# The Physical Pendulum

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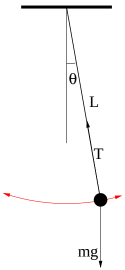



$$\dot{q} = p,$$
$$\dot{p} = -\sin q.$$

We choose to work with a generalized coordinate $q$ which is the angle of displacement from the vertical, with the bob hanging down. The potential energy is taken to be zero in the position with $q = 0$. The Hamiltonian of the physical pendulum is

$$H(p,q) = mgL\left[\frac{1}{2}\frac{L}{g}p^2 + (1-\cos q)\right].$$

Choose the units of energy to be $mgL$, and the units of time to be $\sqrt{L/g}$. Then the Hamiltonian can be written in its reduced form

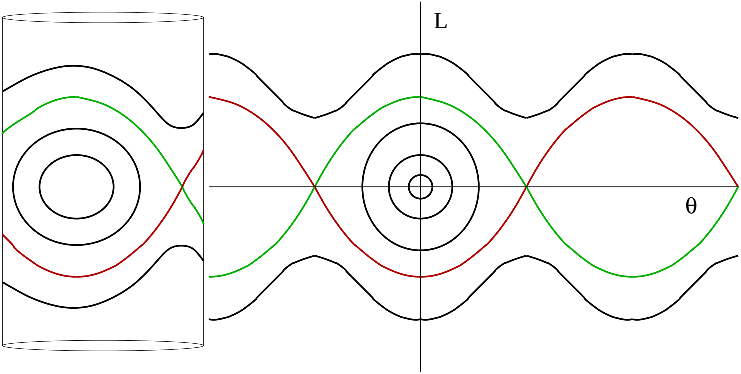
$$H(p,q) = \frac{1}{2}p^2 + (1-\cos q),$$

with the EoM shown alongside.

# Understanding the problem

- The configuration space for the problem is the circle: $S_1$. Coordinates on the circle are the angle from the vertical, which lies in the interval $-\pi \leq q \leq \pi$. If the pendulum rotates around its pivot more than once, then the phase angle can be moved back into the same range by subtracting out integer multiples of $2\pi$, since no measurable quantity depends on how many times the pendulum has gone round the circle.
- Phase space is a cylinder: $S_1 \times R$. Phase space trajectories either wind round the cylinder or they don't. Trajectories which wind round the cylinder have energy $E > 2$ (where $E = 0$ when the bob is at rest at the lowest point of the circle). In this case the pendulum can go all the way round the pivot. When $E < 2$ the motion is restricted to $|q| \leq \cos^{-1}(E/2)$.

# Phase space trajectories




Trajectories labelled by $E$. Seperatrix: $E = 2$.

# Other examples

1. Spherical pendulum: configuration space is the sphere $S_2$, generalized coordinates are angles $\theta$ and $\phi$.
2. Double pendulum: configuration space is $S_1 \times S_1$, generalized coordinates are two angles $\theta_1$ and $\theta_2$
3. Double spherical pendulum: configuration space is $S_2 \times S_2$

## Problem 13: Jointed rods

Take a joint between two rigid rods which allows complete rotational freedom of one rod when the other is fixed. How many degrees of freedom are there in a system of $N$ rigid rods joined end to end? How many degrees of freedom if the free ends are joined back? What are the configuration space and phase space in each case?

# Using the first integral

The trajectory of the pendulum is found most easily by using the first integral

$$p = \dot{q} = \sqrt{2(E - 1 + \cos q)}, \quad \text{so} \quad \frac{dt}{dq} = \frac{1}{\sqrt{2(E - 1 + \cos q)}}.$$

If the initial conditions are $q(0) = 0$ and $p(0) = \sqrt{2E}$, then for $E < 2$ the amplitude $\Theta$ is $E = 1 - \cos\Theta = 2\sin^2(\Theta/2)$, and

$$t = \int_0^{q(t)} \frac{dq}{\sqrt{2(E - 2\sin^2(q/2))}} = \sqrt{\frac{2}{E}} \int_0^{q(t)/2} \frac{dz}{\sqrt{1 - (2/E)\sin^2 z}}$$

where we used the variable $z = q/2$. In terms of a phase variable for the oscillator: $\sin u = \sqrt{2/E}\,\sin z$, the denominator is $\cos u$.

# The trajectory of the pendulum

Since $\sin z = \sqrt{E/2}\sin u$, we find

$$\frac{dz}{\cos u} = \sqrt{\frac{E}{2}}\,\frac{du}{\cos z}.$$

Now $\cos^2 u = 1 - (2/E)\sin^2 z$ and $\cos^2 z = 1 - (E/2)\sin^2 u$. This transforms the integral into the form

$$t = \int_0^{\sin^{-1}\sqrt{2/E}\sin q/2} \frac{du}{\sqrt{1-(E/2)\sin^2 u}} = F\left(\sqrt{\frac{E}{2}}, \sqrt{\frac{2}{E}}\sin\frac{q}{2}\right).$$

The integral defines the incomplete elliptic integral of the first kind. This completes the solution of the problem. Does it?

# Defining the Jacobi Elliptic Functions

The Jacobi Elliptic Functions are inverses of the elliptic integral of the first kind. Given the elliptic integral

$$t = \int_0^\phi \frac{du}{\sqrt{1 - k^2 \sin^2 u}},$$

define the Jacobi Elliptic Functions

$$\mathrm{sn}\, t = \sin\phi, \quad \mathrm{cn}\, t = \cos\phi, \quad \mathrm{dn}\, t = \sqrt{1 - k^2 \sin^2 \phi}.$$

A more complete notation for the above functions is $\mathrm{sn}\,(t,k)$, $\mathrm{cn}\,(t,k)$ and $\mathrm{dn}\,(t,k)$.
Note that the integral gives the special values $\mathrm{sn}\,(t,0) = \sin t$, $\mathrm{cn}\,(t,0) = \cos t$ and $\mathrm{dn}\,(t,0) = 1$.

# Some Elementary Properties

Two elementary properties immediately follow—

$$\mathrm{sn}^2 t + \mathrm{cn}^2 t = 1 \qquad \text{and} \qquad \mathrm{dn}^2 t + k^2 \mathrm{sn}^2 t = 1.$$

The incomplete elliptic integral gives

$$\frac{dt}{d\phi} = \frac{1}{\sqrt{1 - k^2 \sin^2 \phi}}, \qquad ie, \qquad \frac{d\phi}{dt} = \mathrm{dn}\, t.$$

From this the elementary derivatives follow—

$$\frac{d}{dt}\mathrm{sn}\, t = \mathrm{cn}\, t\, \mathrm{dn}\, t, \quad \frac{d}{dt}\mathrm{cn}\, t = -\mathrm{sn}\, t\, \mathrm{dn}\, t, \quad \frac{d}{dt}\mathrm{dn}\, t = -k^2 \mathrm{sn}\, t.$$

**Problem 14: Maclaurin series expansion**

Find the values of the three functions at $t = 0$. Develop the Maclaurin series expansion up to the 10th order for each of the functions.

# Alternative forms of the elliptic integral

Another definition of the incomplete elliptic integral of the first kind is useful in practice—

$$F(k,y) = \int_0^y \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = \int_0^\phi \frac{du}{\sqrt{1-k^2\sin^2 u}}.$$

The substitution $\sin u = z$ can be used to show the equality, with $\sin\phi = y$. $F(k,y)$ is real when the modulus $k$ lies in the interval $(0,1)$. One has the special values $F(k,0)=0$ and $F(0,y)=y$, and $F(1,1)$ diverges.

## Problem 15: Complementary modulus

Show that the elliptic integral for complementary modulus, $\ell$, such that $k^2+\ell^2=1$, is

$$F(\ell,y) = \int_1^f \frac{dx}{\sqrt{(x^2-1)(1-k^2x^2)}}, \quad \text{where} \quad f = \frac{1}{\sqrt{1-\ell^2y^2}}.$$

# Periodicity: the argument

In order to establish the periodicity of the Jacobi elliptic functions, it is enough to prove it for any one of the functions. We will choose to prove it for $\mathrm{sn}$.

## Why is a proof needed?

A functional relation of the form $f(t) = \sin\phi$ is not sufficient to show that $f(t)$ is periodic. Consider $f \equiv \tanh$, for example. So a proof is needed.

## The idea of the proof

In order to prove periodicity, one needs to prove two things: (a) that $t$ is finite for finite $\phi$, (b) increasing $\phi$ by $2\pi$ should increase $t$ by a fixed amount, for every value of $\phi$.

For the counter-example above, the condition (a) failed.

# Periodicity: the proof

One defines the complete elliptic integral of the first kind as the special value

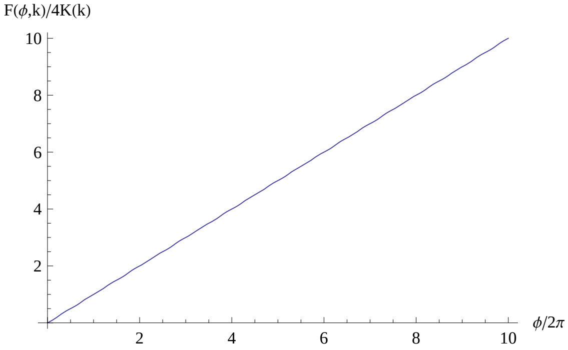
$$K(k) = \int_0^{\pi/2} \frac{du}{\sqrt{1 - k^2 \sin^2 u}},$$

Since this integral does not diverge for $k < 1$, $K$ is finite. This is a proof of step (a) in the plan.
Step (b) follows from the fact that the integrand of the incomplete elliptic integral is periodic with periodicity $\pi$, and symmetric about $u = \pi/2$—

$$F(k, \phi) = \int_0^{\phi} \frac{du}{\sqrt{1 - k^2 \sin^2 u}},$$

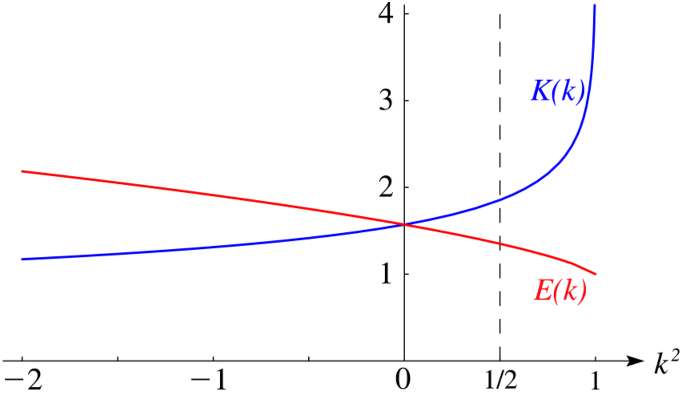
As a result, $F(k, \phi + \pi/2) = K(k) + F(k, \phi)$, so that $\mathrm{sn}\,(t + 4nK) = \mathrm{sn}\, t$.

# Periodicity: visualizing the proof




This is a graph of the elliptic integral (with $k < 1$) which shows that every time $\psi$ increases by $2\pi$, the incomplete elliptic integral increases by $4K$.

# Variation of period with modulus



# Addition theorems

## Problem 16: Addition theorems

1. Prove the addition theorem

$$\mathrm{sn}\,(u+v) = \frac{\mathrm{sn}\,u\,\mathrm{cn}\,v\,\mathrm{dn}\,v + \mathrm{sn}\,v\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{1-k^2\mathrm{sn}^2u\,\mathrm{sn}^2v}.$$

   Do this by showing that if one varies $u$ and $v$ while keeping the sum $u+v$ fixed, then $(s_1's_2 - s_2's_1)/(1-k^2s_1^2s_2^2)$ is a constant (the prime denotes differentiation with respect to $u$).

2. Using the above, prove the addition theorems

$$\mathrm{cn}\,(u+v) = \frac{\mathrm{cn}\,u\,\mathrm{cn}\,v - \mathrm{sn}\,u\,\mathrm{sn}\,v\,\mathrm{dn}\,u\,\mathrm{dn}\,v}{1-k^2\mathrm{sn}^2u\,\mathrm{sn}^2v}$$

$$\mathrm{dn}\,(u+v) = \frac{\mathrm{dn}\,u\,\mathrm{dn}\,v - k^2\mathrm{sn}\,u\,\mathrm{sn}\,v\,\mathrm{cn}\,u\,\mathrm{cn}\,v}{1-k^2\mathrm{sn}^2u\,\mathrm{sn}^2v}$$

# Special values and periodicity

From the integral representation one find

$$\operatorname{sn} K = 1, \qquad \operatorname{cn} K = 0 \qquad \text{and} \qquad \operatorname{dn} K = \ell.$$

Then, using the addition theorems, one finds

$$\operatorname{sn} 2K = 0, \qquad \operatorname{cn} 2K = -1 \qquad \text{and} \qquad \operatorname{dn} 2K = 1.$$

Further use of the addition theorems gives

$$\operatorname{sn} 3K = -1, \qquad \operatorname{cn} 3K = 0 \qquad \text{and} \qquad \operatorname{dn} 3K = \ell.$$

**Problem 17: Quarter and half period identities**

Express the Jacobi elliptic functions for values of arguments $t \pm K$, $t \pm 2K$ and $t \pm 3K$ in terms of the functions evaluated at $t$.

# Complex periodicity

Introduce the notation for the complete elliptic integral of the complementary modulus, $K(\ell) = K'$. Then, from the expressions derived earlier, we can write

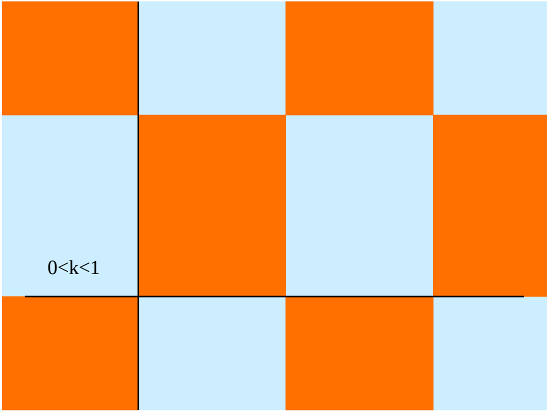
$$L = K + iK' = \int_0^{1/k} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

This implies that $\mathrm{sn}\, L = 1/k$ and $\mathrm{dn}\, L = 0$, whereas $\mathrm{cn}\, L = -i\ell/k$. By using the addition theorems we have $\mathrm{sn}\, 2L = 0$, $\mathrm{cn}\, 2L = 1$ and $\mathrm{dn}\, 2L = -1$. Further, $\mathrm{sn}\, 4L = 0$, $\mathrm{cn}\, 4L = 1$ and $\mathrm{dn}\, 4L = 1$.
As a result, we have the complex periodicity relations

$$\begin{aligned}
\mathrm{sn}\,(t+4L) &= \frac{\mathrm{sn}\, t\, \mathrm{cn}\, 4L\, \mathrm{dn}\, 4L + \mathrm{sn}\, 4L\, \mathrm{cn}\, t\, \mathrm{dn}\, t}{1 - k^2 \mathrm{sn}^2 t\, \mathrm{sn}^2 4L} = \mathrm{sn}\, t, \\
\mathrm{cn}\,(t+4L) &= = \mathrm{cn}\, t, \qquad \text{and} \qquad \mathrm{dn}\,(t+4L) = \mathrm{dn}\, t.
\end{aligned}$$

Utilizing the periodicity in $4K$, we find that the functions have periodicity of $4iK'$.

# Fundamental modular domains




Jacobi elliptic functions are doubly periodic in the complex plane. As a result, they can be defined completely by their behaviour inside a parallelogram called the fundamental modular domain.

# Poles

Using the addition theorem we can write

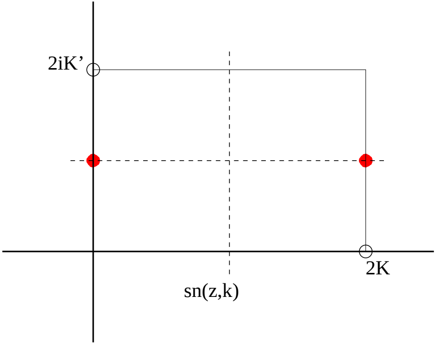
$$\mathrm{sn}\,(t+iK') = \mathrm{sn}\,(t+L-K) = \frac{\mathrm{cn}\,(t+L)\mathrm{dn}\,(t+L)}{1-k^2\mathrm{sn}^2(t+L)}.$$

Again, using the addition theorem we have,

$$\begin{aligned}
\mathrm{sn}\,(t+L) &= \frac{\mathrm{cn}\,t\,\mathrm{dn}\,t}{k(1-\mathrm{sn}^2 t)} = \frac{\mathrm{dn}\,t}{k\mathrm{cn}\,t}\\
\mathrm{cn}\,(t+L) &= -\frac{i\ell\mathrm{cn}\,t}{k(1-\mathrm{sn}^2 t)} = -\frac{i\ell}{k\mathrm{cn}\,t}\\
\mathrm{dn}\,(t+L) &= \frac{i\ell\mathrm{sn}\,t\,\mathrm{cn}\,t}{1-\mathrm{sn}^2 t} = \frac{i\ell\mathrm{sn}\,t}{\mathrm{cn}\,t}.
\end{aligned}$$

Putting these together, we find $\mathrm{sn}\,(t+iK') = 1/\mathrm{sn}\,t$. Since the Taylor expansion gives $\mathrm{sn}\,t = t + \mathcal{O}(t^3)$, $\mathrm{sn}$ has a simple pole at $iK'$. As a result, $\mathrm{cn}$ and $\mathrm{dn}$ also have simple poles at the same point. By the periodicity relations, there is also a pole at $2K+iK'$.

# Zeroes and Poles




Red circles: poles, white circles: zeroes.

# The time period of the pendulum

The time period, $T$, of the pendulum in the oscillatory mode is given by

$$\frac{T}{4} = \begin{cases} K\left(\sqrt{\frac{E}{2}}\right) & (E \leq 2), \\ \sqrt{\frac{2}{E}}K\left(\sqrt{\frac{2}{E}}\right) & (E > 2). \end{cases}$$

For small $E$ the period is $2\pi$ and as $E \to 2$, the period diverges. When $E > 2$ the period decreases with $E$, asymptotically as $1/\sqrt{E}$. The large and small energy limits are amenable to elementary analysis.

The trajectory of the pendulum is

$$\sin\left(\frac{q}{2}\right) = \begin{cases} \sqrt{\frac{E}{2}}\mathrm{sn}\left(t, \sqrt{\frac{E}{2}}\right) & (E \leq 2), \\ \mathrm{sn}\left(\sqrt{\frac{E}{2}}t, \sqrt{\frac{2}{E}}\right) & (E > 2). \end{cases}$$

# Changing initial conditions

**Problem 18: Changed initial conditions**

Suppose that the initial conditions are $q(0) = \Theta$ and $p(0) = 0$. Set up the solution of the problem and see how it differs from the standard solution discussed in this lecture.

**Problem 19: Fourier modes**

In the limit $E \ll 1$, the trajectory contains a single harmonic. When the amplitude is larger, the solution contains many Fourier components. Find the power spectrum of the solution.

**Problem 20: Phase space area**

Find the phase space area, $S$, enclosed by an orbit of energy $E \leq 2$. Check what is the relation between the period $T$ and the $dS/dE$.

# Keywords and References

## Keywords

seperatrix, period, incomplete elliptic integral of the first kind, complete elliptic integral of the first kind, Jacobi elliptic functions, modulus, complementary modulus, fundamental modular domain, poles and zeroes

## References

Whittaker and Watson, Chapter XXII.

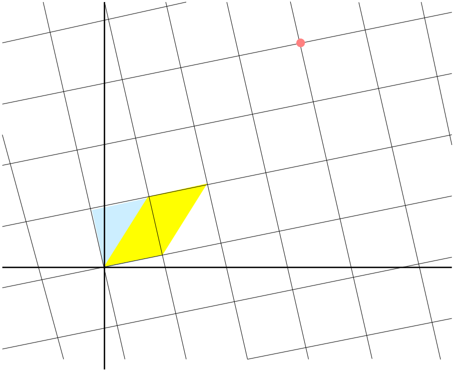
# What is an Elliptic Function?

## A general definition

A function of a single complex variable which is analytic everywhere in the complex plane except for isolated poles (*i.e.*, a meromorphic function) which is doubly periodic, *i.e.*, $f(z+w_1) = f(z+w_2) = f(z)$ for two complex numbers $w_1$ and $w_2$, such that $w_1/w_2$ is not real, is called an elliptic function.

If $w_1/w_2$ is an integer, then the function is periodic. If $w_1/w_2$ is an irrational number, then the function must be constant.
Since the function is doubly periodic, it is enough to understand its behaviour within one period, *i.e.*, a parallelogram in the complex plane with vertices at the origin, $w_1$, $w_2$ and $w_1 + w_2$. Such a period is called a fundamental modular domain.

# Fundamental Modular Domains



The FMD is not unique. Any choice of $w_1$ and $w_2$ which reproduces all the lattice points as vertices of the corresponding FMD are allowed. Choose one according to convenience: for example no singularities on the boundaries.

# Analytic properties

The following theorems give fundamental properties of the elliptic functions:

1. **The number of poles in any FMD is finite**: otherwise a limit point would exist, which would be an essential singularity of the function.
2. **The number of zeroes in any FMD is finite**: otherwise there would be an essential singularity of the reciprocal of the function.
3. **The sum over residues in any FMD vanishes**: proven choosing an FMD which contains no poles on the boundaries and then using periodicity along with the Cauchy theorem.
4. **An elliptic function without poles is constant**: a special case of a more general theorem on meromorphic functions.

# The order of an elliptic function

## Defining the order

The order of an elliptic function $f(z)$ is the number of roots of $f(z) = c$ within one fundamental modular domain.

The Cauchy theorem can be used to show that the number of zeroes of the above equation is equal to the number of poles. Since every pole of $f(z) - c$ is also a pole of $f(z)$, the definition of the order does not depend on $c$.
The order of an elliptic function must be at least 2, otherwise the sum over residues cannot vanish.
The Jacobi elliptic functions are of order 2 and consist of two simple poles in each FMD. The Weierstrass elliptic functions are also of order 2 and contain an irreducible double pole. These are the only elliptic functions of order 2.