The special theory of relativity

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The constancy of the speed of light

There is experimental evidence that the speed of light, c, is the same in all inertial frames. Since the distance travelled by light is related to the time of flight, in two different inertial frames, one must have

$$c^2t^2 - x^2 - y^2 - z^2 = 0 = c^2(t')^2 - (x')^2 - (y')^2 - (z')^2.$$

The transformation from one inertial frame to another cannot have the form required in Galilean relativity, but must mix space and time.

Problem 37: The Michelson-Morley Experiment

The Michelson-Morley experiment consists of measuring the speed of light in two orthogonal directions in an earth-fixed frame. Since the earth is not exactly an inertial frame (see Problem 3), find the percent error in the measurement of the constancy of the speed of light through such an experiment.

Four vectors

Note that all of mechanics involves the special speed c. In all problems we can measure velocities in units of c. Introduce the notation $\beta = v/c$, which we call a boost. Velocities then become dimensionless, which means that the dimensions of length and time are the same. We take the dimensions of time to be m (sometimes written m$/c$). In these units, the frame invariant quantity which we had written down earlier becomes

$$t^2 - x^2 - y^2 - z^2 = 0 = (t')^2 - (x')^2 - (y')^2 - (z')^2.$$

Introduce a 4 dimensional vector so we can write this as

$$\mathbf{x}^T G \mathbf{x} = 0 = \mathbf{x}'^T G \mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \quad G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

$\mathbf{x}$ is called a 4-vector and G is called the metric.

Notation

As a matter of convenience, we will use the notation $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}^T G\mathbf{b}$, *i.e.*, the dot product of two 4-vectors **a** and **b** will be taken with respect to the Minkowski metric, G, instead of the usual Euclidean metric. We will extend the the notation for the norm, $|\mathbf{a}|^2 = \mathbf{a}\cdot\mathbf{a}$ to 4-vectors. In terms of components, we will label the components, $\mathbf{a} = (a_0, a_1, a_2, a_3)$. Therefore, we say $\mathbf{x}_0 = t$. For light we have $\mathbf{x}\cdot\mathbf{x} = 0$; for other particles we have $\mathbf{x}\cdot\mathbf{x} > 0$.

Problem 38: Triangle inequality

One often comes across the condition for norms that for two vectors **a** and **b** one should have

$$|\mathbf{a}-\mathbf{b}|^2 \leq |\mathbf{a}|^2 + |\mathbf{b}|^2.$$

Does this triangle inequality hold for 4-vectors with the Minkowski metric?

Lorentz transformations

We seek transformations between inertial frames which leave the norm of any 4-vector, $\mathbf{x}\cdot\mathbf{x}$ unchanged. These will be called Lorentz transformations. If $\mathbf{x}' = L(\beta)\mathbf{x}$, then the matrix $L(\beta)$ satisfies $L(\beta)^T GL(\beta) = G$ and

1. Since $\det L^T GL = \det G$, we have $\det L = \pm 1$. No transformation, *i.e.*, $\beta = 0$ implies $L = I$, and this has $\det L = 1$.
2. We would clearly like to impose a law of addition of velocities: $L(\beta)L(\beta') = L(\beta'')$.
3. A boost of β can be undone by a boost of $-\beta$, *i.e.*, we would like to have $L(\beta)L(-\beta) = I$.
4. The order in which we add the velocities does not matter; this associative law implies that looking for a matrix representation of L is consistent.

These properties define a group: the Lorentz group.

A simpler problem

If there is a boost only in the x direction, then it is reasonable to assume that orthogonal coordinates do not change, and seek

$$L = \begin{pmatrix} L_2 & 0 \\ 0 & I \end{pmatrix}, \quad L_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad L_2^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The last matrix equality gives 3 scalar equations–

$$a^2 - c^2 = 1, \quad d^2 - b^2 = 1, \quad ab - cd = 0.$$

The first two equations are solved by $a = \cosh\psi$, $c = \sinh\psi$ and $d = \cosh\chi$, $b = \sinh\chi$. The third reduces to $\sinh(\psi - \chi) = 0$, which gives $\chi = \psi$. The determinant is automatically unity.

Problem 39: Symmetry of L_2

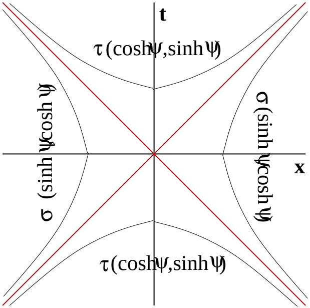
The solution for L_2 turns out to be symmetric. Is there a simple argument which shows that L_2 is not antisymmetric? Is there an argument which shows that L_2 and L_2^T have to be related?

Space-time diagrams

The two-dimensional Lorentz transformation is now,

$$L_2(\beta) = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix}.$$

This is the symmetry group of a hyperbola, *i.e.*, any point on a hyperbola can be parametrized by ψ.



The light cone are those parts of space-time which are reached by a light pulse originating from the origin. Time-like regions of space-time are those which have positive values of $\tau^2 = t^2 - x^2$. τ is called the proper time. Space-like regions of space-time are those which have negative values of $\sigma^2 = t^2 - x^2$.

The velocity

The velocity $\beta = \tanh\psi$, so that one has

$$\cosh\psi = \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \text{and} \quad \sinh\psi = \beta\gamma.$$

Note the composition law for collinear Lorentz transformations is

$$L_2(\psi)L_2(\psi') = \begin{pmatrix} \cosh(\psi+\psi') & \sinh(\psi+\psi') \\ \sinh(\psi+\psi') & \cosh(\psi+\psi') \end{pmatrix} = L(\psi+\psi').$$

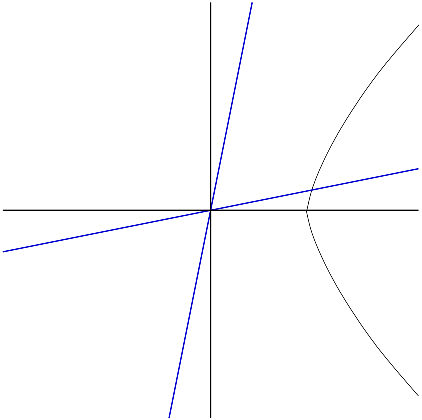
This gives the relativistic law for the addition of velocities.

$$\tanh(\psi+\psi') = \frac{\tanh\psi + \tanh\phi'}{1+\tanh\psi\tanh\psi'} = \frac{\beta+\beta'}{1+\beta\beta'}.$$

Problem 40: Light-cone variables

It is sometimes useful to change to light-cone variables $x_+ = t + x$ and $x_- = t - x$. How do Lorentz transformations change $x_\pm$?

New phenomena



Simultaneous events in one frame may not be simultaneous in another frame. Moving inertial frames see time dilation and Lorentz contraction of lengths.

The Lorentz group

Rotation matrices in space can be written in block diagonal form as

$$L = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & R \end{pmatrix}, \qquad \text{where} \qquad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since $\det R = 1$, one also has $\det L = 1$. Also, these matrices leave $x^2 + y^2 + z^2$ and t seperately unchanged, and hence $L^T G L = G$. Therefore, rotations in space are part of the Lorentz group. A subgroup of the Lorentz group formed by the boosts and rotations is called the proper Lorentz group.

Parity transformations, $P = \mathrm{diag}(1, -1, -1, -1)$, have $\det P = -1$, and give $P^T G P = G$. Parity and the proper Lorentz group together form the orthochronous Lorentz group.

Time-reversal transformations, $T = \mathrm{diag}(-1, 1, 1, 1)$ are part of the Lorentz group. These, together with the orthochronous Lorentz group give the full Lorentz group.

The proper velocity

The 4-vector $\mathbf{v} = d\mathbf{x}/d\tau$, where τ is the proper time, has the same transformation properties as the 4-vector $\mathbf{x}$. $\mathbf{v}$ is called the proper velocity. The position of a particle at rest at the spatial origin in one frame is given by $\mathbf{x} = (\tau, \mathbf{0})$. In a frame boosted by β in the x direction, the position becomes $\mathbf{x}' = L\mathbf{x} = (\gamma\tau, \beta\gamma\tau, 0, 0)$. As a result, its proper velocity is $\mathbf{v} = (\gamma, \beta\gamma, 0, 0)$. Obviously, $|\mathbf{v}|^2 = 1$. If the mass of a particle measured in its rest frame is m, then $\mathbf{p} = m\mathbf{v}$ is a 4-vector. Clearly $|\mathbf{p}|^2 = m^2$; this is called the mass shell condition. We find that

$$p_0 = \sqrt{m^2 + p_1^2 + p_2^2 + p_3^2} = m\sqrt{1 + \beta^2\gamma^2} \simeq m\left(1 + \frac{1}{2}\beta^2 + \cdots\right).$$

Clearly $p_0 - m$ is the non-relativistic kinetic energy in the limit of small β. We will define p_0 to be the relativistic energy. As a special case we find for the rest energy of a particle, the famous equation $\boxed{E = m}$. 4-momentum is conserved.

A particle decaying into two particles

Let the initial particle have 4-momentum $\mathbf{p}_A$, and let the final momenta be $\mathbf{p}_B$ and $\mathbf{p}_C$. We can take $\mathbf{p}_A$ to be given, so that there are 8 unknowns in the final state.

Let the masses of the particles be m_A, m_B and m_C respectively. The two mass shell conditions

$$|\mathbf{p}_B|^2 = m_B^2, \qquad \text{and} \qquad |\mathbf{p}_C|^2 = m_C^2,$$

reduce the number of unknowns by 2.

Then we have 4-momentum conservation,

$$\mathbf{p}_A = \mathbf{p}_B + \mathbf{p}_C.$$

This reduces the number of unknowns by 4.

The remaining unknowns can be absorbed into a boost, to choose the inertial frame in which we work, and the orientation of axes in that frame. So, all the components of the final momenta will be given in terms of known quantities.

Solving the two-body decay kinematics

Work in the rest frame of A, so $\mathbf{p}_A = (m_A, 0, 0, 0)$. Let the final particles come out in the $\hat{\mathbf{x}}$ direction, so $\mathbf{p}_B = (E, p, 0, 0)$ and $\mathbf{p}_C = (E', -p, 0, 0)$. There are 3 unknowns, E, E' and p, and 3 equations, one from conservation two from mass shell—

$$E + E' = m_A, \qquad E^2 = m_B^2 + p^2, \qquad E'^2 = m_C^2 + p^2.$$

Subtract the last two equations to get

$$E^2 - E'^2 = m_B^2 - m_C^2, \qquad \text{so} \qquad E - E' = \frac{m_B^2 - m_C^2}{m_A}.$$

This gives

$$E = \frac{m_A^2 + m_B^2 - m_C^2}{2m_A}, \qquad \text{and} \qquad E' = \frac{m_A^2 - m_B^2 + m_C^2}{2m_A}.$$

Either of these can be used with the corresponding mass-shell condition to give p.

Working with Lorentz invariants

With 3 different momenta, $\mathbf{p}_A$, $\mathbf{p}_B$ and $\mathbf{p}_C$, there are 6 Lorentz invariants— $|\mathbf{p}_i|^2 = m_i^2$ and the 3 dot products. Since $\mathbf{p}_A = \mathbf{p}_B + \mathbf{p}_C$, one finds $\mathbf{p}_B \cdot \mathbf{p}_C = (m_A^2 - m_B^2 - m_C^2)/2$, and similarly for the two remaining dot products.

The total number of vector components that one had to begin with are 12. Of these, 6 are completely determined by the dot products, which are known. The remaining 6 are completely specified by 3 boosts and 3 angles which serve to define an inertial frame completely.

Therefore, all the components of the initial and final momenta are known given any choice of frame and coordinates and the particle masses.

Two particles scattering elastically

Two particles of masses m_A and m_B scatter elastically. The initial 4-momenta are $\mathbf{p}_A$ and $\mathbf{p}_B$, and the final are $\mathbf{p}'_A$ and $\mathbf{p}'_B$. There are 8 components of momenta in the final state, 2 mass-shell conditions, and 4 conservation equations: so 2 unknowns.

Work in the CM frame with initial particles coming in the $\hat{\mathbf{x}}$ direction: $\mathbf{p}_A = (E, p, 0, 0)$ and $\mathbf{p}_B = (E, -p, 0, 0)$. Clearly $s = (\mathbf{p}_A + \mathbf{p}_B)^2 = (E + E')^2$ is Lorentz invariant. Now solve this in terms of the CM energy, $\sqrt{s}$, m_A and m_B exactly as for the previous problem.

Suppose one studies the final particles in a rotated frame with the new $\hat{\mathbf{x}}$ direction in the direction of the momentum of A, then the solution is exactly as above. The only additional information needed is the direction between the initial and final momentum of A, *i.e.*, the scattering angle, χ. So the kinematics is completely specified by the CM energy, $\sqrt{s}$, and the scattering angle, χ.

Counting invariants

Of the 16 components of four 4-momenta, 6 can be removed into a choice of inertial frame: 3 boosts and 3 rotations. There remain 10 more variables, which must be exactly equal in number to the Lorentz invariants. Explicitly, these are the norms of the momenta and 6 different dot products. Since there are only two unknowns, both can be written in terms of Lorentz invariants.
One is the square of the CM energy, s. The other is

$$t = |\mathbf{p}_A - \mathbf{p}'_A|^2 = 2(m_A^2 + \mathbf{p}_A \cdot \mathbf{p}'_A), \qquad \mathbf{p}_A \cdot \mathbf{p}'_A = m_A^2 \gamma^2 (1 - \beta^2 \cos \chi).$$

Here we have used the parametrization for the 4-momentum $\mathbf{p}_A = m_A \gamma(1, \beta, \hat{\mathbf{n}})$, where $\hat{\mathbf{n}}$ is the direction of the velocity of the particle in the chosen frame and β is the boost required to bring it to that frame from its RF. Note also that β is specified as soon as s is give, so one can trade χ for t. The quantities s and t are called Mandelstam variables.

Some problems

Problem 41: Mandelstam variables I

Show that in the elastic scattering of two particles the 6 dot products can all be written in terms of the s and t and the two masses. Write explicit expressions for each of the dot products.

Problem 42: Mandelstam variables I

For the inelastic scattering $A + B \to C + D$, count the number of variables. Solve the scattering problem. Write the results in terms of Lorentz invariant quantities.

Problem 43: Particle decay

For the decay process $A \to B + C + D$, count the number of variables. Solve the problem. Write the results in terms of Lorentz invariant quantities.

Keywords and References

Keywords

Galilean relativity, Michelson-Morley experiment, boost, dimensions of time, 4-vector, metric, Minkowski metric, Euclidean metric, triangle inequality, addition of velocities, light cone, time-like regions, proper time, space-like regions, light-cone variables, simultaneous events, time dilation, Lorentz contraction, Lorentz group, proper Lorentz group, Parity transformations, orthochronous Lorentz group, Time-reversal transformations, proper velocity, mass shell, relativistic energy, rest energy, Mandelstam variables.

References

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