Hamilton's principle and Symmetries

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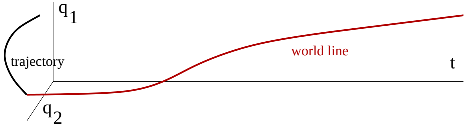
Action

For a mechanical system with generalized coordinates q_1, q_2, in motion between times t_1 and t_2, the action is defined as the integral

$$S[t_1, t_2] = \int_{t_1}^{t_2} dt L(q_1, q_2, \cdots, \dot{q}_1, \dot{q}_2, \cdots, t)$$

The value of the action depends on the world line of the particle $\{q_1(t), q_2(t), \cdots\}$: it is a functional of the world line.



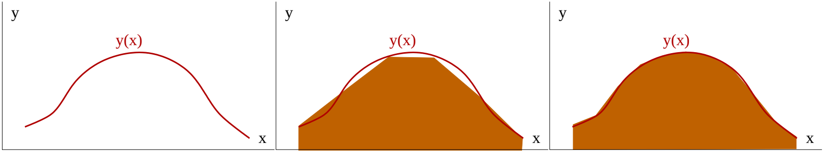
Hamilton's principle

The particle's actual trajectory is the one that minimizes the action subject to the boundary conditions imposed on it.

Functionals: easy as π

An example of a functional of a curve: the area under a curve!

$$A[y] = \int_{x_i}^{x_f} dx y(x).$$



The area is a functional of the curve. Use Riemann sums to approximate the area in terms of N variables $y(x_1)$, $y(x_2)$, *etc.*,

$$A[y] = h_N \sum_{i=1}^{N} y(x_i), \qquad \text{where} \qquad h_N = \frac{x_f - x_i}{N}.$$

Finding variations of the area with the curve is just calculus of many variables.

Calculus of variations

Finding an extremum of a function involves setting its derivatives to zero and then solving the resulting equations. Checking whether an extremum is a maximum or minimum involves checking the sign of the second derivative. The minimum of a functional is similar. Make small variations $\delta q_k(t)$ around a trajectory specified by $q_k(t)$ and $\dot{q}_k(t)$. Since the boundary conditions are fixed, so $\delta q_k(t_1) = \delta q_k(t_2) = 0$. The variation of L under such a variation of the trajectory is

$$\delta L = \sum_k \delta q_k \frac{\partial L}{\partial q_k} + \frac{d}{dt}\left(\delta q_k \frac{\partial L}{\partial \dot{q}_k}\right) - \delta q_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)$$

Now we put this inside the integral to evaluate δS. Since

$$\int_{t_1}^{t_2} dt \frac{d}{dt}\left(\delta q_k \frac{\partial L}{\partial \dot{q}_k}\right) = \delta q_k \frac{\partial L}{\partial \dot{q}_k}\bigg|_{t_1}^{t_2} = 0,$$

we recover the Euler-Lagrange equations.

Two examples

	Quartic oscillator	Physical pendulum
L	$\frac{1}{2}m\dot{q}^2 - \frac{1}{4}V_0 q^4$	$\frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1-\cos\theta)$
Q	$V_0 q^3$	$-mg\ell\sin\theta$
p	$m\dot{q}$	$m\ell^2\dot{\theta}$
EoM	$m\ddot{q} + V_0 q^3 = 0$	$m\ell^2\ddot{\theta} + mg\ell\sin\theta = 0$
	$\ddot{q} + \zeta^2 q^3 = 0$	$\ddot{\theta} + \Omega^2\sin\theta = 0$
	where $\zeta^2 = \frac{V_0}{m}$	where $\Omega^2 = \sqrt{\frac{g}{L}}$
$t \to \Omega t$		$\ddot{\theta} + \sin\theta = 0$
$q \to q/\zeta$	$\ddot{q} + q^3 = 0$	

Numerical solutions

Minimization of the action can be carried out numerically very simply using a lattice discretization. Work with D degrees of freedom, $q_1, q_2, \cdots q_D$. Divide the time interval into N equal pieces. At times $t_j = t_0 + jh_N$ use the notation $q_k^j = q_k(t_j)$, so that the action is a function of $D(N-1)$ variables—

$$
\begin{aligned}
S &= \frac{m_k}{2h_N^2} \sum_{k=1,j=1}^{D,N} \left(q_k^j - q_k^{j-1}\right)^2 - \sum_{j=1}^{N} V(q_1^j, q_2^j, \cdots, q_D^j) \\
&= N^2 \sum_{j=1}^{N} \left[\sum_{k=1}^{D} q_k^j q_k^{j-1} - \overline{V}(q_1^j, q_2^j, \cdots, q_D^j)\right].
\end{aligned}
$$

In the second line we have assumed that units have been chosen so that $m_k = 1$ and $t_N - t_0 = 1$. Since V depends only the q_k at one fixed time, we have defined $\overline{V} = V - N^2 \sum_k (q_k^j)^2$. The only connection between different times is the hopping term $q_k^j q_k^{j-1}$.

An idiotically simple algorithm for minimization

There are many ways to minimize a function of many variables. A mindlessly simple algorithm is:

1. Choose a stopping criterion ϵ.
2. Start with a trajectory $\{q_k^j\}$. The corresponding value of the action is S.
3. Select a random set of values $\{{q'}_k^j\}$ and compute the corresponding value of the action, S'.
4. $|S - S'| < \epsilon$ then stop. The trajectory is now $\{q_k^j\}$.
5. If $S' < S$ set $q_k^j = {q'}_k^j$. Repeat from step 3.

Minimization of a function of many variables

Can you think of other ways of minimizing the action which do not use the Euler-Lagrange equations?

Two Problems

Problem 6: Simple harmonic motion

The Lagrangian for a simple harmonic oscillator becomes

$$L = \frac{1}{2}\dot{\theta}^2 - \frac{1}{2}\theta^2, \qquad \theta(0) = \frac{1}{2},\ \theta(1) = -\frac{1}{2}.$$

Find the trajectory by numerical minimization.

Problem 7: The physical pendulum

The Lagrangian for the physical pendulum is

$$L = \frac{1}{2}\dot{\theta}^2 - (1 - \cos\theta), \qquad \theta(0) = \frac{1}{2},\ \theta(1) = -\frac{1}{2}.$$

Find the trajectory by numerical minimization.

Integrals of motion

Some constraints on the solutions of the equations of motion can be found without solving the full problem, *i.e.*, without giving q's as functions of time. One example is of a free particle— in this case the integral of motion is the conserved value of the momentum. Similarly for a freely rotating body the conserved angular momentum is an integral of motion.

First integrals of motion

In general a first integral of motion is some relation

$$f(q_1, q_2, \cdots, \dot{q}_1, \dot{q}_2, \cdots, t) = \text{constant}.$$

They are called first integrals because the equations of motion involve $\ddot{q}_k$, *i.e.*, differential equations of the second order. However, these conditions "integrate" the equations once, so that they provide differential equations of first order.

Symmetries of a system

For a free particle, any point in space looks the same as any other, because every point is free of forces. As a result, the Lagrangian is independent of the coordinates, *i.e.*, $V = \text{constant}$.

For a freely rotating body, any orientation is the same as any other, since there is no moment that arises as it rotates. As a result, the Lagrangian is independent of the angle the body makes with respect to a fixed frame.

Symmetries of mechanical systems

These notions of symmetry permit immediate generalization. We say that a system posseses a symmetry (or symmetries) if one (or more) of the generalized coordinates do not appear in the Lagrangian.

Symmetries and integrals of motion

Write the Euler-Lagrange equations in the form

$$\dot{p}_k = \frac{\partial L}{\partial q_k}.$$

If a Lagrangian possesses a symmetry with respect to one of the generalized coordinates, q_k, then the right hand side vanishes. The symmetry leads to a first integral of motion, the conservation of the corresponding momentum.

Noether's Theorem

Every symmetry leads to a conservation law for the corresponding momentum. Generalized coordinates associated with such integrals of motion are called cyclic coordinates.

Motion of a charged particle

Problem 8: Motion of a charged particle

What is the canonical momentum of a charged particle in a EM field? If the external fields are time-independent, then L does not depend explicitly on the time. In this case check whether $T+V$ is conserved. system?

Problem 9: A charged particle in a constant magnetic field

Are there any conserved quantities when a charged particle moves in a constant magnetic field and zero electric field? What does your analysis show about the general character of the trajectories?

Problem 10: A charged particle around a magnetic monopole

A magnetic monopole is any source which produces a radial magnetic field. For a charged particle moving in the field of a fixed magnetic monopole, what are the conserved quantities?

Scaling symmetries

Under a scaling $L \to \lambda L$ the equations of motion and its solutions remain unchanged.
Assume that under a scaling of coordinates $q_k \to \lambda q_k$, the potential scales as $V \to \lambda^\alpha V$. Scale the time as $t \to \lambda^\beta t$; the kinetic energy scales as $T \to \lambda^{2(1-\beta)} T$. For the Lagrangian to scale uniformly, one must have

$$2(1-\beta) = \alpha, \qquad i.e. \qquad \beta = 1 - \frac{\alpha}{2}.$$

As a result, two trajectories related by scaling must obey the law

$$\frac{t'}{t} = \left(\frac{l'}{l}\right)^{(1-\alpha/2)} .$$

Then Kepler's third law that the cube of the distance of a planet from the sun is proportional to the square of its period of revolution implies $\alpha = -1$, *i.e.*, Newton's law of gravity.

Keywords and References

Keywords

action, world line, functional, lattice discretization, hopping term, algorithm, stopping criterion, first integral of motion, symmetry, conservation law, cyclic coordinates, Noether's theorem, scaling, Kepler's third law, Newton's law of gravity

References

Goldstein, Chapter 2
Landau, Chapter II