Hamilton's principle and Symmetries

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The Hamiltonian

The change in the Lagrangian due to a virtual change of coordinates is

$$dL = \sum_k \dot{p}_k dq_k + p_k d\dot{q}_k.$$

Using this, one can define a function, called the Hamiltonian, $H(p_k, q_k)$, by eliminating $\dot{q}_k$ to write

$$dH = d\left[\sum_k p_k \dot{q}_k - L\right] = \sum_k \dot{q}_k dp_k - \dot{p}_k dq_k.$$

Since p_k and q_k are independent variables, one has Hamilton's equations

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \qquad \text{and} \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k}.$$

An example

For a single particle moving in a potential, one can write

$$L = \frac{1}{2} m |\dot{\mathbf{x}}|^2 - V(\mathbf{x}), \qquad \text{and} \qquad \mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = m\dot{\mathbf{x}}.$$

Eliminating $\dot{\mathbf{x}}$, the Hamiltonian is

$$H = \mathbf{p} \cdot \dot{\mathbf{x}} - L = \frac{1}{2m} |\mathbf{p}|^2 + V(\mathbf{x}).$$

Hamilton's equations are

$$\dot{\mathbf{x}} = \frac{1}{m}\mathbf{p}, \qquad \text{and} \qquad \dot{\mathbf{p}} = -\nabla V(\mathbf{x}).$$

These are exactly the usual equations of motion.

Poisson brackets

Take any two functions on phase space, $f(q_k, p_k)$ and $g(q_k, p_k)$. Then we define the Poisson bracket of these two functions as

$$[f, g] = \sum_k \left[\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k} \frac{\partial f}{\partial p_k} \right].$$

Clearly, interchanging f and g in the expression on the right changes the sign. So $[f, g] = -[g, f]$. Also, because of this antisymmetry, $[f, f] = 0$.
Clearly, the time derivative of f is given by the expression

$$\begin{aligned} \frac{df}{dt} &= \sum_k \left[\frac{\partial f}{\partial q_k} \dot{q}_k + \frac{\partial f}{\partial p_k} \dot{p}_k \right] \\ &= \sum_k \left[\frac{\partial f}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial H}{\partial q_k} \right] = [f, H]. \end{aligned}$$

We have used the EoM to get the second line from the first.

Time independent Hamiltonians are conserved

More generally, for any time-dependent function on phase space $f(q_k, p_k, t)$, one has

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [f, H].$$

It follows trivially that, for the Hamiltonian itself we have the identity

$$\frac{dH}{dt} = \frac{\partial H}{\partial t},$$

so that if the Hamiltonian is time independent, then it is conserved.
Other identities for the Poisson bracket include the Jacobi identity

$$[[f, g], h] + [[g, h], f] + [[h, f], g] = 0.$$

Problem 11

Prove the Jacobi identity.

Elementary Poisson brackets

Since all the q_k and p_k are independent variables, their derivatives with respect to each other vanish. So we have $[q_j, q_k] = 0$ and $[p_j, p_k] = 0$. Also,

$$[q_i, p_j] = \sum_k \left[\frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial p_j}{\partial q_k} \frac{\partial q_i}{\partial p_k} \right] = \sum_k \delta_{ik}\delta_{jk} = \delta_{ij}.$$

The EoM can be written in the form $\dot{q}_i = [q_i, H]$ and $\dot{p}_i = [p_i, H]$. Using the definition of the Poisson bracket, one sees that these reproduce the canonical equations. If some momenta are conserved, then one clearly has $\partial H/\partial q_i = 0$. The Hamiltonian does not depend on the corresponding coordinates, *i.e.*, we have a symmetry.

Angular momenta rotate vectors

We write the components of the angular momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ using the Levi-Civita tensor as $L_i = \epsilon_{ijk} x_j p_k$. In order to evaluate the Poisson bracket $[L_i, x_j]$ we note that for any three functions on phase space, f, g, h,

$$[fg, h] = f[g, h] + g[f, h].$$

Using this we find

$$\begin{aligned} [L_i, x_j] &= \epsilon_{i\alpha\beta}[x_\alpha p_\beta, x_j] \\ &= \epsilon_{i\alpha\beta}\left\{x_\alpha[p_\beta, x_j] + p_\beta[x_\alpha, x_j]\right\} \\ &= -\epsilon_{i\alpha\beta} x_\alpha \delta_{\beta j} = \epsilon_{ijk} x_k. \end{aligned}$$

This is the definition of a vector function on phase space.

Problem 12

Prove that the momentum $\mathbf{p}$ is a vector function on phase space.

Poisson brackets of angular momenta

The Poisson bracket of two components can be easily worked out

$$[L_i, L_j] = \epsilon_{i\alpha\beta}\epsilon_{j\gamma\delta}[x_\alpha p_\beta, x_\gamma p_\delta].$$

Using the identity for Poisson brackets of products, we can write

$$[L_i, L_j] = \epsilon_{i\alpha\beta}\epsilon_{j\gamma\delta}\left\{x_\alpha[p_\beta, x_\gamma p_\delta] + p_\beta[x_\alpha, x_\gamma p_\delta]\right\}.$$

Using the identity a second time we can write

$$\begin{aligned}[L_i, L_j] &= \epsilon_{i\alpha\beta}\epsilon_{j\gamma\delta}\left\{x_\alpha x_\gamma[p_\beta, p_\delta] + x_\alpha p_\delta[p_\beta, x_\gamma] \right.\\ &\qquad \left. + p_\beta x_\gamma[x_\alpha, p_\delta] + p_\beta p_\delta[x_\alpha, x_\gamma]\right\} \\ &= \epsilon_{i\alpha\beta}\epsilon_{j\gamma\delta}\left\{-x_\alpha p_\delta \delta_{\beta\gamma} + p_\beta x_\gamma \delta_{\alpha\delta}\right\}\end{aligned}$$

The angular momentum is a vector

We have the tensor identity

$$\epsilon_{i\alpha\beta}\epsilon_{j\gamma\beta} = \delta_{ij}\delta_{\alpha\gamma} - \delta_{i\gamma}\delta_{j\alpha},$$

where we have used the summation convention— repeated indices are summed. Using this we reduce the Poisson bracket

$$\begin{aligned}
[L_i, L_j] &= \epsilon_{i\alpha\beta}\epsilon_{j\gamma\delta}\left\{-x_\alpha p_\delta \delta_{\beta\gamma} + p_\beta x_\gamma \delta_{\alpha\delta}\right\} \\
&= \epsilon_{i\alpha\beta}\epsilon_{j\gamma\beta}\left\{x_\alpha p_\gamma - x_\gamma p_\alpha\right\} \qquad \text{renaming dummy indices} \\
&= \left\{\delta_{ij}\delta_{\alpha\gamma} - \delta_{i\gamma}\delta_{j\alpha}\right\}\left\{x_\alpha p_\gamma - x_\gamma p_\alpha\right\} \\
&= x_i p_j - x_j p_i = \epsilon_{ijk} L_k.
\end{aligned}$$

This completes the demonstration that the angular momentum itself is a vector.

Keywords and References

Keywords

Hamiltonian, Hamilton's equations, Poisson bracket, antisymmetry, Jacobi identity, conserved, symmetry, Levi-Civita tensor, vector function

References

Goldstein, Chapters 8, 9
Landau, Sections 40, 42