Then $\mathbf{A}\cdot\mathbf{x} = Ar\cos\theta$. But, $\mathbf{A}\cdot\mathbf{x} = \mathbf{x}\cdot(\mathbf{p}\times\mathbf{L}) - m\kappa r$, and $\mathbf{x}\cdot\mathbf{p}\times\mathbf{L} = \mathbf{L}\cdot\mathbf{x}\times\mathbf{p} = |\mathbf{L}|^2$, then gives $Ar\cos\theta = |\mathbf{L}|^2 - m\kappa r$. Clearly this gives

$$\frac{1}{r} = \frac{m\kappa}{|\mathbf{L}|^2}\left(1 + \frac{A}{m\kappa}\cos\theta\right).$$

This is the equation of the orbit and immediately tells us that $A = m\kappa e$, so it is given in terms of κ, E and $|\mathbf{L}|$. Also, it tells us that the conserved vector $\mathbf{A}$ points to the periapsidal point.



The geometrical meaning of the various conserved quantities is clear. The direction $\hat{\mathbf{L}}$ specifies the plane of the orbit. E specifies the semi-major axis, *i.e.*, the size of the orbit. $|\mathbf{L}|$, together with E specify the eccentricity. $\hat{\mathbf{A}}$ defines the apsidal direction.

Problem 36: Inverse square potential

Two particles move in their mutual interaction potential $V(r) = -\kappa/r^2$ where $\kappa > 0$.

1. Find when the orbits are open and closed.
2. Check whether there are circular orbits.
3. Integrate the equations of motion to find the equations of the orbits.
4. Find the differential cross section for scattering of such particles.

Repeat all these steps for a potential $V(r) = -\kappa/r^3$ with $\kappa > 0$.

Keywords and References

Keywords

Kepler's second law of planetary motion, virial, Virial Theorem, apses, apsidal distances, orbit, conic section, semi-major axis of the ellipse, eccentric anomaly, Kepler's third law of planetary motion, Kepler's equation, mean anomaly, scattering angle, impact parameter, Rutherford scattering, periapsidal point Laplace-Hamilton-Gibbs-Runge-Lenz vector,

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